Gravitational waves from phase transitions

1. Thermodynamics and hydrodynamics in the early Universe

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Outline

Phase transitions & cosmology

Phase transitions in early Universe:

Thermal Changing $T(t)$ | Vacuum Changing field $\sigma(t)$

- **QCD phase transition**
  - Thermal (Confinement of strong interactions: quarks & gluons $\rightarrow$ hadrons)
- **Electroweak phase transition**
  - Thermal (First order: electroweak baryogenesis[1])
  - Vacuum: cold electroweak baryogenesis[2]
- **Grand Unified Theory & other high-scale phase transitions**
  - Thermal: topological defects[3]
  - Vacuum: hybrid inflation, topological defects, ... [4]

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(1) Kuzmin, Rubakov, Shaposhnikov 1988
(2) Smit and Tranberg 2002-6; Smit, Tranberg & Hindmarsh 2007
(3) Kibble 1976; Zurek 1985, 1996; Hindmarsh & Rajantie 2000
(4) Copeland et al 1994; Kofman, Linde, Starobinsky 1996

Phase transitions and gravitational waves

- GWs require shear stress $\implies$ departure from equilibrium[5]
- e.g. 1st order phase transition – c.f. water boiling
- What frequency GWs can we expect from a phase transition?
- Suppose process happens at a rate β at time t. Causality: $(H/\beta) \lesssim 1$

$$\text{Frequency today:} \qquad f_0 \simeq \frac{a(t_0)}{a(t)}\beta$$

Event	T	t	f_0
QCD transition	100 MeV	10^{-3} s	$10^{-8}(\beta/H)$ Hz
Electroweak transition	100 GeV	10^{-11} s	$10^{-5}(\beta/H))$ Hz
GUT/Hybrid inflation	$< 10^{16}$ GeV	$> 10^{-36}$ s	$< 10^{8}(\beta/H)$ Hz

- Electroweak transition most interesting for LISA
- QCD transition most interesting for Pulsar Timing Arrays

[5] Eqm g-wave production is small (Ghiglieri, Laine 2015)

Conventions

- Natural Units: $\hbar = 1$, $c = 1$, $k_B = 1$
- Natural Unit converter:

Quantity	Nat. U.	S.I. Conversion	
Energy:	GeV	1.6022×10^{-10}	Joule
Temperature:	GeV	1.1605×10^{13}	K
Mass:	GeV	1.7827×10^{-27}	kg
Length:	GeV^{-1}	1.9733×10^{-16}	m
Time:	GeV^{-1}	6.5822×10^{-25}	s

- Planck Mass (Energy): $M_P = \sqrt{\hbar c^5/G} = 1.2211 \times 10^{19}$ GeV
- Reduced Planck Mass $m_P = \sqrt{\hbar c^5/8\pi G} = 2.436 \times 10^{18}$ GeV
- $\bar{d}p = \frac{dp}{2\pi}$
- $\bar{\delta}(p) = 2\pi\delta(p)$
- Metric $- + + +$

Thermodynamics of harmonic oscillators 1: bosons

Partition function:

$$Z = \mathrm{Tr}[e^{-\beta\hat{H}}]$$

Leading to:

$$\begin{array}{rrcl} \text{free energy} & F & = & -T\ln Z \\ \text{entropy} & S & = & -\partial F/\partial T \\ \text{energy} & E & = & Z^{-1}\,\mathrm{Tr}[\hat{H}e^{-\beta\hat{H}}] = F + TS \end{array}$$

Bosonic harmonic oscillator

- $\hat{H} = \frac{1}{2}\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$
- $[\hat{a}, \hat{a}^\dagger] = 1$
- $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$
- $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$,

B.h.o. partition function

$$\begin{aligned} Z_{\mathrm{Bho}} &= \textstyle\sum_{n=0}^{\infty}\langle n|e^{-\beta\hat{H}}|n\rangle \\ &= \textstyle\sum_{n=0}^{\infty}\exp[-\beta\omega(n+\frac{1}{2})] \\ &= e^{-\beta\omega/2}/(1-e^{-\beta\omega}) \end{aligned}$$

$$\boxed{F_{\mathrm{Bho}} = \frac{1}{2}\omega + T\ln(1-e^{-\beta\omega})}$$

Free scalar field

Field operator:

$$\hat{\phi}(x) = \int \frac{\bar{d}^3 k}{2\omega_{\mathbf{k}}} \left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}^\dagger_{\mathbf{k}} e^{ik\cdot x} \right), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 2\omega_{\mathbf{k}}\, \bar{\delta}^3(\mathbf{k} - \mathbf{k}').$$

Field equation:

$$(\Box - m^2)\hat{\phi}(x) = 0 \qquad \Longrightarrow \qquad (k^0)^2 = {\omega_{\mathbf{k}}}^2 = k^2 + m^2$$

$\boxed{\text{Free scalar field is a collection of harmonic oscillators, one for each momentum } \mathbf{k}}$

Partition function: $Z_B = \prod_{\mathbf{k}} Z_{\mathrm{Bho}}$
Free energy: $F_B = -T \sum_{\mathbf{k}} \ln Z_{\mathrm{Bho}} \implies \boxed{F_B = \sum_{\mathbf{k}} \left(\frac{1}{2}\omega_{\mathbf{k}} + T \ln(1 - e^{-\beta\omega_{\mathbf{k}}}) \right)}$
Quantum statistics of fields: $\sum_{\mathbf{k}} \to V \int \bar{d}^3 k$

Thermodynamics of harmonic oscillators 2: fermions

Partition function:

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$$

Fermionic harmonic oscillator

- $\hat{H} = \frac{1}{2}\omega(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)$
- $\{\hat{a}, \hat{a}^\dagger\} = 1$
- $\hat{a}|0\rangle = 0, \hat{a}|1\rangle = |0\rangle$
- $\hat{a}^\dagger|0\rangle = |1\rangle, \hat{a}^\dagger|1\rangle = 0,$

F.h.o. partition function

$$\begin{aligned} Z_{\mathrm{Fho}} &= \sum_{n=0}^{1} \langle n| e^{-\beta \hat{H}} |n\rangle \\ &= \sum_{n=0}^{1} \exp[-\beta\omega(n + \tfrac{1}{2})] \\ &= e^{\beta\omega/2}/(1 + e^{-\beta\omega}) \end{aligned}$$

$$\boxed{F_{\mathrm{Fho}} = -\tfrac{1}{2}\omega - T\ln(1 + e^{-\beta\omega})}$$

Free fermionic field

Field operator (Dirac 4-component field):

$$\hat{\psi}(x) = \int \frac{\bar{d}^3 k}{2\omega_{\mathbf{k}}} \left(u_A(\mathbf{k}) \hat{b}^A_{\mathbf{k}} e^{-ik\cdot x} + \bar{v}_A(\mathbf{k}) \hat{d}^{A\dagger}_{\mathbf{k}} e^{ik\cdot x} \right), \quad \begin{array}{ll} \{\hat{b}^A_{\mathbf{k}}, \hat{b}^{B\dagger}_{\mathbf{k}'}\} & = 2\omega_{\mathbf{k}} \delta^{AB}\, \bar{\delta}^3(\mathbf{k}-\mathbf{k}') \\ \{\hat{d}^A_{\mathbf{k}}, \hat{d}^{B\dagger}_{\mathbf{k}'}\} & = 2\omega_{\mathbf{k}} \delta^{AB}\, \bar{\delta}^3(\mathbf{k}-\mathbf{k}') \end{array}$$

Field equation:

$$(i\gamma^\mu \partial_\mu + m)\hat{\psi}(x) = 0 \qquad \Longrightarrow \qquad \begin{array}{l} (k^0)^2 = {\omega_{\mathbf{k}}}^2 = k^2 + m^2 \\ (\not{k} - m) u_A(\mathbf{k}) = 0 \\ (\not{k} + m) \bar{v}_A(\mathbf{k}) = 0 \end{array}$$

Free fermionic field is a collection of harmonic oscillators, 4 for each momentum **k**

Partition function: $Z_F = \prod_{\mathbf{k}} Z_{\mathrm{Fho}}$

Free energy: $F_F = -T \sum_{\mathbf{k}} \ln Z_{\mathrm{Fho}} \implies \boxed{F = \sum_{\mathbf{k}} \left(-\frac{1}{2}\omega_{\mathbf{k}} - T\ln(1 + e^{-\beta\omega_{\mathbf{k}}})\right)}$

Quantum statistics of fields: $\sum_{\mathbf{k}} \to V \int \bar{d}^3 k$

Free energy (density) of an ideal gas

Free relativistic particles of mass m in equilibrium (zero chemical potential)

$$f = -\eta T \int \bar{d}^3 k \ln(1 + \eta e^{-E/T})$$

where $\eta = \pm 1$ (Fermi-Dirac/Bose-Einstein).

- Entropy density: $s = -\frac{\partial f}{\partial T}$
- Energy density: $e = f + Ts$
- Thermodynamic pressure: $p = Ts - e$ (Note $p = -f$)

To find equilibrium state we minimise free energy

- Dimensions: $f = T^4 \phi(m/T)$ with $\phi(0) = -g_{\text{eff}} \pi^2/90$.

Defines effective number of relativistic degrees of freedom g_{eff}.

Free energy: exact formulae in high T expansion

Bosons:

$$f_B = -\frac{\pi^2}{90}T^4 + \frac{m^2T^2}{24} - \frac{(m^2)^{\frac{3}{2}}T}{12\pi} - \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{a_bT^2}\right) - \frac{m^4}{16\pi^{\frac{5}{2}}}\sum_\ell(-1)^\ell\frac{\zeta(2\ell+1)}{(\ell+1)!}\left(\frac{m^2}{4\pi^2T^2}\right)^\ell$$

Fermions:

$$f_F = -\frac{\pi^2}{90}\frac{7}{8}T^4 + \frac{m^2T^2}{48} + \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{a_fT^2}\right) + \frac{m^4}{16\pi^{\frac{5}{2}}}\sum_\ell(-1)^\ell\frac{\zeta(2\ell+1)}{(\ell+1)!}(1-2^{-2\ell-1})\Gamma(\ell+\tfrac{1}{2})\left(\frac{m^2}{4\pi^2T^2}\right)^\ell$$

$a_b = 16\pi^2\ln(\frac{3}{2}-2\gamma_E)$, $a_f = a_b/16$, $\gamma_E = 0.5772\ldots$ (Euler's constant)

Effective potential for scalar field with gauge fields and fermions

Let scalar field give masses to

- scalars ($M_S(\bar{\phi})$),
- vectors ($M_V(\bar{\phi})$)
- (Dirac) fermions ($M_F(\bar{\phi})$)

Define effective potential $V_T(\bar{\phi}) = V_0(\bar{\phi}) + f(\bar{\phi}) + g_{\text{eff}}\pi^2 T^4/90$

$$\begin{aligned}
V_T(\bar{\phi}) &= V_0(\bar{\phi}) + \frac{T^2}{24}\left(\sum_S M_S^2(\bar{\phi}) + 3\sum_V M_V^2(\bar{\phi}) + 2\sum_F M_F^2(\bar{\phi})\right) \\
&- \frac{T}{12\pi}\left(\sum_S (M_S^2(\bar{\phi}))^{\frac{3}{2}} + 3\sum_V (M_V^2(\bar{\phi}))^{\frac{3}{2}}\right) \\
&+ \frac{1}{64\pi^2}\sum_S M_S^4(\bar{\phi}) \ln\left(\frac{M_S^2}{a_b T^2}\right) + \frac{3}{64\pi^2}\sum_V M_V^4(\bar{\phi}) \ln\left(\frac{M_V^2}{a_b T^2}\right) \\
&- \frac{2}{64\pi^2}\sum_F M_F^4(\bar{\phi}) \ln\left(\frac{M_F^2}{a_f T^2}\right) + \cdots
\end{aligned}$$

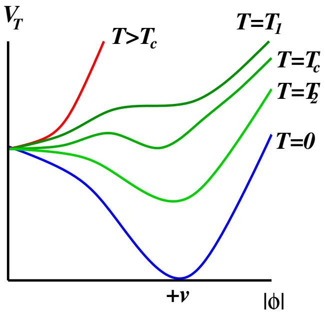
Neglect higher order terms where $M^2(\phi)/T^2 \ll 1$.

Phase transition (weakly coupled field theory)

Effective potential: expand in $\bar{\phi}/T$

$$V_T \simeq \frac{D}{2}(T^2 - T_0^2)|\bar{\phi}|^2 - \frac{A}{3}T|\bar{\phi}|^3 + \frac{\lambda_T}{4!}|\bar{\phi}|^4$$

- High temperature: equilibrium at $\bar{\phi} = 0$.
- Second minimum develops at T_1, $\phi_b(T)$.
- Critical temperature T_c: $f(0) = f(\bar{\phi}_b)$.
- System can supercool below T_c (until T_0).
- First order transition (apparently)
- Latent heat $\mathcal{L} = T_c \Delta s(T_c)$
- 1st order from cubic term (bosons only)



Degrees of freedom of SM: mostly coloured

		$M(T=0)$	g		$M(T=0)$	g	
	γ	0	2	g	0	16	
	ν_e	$\lesssim$ 1 eV	2	u	3 MeV	12	
	ν_μ	$\lesssim$ 1 eV	2	d	7MeV	12	
	ν_τ	$\lesssim$ 1 eV	2	s	76 MeV	12	
	e	0.5 MeV	4	c	1.2 GeV	12	
	μ	106 MeV	4	b	4.2 GeV	12	
	τ	1.7 GeV	4	t	174 GeV	12	
	W	80 GeV	6				
	Z	91 GeV	3				
	h	125 GeV	1				
>1 TeV:			$\frac{7}{8}18+8$			$\frac{7}{8}72+16$	72/106.75
40 GeV:			$\frac{7}{8}18+2$			$\frac{7}{8}60+16$	68.5/84.25
0.4 GeV:			$\frac{7}{8}14+2$			$\frac{7}{8}36+16$	47.5/61.75

QCD interactions important, especially around 1GeV

W, Z, t, h contribute most to V_T around 100GeV: largest mass change

Standard Model effective potential in weak coupling approximation

	h	$W^{\pm}$	Z	t
M/GeV	125	80.4	91.2	174
d.o.f.	1	6	3	$\frac{7}{8}12$
$M(\bar{\phi})$	$\sqrt{V_0''(\bar{\phi})}$	$\frac{1}{2}g_w\bar{\phi}$	$\frac{1}{2}\sqrt{g_w^2+g'^2}\bar{\phi}$	$\sqrt{2}y_t\bar{\phi}$

Form of effective potential: $V_T \simeq \frac{D}{2}(T^2 - T_0^2)|\bar{\phi}|^2 - \frac{A}{3}T|\bar{\phi}|^3 + \frac{\lambda_T}{4!}|\bar{\phi}|^4$

$$D = \frac{1}{12\bar{\phi}^2}\left(6M_W^2 + 3M_Z^2 + 6M_t^2\right) \qquad A = \frac{1}{12\pi\bar{\phi}^2}\left(6M_W^3 + 3M_Z^3\right)$$

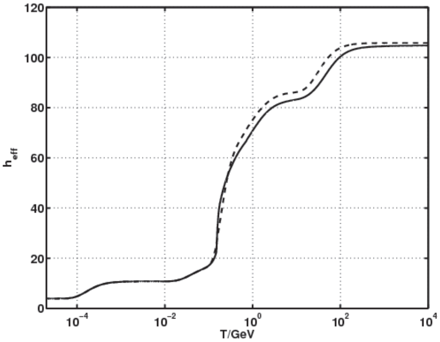
$$\lambda_T = \lambda - \frac{1}{16\pi^2\bar{\phi}^4}\left(6M_W^4\ln\left(\frac{M_W^2}{a_bT^2}\right) + 3M_Z^4\ln\left(\frac{M_Z^2}{a_bT^2}\right) - 4M_t^4\ln\left(\frac{M_t^2}{a_fT^2}\right)\right)$$

$\boxed{\text{Predicts: } T_c = 166\text{ GeV},\ T_0 = 165\text{ GeV}}$

Transition is very weak.

Standard Model effective degrees of freedom

Ideal gas, model QCD transition[(6)] (dashed)
With interactions, lattice QCD[(7)] (solid)



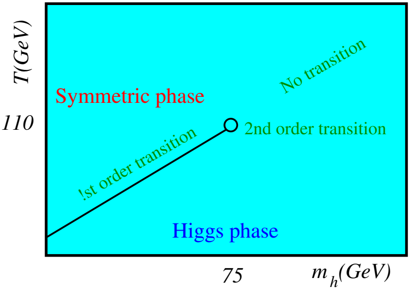
Temp.	Event
100 GeV	t non-relativistic
1 GeV	b non-relativistic
500 GeV	c, τ non-relativistic
200 MeV	QCD phase transition
30 MeV	μ non-relativistic
2 MeV	ν freeze-out
0.2 MeV	e non-relativistic
1 eV	matter = radiation
0.1 eV	photon decoupling

(6) Olive 1981
(7) Hindmarsh & Philipsen 2005, Laine & Schroder 2006, Borsanyi et al 2016

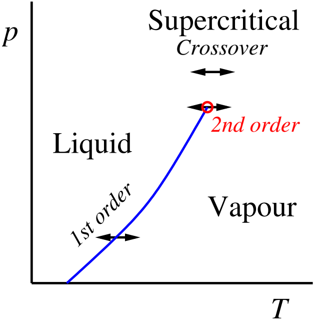
Electroweak phase transition in the Standard Model

Interactions are important!

Standard Model phase diagram



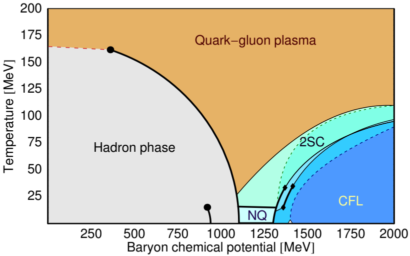
Kajantie et al 1996
SM is cross-over



Water phase diagram (sketch)

QCD phase diagram

- $\eta_B = n_B/n_\gamma = (6.10 \pm 0.04) \times 10^{-10}$ (Planck)[8]
- Low $\eta_B \implies$ low chemical potential

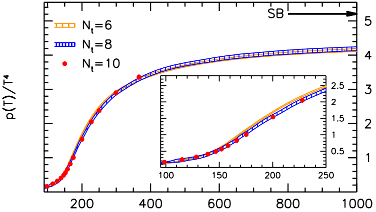


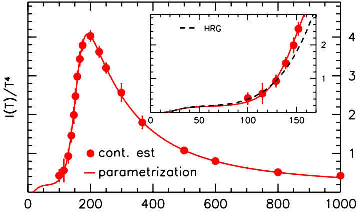
Ruester et al hep-ph/0503184

[8] Ade et al 2015

QCD equation of state

- Budapest-Marseille-Wuppertal lattice (physical quark masses)[9]
- Shown: pressure and trace anomaly $I(T) = \rho(T) - 3p(T)$ (with fit)





- Can model with hadronic resonance gas at low T

[9] Borsányi et al. (2010)

1st order phase transitions in SM extensions

- 2HDM (2 Higgs doublet model)
  - Extra scalars (A^0, H^0, $H^\pm$) increase strength of cubic term.
  - Strong phase transition when $m_{A^0} \gtrsim 400$ GeV[10]
- Extra singlet scalars
  - Tree level first order phase transition
  - Strong phase transition with SM-like phenomenology allowed[11]
- Effective field theory with h^6 operator[12]
  - e.g. by integrating out singlet[13]
  - $V_T(\phi) \simeq c_0 + c_1(T)h^2 + c_2 h^4 + c_3 h^6 + \cdots$
  - $c_2 < 0$ gives 1st order transition at tree level.
- etc. etc. etc.

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(10) Dorsch, Huber, No (2015)
(11) Ashoorioon, Konstandin (2009)
(12) Grojean, Servant, Wells (2005)
(13) Huber et al (2006)

Standard Model plasma: semiclassical approximation

	h	$W^{\pm}$	Z	t
M/GeV	125	80.4	91.2	174
Γ/GeV	4×10^{-3} (*)	2.1	2.5	1.4
d.o.f.	1	6	3	$\frac{7}{8}12$

(*) calculated from SM, not yet measured

- W, Z, t, h have largest mass change: $g_{\text{eff}} = 20.5$
- Each have frequent scatterings with "light" particles $g_{\text{eff}} = 86.25$
- Relatively narrow width of important particles
- Scattering more rapid than decays: semi-classical particles

Relativistic Boltzmann equation

- Distribution function[14] (Lorentz scalar): $f(p, x)$
- Average number of particles in phase space volume element at $(\mathbf{p}, \mathbf{x}, t)$
- $p^0 = E_\mathbf{p} = \sqrt{(\mathbf{p}^2 + m^2)}$ is not independent

number density	$n(x)$	$\int \bar{d}^3 p f(p, x)$
particle flux	$j^i(x)$	$\int \bar{d}^3 p \frac{p^i}{E} f(p, x)$
energy density	$e(x)$	$\int \bar{d}^3 p E f(p, x)$
momentum density	$\Pi^i(x)$	$\int \bar{d}^3 p\, p^i f(p, x)$
momentum flux (j direction)	$\Pi^{ij}(x)$	$\int \bar{d}^3 p\, p^i \frac{p^j}{E} f(p, x)$

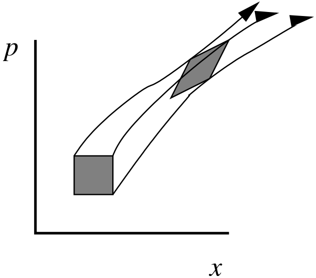
Organise into 4-vector and 4-tensor:

$$j^\mu = \int \frac{\bar{d}^3 p}{2E} 2p^\mu f(p, x) \qquad T^{\mu\nu} = \int \frac{\bar{d}^3 p}{2E} 2p^\mu p^\nu f(p, x)$$

Manifestly covariant form: $\int \frac{\bar{d}^3 p}{2E} = \int \bar{d}^4 p \theta(p^0)\, \bar{\delta}(p^2 + m^2)$

[14] Bad notation: not to be confused with free energy density

Particle flow in phase space with forces



$$\begin{aligned} x^\mu &\rightarrow x^\mu + \frac{dX^\mu}{d\tau}\Delta\tau \\ p^\mu &\rightarrow p^\mu + F^\mu \Delta\tau \end{aligned}$$

Force must preserve $p^2 + m^2 = 0$

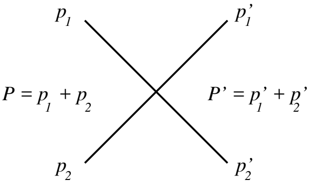
- $F^\mu p_\mu = 0$
- or $F^\mu + \partial^\mu m(x) = 0$

Without collisions: $f(p + F\Delta\tau, x + \frac{1}{m}p\Delta\tau) = f(p, x)$
Hence

$$\boxed{\left(p^\mu \partial_\mu + mF^\mu \frac{\partial}{\partial p^\mu}\right)\theta(p^0)\delta(p^2 + m^2)f(p, x) = 0}$$

where p^μ are independent in $f(p, x)$.

Particle flow in phase space with collisions



Described by scattering function:

$$W(p_1, p_2|p_1', p_2') = s\sigma(s, \Theta)\delta(P' - P)$$

$$\cos\Theta = 1 + 2t/(s - 4m^2)$$
$$s = (p_1 + p_2)^2,\ t = (p_1 - p_1')^2$$
$$W(p_1, p_2|p_1', p_2') = W(p_1', p_2'|p_1, p_2)$$

$R(p, x) d^4x \dfrac{\bar{d}^3 p}{2E}$ – Scatterings in which one of the initial particles has momentum p at space-time point x

$R'(p, x) d^4x \dfrac{\bar{d}^3 p}{2E}$ – Scatterings in which one of the final particles has momentum p at space-time point x

$$\boxed{p^\mu \partial_\mu f(p, x) = C[f]} = R'(p, x) - R(p, x) =$$

Classical statistics:

$$R(p, x) = \int \frac{\bar{d}^3 p_2}{2E_2} \frac{\bar{d}^3 p_1'}{2E_1'} \frac{\bar{d}^3 p_2'}{2E_2'} f(p_1, x) f(p_2, x) W(p_1, p_2|p_1', p_2')$$

Collision invariants and conservation laws

2-body collisions conserve

- Particle number
- Momentum

Can show

$$\psi(x) = a(x) + b_\mu(x) p^\mu$$

$$\int \frac{\bar{d}^3 p}{2E} \psi(x) C[f] = 0$$

for arbitrary $a(x)$, $b(x)$.

$\times$ both sides of $p^\mu \partial_\mu f(p,x) = C[f]$ by ψ and integrate over momentum space

$$b_\mu = 0 \implies \int \frac{\bar{d}^3 p}{2E} p^\mu \partial_\mu f = 0 \implies \tfrac{1}{2}\partial_\mu \int \bar{d}^3 p \frac{p^\mu}{E} f = 0 \implies \boxed{\partial_\mu j^\mu = 0}$$

$$a = 0 \implies \int \frac{\bar{d}^3 p}{2E} p^\nu p^\mu \partial_\mu f = 0 \implies \tfrac{1}{2}\partial_\mu \int \bar{d}^3 p \, p^\nu \frac{p^\mu}{E} f = 0 \implies \boxed{\partial_\mu T^{\mu\nu} = 0}$$

Equilibrium distribution (classical statistics)

Recall $p^\mu \partial_\mu f(p,x) = R'(p,x) - R(p,x)$ with:

$$R(p,x) = \int \frac{\bar{d}^3 p_2}{2E_2} \frac{\bar{d}^3 p'_1}{2E'_1} \frac{\bar{d}^3 p'_2}{2E'_2} f(p_1,x) f(p_2,x) W(p_1,p_2|p'_1,p'_2)$$

$W(p_1,p_2|p'_1,p'_2) = W(p'_1,p'_2|p_1,p_2)$

Local equilibrium (vanishing collision term) is established if

$$f(p_1,x)f(p_2,x) = f(p'_1,x)f(p'_2,x) \quad \text{for } \textbf{all} \quad (p_a,p'_a)$$

Hence

$$\log f_1 + \log f_2 = \log f'_1 + \log f'_2 \quad \text{for } \textbf{all} \quad (p_a,p'_a)$$

$\log f_1 + \log f_2$ is a conserved quantity, and must be $\propto \psi(x) = a(x) + b_\mu(x) p^\mu$

$$f^{\mathrm{eq}}(p,x) = \exp[a(x) + b_\mu(x) p^\mu]$$

Identify: $a = \beta(x)\mu(x)$, $b_\mu = \beta(x) U_\mu(x)$

μ chemical potential – β inverse temperature – U^μ 4-velocity

Equilibrium distribution (quantum statistics)

With quantum statistics:

$$R(p,x) = \int \frac{\bar{d}^3 p_2}{2E_2}\frac{\bar{d}^3 p_1'}{2E_1'}\frac{\bar{d}^3 p_2'}{2E_2'} f_1 f_2 (1 \pm f_1)(1 \pm f_2) W(p_1, p_2 | p_1', p_2')$$

Bose enhancement
Fermi blocking

Local equilibrium (vanishing collision term) is established if

$$f_1 f_2 (1 \pm f_1)(1 \pm f_2) = f_1' f_2' (1 \pm f_1')(1 \pm f_2') \quad \text{for } \mathbf{all} \quad (p_a, p_a')$$

Hence

$$\log f_1 (1 \pm f_1) + \log f_2 (1 \pm f_2) = \log f_1'(1 \pm f_1') + \log f_2'(1 \pm f_2') \quad \text{for } \mathbf{all} \quad (p_a, p_a')$$

Now $\log f(1 \pm f)$ is conserved quantity $\propto \psi(x) = a(x) + b_\mu(x) p^\mu$

$$f^{\text{eq}}(p,x) = (\exp[a(x) + b_\mu(x) p^\mu \pm 1)^{-1}$$

Identify: $a = \beta(x)\mu(x)$, $b_\mu = \beta(x) U_\mu(x)$
μ chemical potential – β inverse temperature – U^μ 4-velocity

Fluid energy-momentum tensor

Distribution function for system in local equilibrium:

$$f^{\mathrm{eq}}(p,x) = \frac{1}{e^{\beta(U_\mu p^\mu - \mu)} \pm 1}$$

Energy-momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= \int \frac{\bar{d}^3 p}{2E} 2p^\mu p^\nu f^{\mathrm{eq}}(p,x) \\ T^{\mu\nu} &= (e+p)U^\mu U^\nu + p g^{\mu\nu} \end{aligned}$$

where

$$\begin{aligned} e &= \int \bar{d}^3 p \, E \, f_0^{\mathrm{eq}}(p,x) \quad \text{rest frame energy density} \\ p &= \int \bar{d}^3 p \, \frac{\mathbf{p}^2}{3E} \, f_0^{\mathrm{eq}}(p,x) \quad \text{rest frame (kinetic) pressure} \end{aligned}$$

EM (non)-conservation for particles with field-dependent mass

$$\left(p^\mu\partial_\mu + mF^\mu\frac{\partial}{\partial p^\mu}\right)\theta(p^0)\delta(p^2+m^2)f(p,x) = C[f]$$

- $\times$ both sides by p^ν and integrate over momenta
- Assume collisions occur "at a point" and still conserve momentum

$$\tfrac{1}{2}\partial_\mu T^{\mu\nu} + mF^\mu \int \bar{d}^4p\, p\, p^\nu \tfrac{\partial}{\partial p^\mu}\theta(p^0)\delta(p^2+m^2)f(p,x) = 0$$

Integration by parts, $F^\mu = -\partial^\mu m = \partial^\mu\bar\phi\, dm/d\bar\phi$

$$\partial_\mu T^{\mu\nu} = -\partial^\nu\bar\phi\,\frac{dm^2}{d\bar\phi}\int\frac{\bar{d}^3p}{2E}f(p,x)$$

Fluid coupled to scalar field through mass 1

Model for the system near phase transition[15]

$$\begin{aligned} \text{fluid} \quad T_{\mathrm{f}}^{\mu\nu} &= (e+p)U^\mu U^\nu + p g^{\mu\nu} \\ \text{field} \quad T_{\phi}^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(\tfrac{1}{2}(\partial\phi)^2 + V_0(\phi)\right) \end{aligned}$$

- Note: $p = g_{\mathrm{eff}}\pi^2 T^4/90 - \Delta V_T(\phi)$ i.e. minus free energy of the fluid
- Conservation of energy-momentum: $\partial_\mu \left(T_{\mathrm{f}}^{\mu\nu} + T_{\phi}^{\mu\nu} \right) = 0$

Hence non-conservation of $T_{\mathrm{f}}^{\mu\nu}$ must appear in $T_{\phi}^{\mu\nu}$

$$\partial_\mu T_{\phi}^{\mu\nu} = +\partial^\nu\bar{\phi}\,\frac{dm^2}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} f(p,x)$$

Implies for scalar field equation[16]

$$\Box\phi - V_0'(\phi) = \frac{dm}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} f(p,x)$$

[15] Ignatius, Kajantie, Kurki-Suonio, Rummukainen 1991
[16] Also derivable from field theory, see Moore & Prokopec 1996

Fluid coupled to scalar field through mass 2

$$\Box\phi - V_0'(\phi) = \frac{dm}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} f(p,x)$$

Write $f = f^{\mathrm{eq}} + \Delta f$

$$\Box\phi - V_0'(\phi) = \Delta V_T(\phi) + \frac{dm}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} \Delta f(p,x)$$

Put equilibrium part on LHS:

$$\boxed{\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} \Delta f(p,x)}$$

Repackage all effective potential into fluid EM: $p \to p = g_{\mathrm{eff}}\pi^2 T^4/90 - V_T(\phi)$

$$\boxed{\partial_\mu T_{\mathrm{f}}^{\mu\nu} + \partial^\nu\phi \frac{\partial V_T(\phi)}{\partial\phi} = -\partial^\nu\phi \frac{dm^2}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} \Delta f(p,x)}$$

IKKR model and entropy generation

$$\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} \Delta f(p, x)$$

- Near equilibrium RHS a function of dynamical variables β, U^μ, (μ), ϕ
- Field gradients disturb eqm: expect RHS $\sim \partial_\mu \phi$
- Isotropy: expect RHS $\sim U^\mu \partial_\mu \phi$
- Field comes from $m^2(\phi)$ so $\partial_\mu \phi \to \beta \partial_\mu m^2$

Suggests:

$$\boxed{\Box\phi - V_T'(\phi) = \eta_T(\phi) U \cdot \partial\phi \qquad \text{with} \qquad \eta_T(\phi) = \tilde{\eta}\beta\phi^2}$$

Can show that entropy generation is always positive Exercise!:

$$\partial^\mu S^\mu = \tilde{\eta}(\beta\phi)^2 (U \cdot \partial\phi)^2 \geq 0$$

Entropy current $S^\mu = sU^\mu$, $s = dp/dT$

Summary

- Electroweak symmetry is broken at $T \simeq 100$ GeV
- Standard Model plasma at $T \simeq 100$ GeV:
  weakly-interacting and long-lived W, Z, t, h + “bath" of light particles
- In semi-classical picture SM phase transition is 1st order (just)
- Interactions (non-Abelian gauge bosons) $\to$ cross-over
- Beyond the Standard Model: more scalars $\to$ 1st order phase transition
- Model of coupled order-parameter ϕ and fluid $T_{\mathrm{f}}^{\mu\nu}$

$$\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} \Delta f(p, x) \qquad \simeq \tilde{\eta} \frac{\phi^2}{T}(U \cdot \partial\phi)$$

$$\partial_\mu T_{\mathrm{f}}^{\mu\nu} + \partial^\nu\phi \frac{\partial V_T(\phi)}{\partial\phi} = -\partial^\nu\phi \frac{dm^2}{d\bar{\phi}} \int \frac{\bar{d}^3 p}{2E} \Delta f(p, x) \simeq \tilde{\eta} \frac{\phi^2}{T}(U \cdot \partial\phi)\partial^\nu\phi$$

Where $p = g_{\mathrm{eff}}\pi^2 T^4/90 - V_T(\phi)$, $\Delta f(p, x) = f(p, x) - f^{\mathrm{eq}}(p, x)$

Reading

Statistical physics

- *Statistical Mechanics*, K Huang (Wiley, 1987)

Thermal quantum field theory

- *Basics of Thermal Field Theory*, M. Laine and A. Vuorinen (Springer, 2016) [arXiv:1701.01554]

Relativistic hydrodynamics

- *Relativistic Hydrodynamics*, L. Rezzolla and O. Zanotti (OUP, 2013)

Scalar field coupled to a fluid

- *From Boltzmann equations to steady wall velocities*, T. Konstantin, G. Nardini, I. Rues [arXiv:1407.3132]
- *Energy Budget of Cosmological First-order Phase Transitions*, J.R. Espinosa, T. Konstantin, J.M. No, G. Servant [arXiv:1004.4187]
- *Growth of bubbles in cosmological phase transitions*, J. Ignatius, K. Kajantie, H. Kurki-Suonio, M. Laine [arXiv:astro-ph/9309059]