The reachable space of the heat equation and spaces of analytic functions in a square

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Problem Statement

$$(BCH)\quad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geqslant 0,\ \ x \in (0,\pi), \\ w(t,0) = u_0(t), \quad w\left(t,\pi\right) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in \left(0,\pi\right), \end{cases}$$

Given $\tau > 0$, define the *input to state map*

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau,\cdot) \qquad (\tau > 0,\ \ u_0,\ \ u_\pi \in L^2[0,\tau]),$$

Basic question: what can we say about $\operatorname{Ran}\Phi_\tau$?

Existing results

Given $\tau > 0$ it is known that:

- $\operatorname{Ran}\Phi_\tau \subset W^{-1,2}(0,\pi)$;
- $\operatorname{Ran}\Phi_\tau \subset \mathrm{HOL}(D)$, where $D = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\}$;
- $\operatorname{Ran}\Phi_\tau \supset \{\psi \in \mathrm{HOL}(S) \mid \psi^{(2k)}(0) = \psi^{(2k)}(\pi) = 0 \text{ for } k \in \mathbb{N}\}$, where $S = \{s = x + iy \in \mathbb{C} \mid |y| < \pi\}$ (Fattorini and Russell, 1971);
- $\operatorname{Ran}\Phi_\tau \supset \mathrm{HOL}(B)$, where $B = \left\{s \in \mathbb{C} \mid \left|s - \frac{\pi}{2}\right| < \frac{\pi}{2}\mathrm{e}^{(2\mathrm{e})^{-1}}\right\}$ (Martin, Rosier and Rouchon, 2016);
- For every $\varepsilon > 0$ we have $\operatorname{Ran}\Phi_\tau \supset \mathrm{HOL}(D_\varepsilon)$, where D_ε is an ε-neighbourhood of the square D (Dardé and Ervedoza, 2016).

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Main new results

Proposition 1. *For every* $\tau > 0$ *we have* $\mathrm{Ran}\,\Phi_\tau \subset A^2(D)$, *where*

$$A^2(D) = \{f \in \mathrm{HOL}(D) \ \mid \ \int_D |f(x+iy)|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty\}.$$

Theorem 1. *For every* $\tau > 0$ *we have* $\mathrm{Ran}\,\Phi_\tau \supset E^2(D)$, *where*

$$E^2(D) = \{f \in \mathrm{HOL}(D) \ \mid \ \int_{\partial D} |f(\zeta)|^2 \,|\mathrm{d}\zeta| < \infty\}.$$

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Outline

- Some background on control operators
- The 1D heat equation with boundary control: various representations of the solution
- Proof of the regularity result
  - A result of Aikawa, Hayashi and Saitoh
  - Proof of Proposition 1
- Proof of the main result
  - A result of Levin and Ljubarskii
  - Proof of Theorem 1
- Extensions and comments

Some background on control operators

Some notation

We consider control systems described by equations of the form

$$(SE) \qquad \dot{z}(t) = Az(t) + Bu(t), \qquad \text{with}$$

- X (the state space) and U (the input space) are complex Hilbert spaces. We have $X = \mathbb{C}^n$ and $U = \mathbb{C}^m$ for finite-dimensional control systems.

- $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a strongly continuous semigroup on X generated by A. We have $\mathbb{T}_t = \mathrm{e}^{tA}$ for for finite-dimensional control systems. X_1 is $\mathcal{D}(A)$ endowed with the graph norm and X_{-1} is the dual of $\mathcal{D}(A^*)$ with respect to the pivot space X.

- $B \in \mathcal{L}(U; X_{-1})$ is the control operator.

Admissible control operators

The solution of (SE) writes:

$$z(t) = \mathbb{T}_t z(0) + \Phi_t u,$$

where $\mathbb{T}$ is the semigroup generated by A and

$$\Phi_t \in \mathcal{L}(L^2([0,\infty);U),X_{-1}), \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\,\mathrm{d}\sigma.$$

Definition. B *is called an admissble control operator for* $\mathbb{T}$ *if* $\operatorname{Ran}\Phi_t \subset X$ *for one (and hence all)* $t > 0$.

Example. *Take* $A = -A_0$ *with* $A_0 > 0$. *For* $\alpha > 0$, *denote* $X_\alpha = \mathcal{D}(A_0^\alpha)$ *and* $X_{-\alpha}$ *is the dual of* X_α *with respect to the pivot space* X. *Then every operator* $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$ *is admissible.*

Controllability types

(A, B) is said *exactly controllable in time* τ if $\operatorname{Ran} \Phi_\tau = X$.

(A, B) is said *null controllable in time* τ if $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$. This is equivalent to the existence, for eaxh $z_0 \in X$ of $u \in L^2[0, \tau]; U)$ such that the solution of

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0,$$

satisfies $z(\tau) = 0$.

(A, B) is *approximatively controllable in time* τ if $\overline{\operatorname{Ran} \Phi_\tau} = X$.

The three above concepts coincide with the usual controllability concept in the case of finite dimensional LTIs.

Null controllability and reachable space

Proposition. (Fattorini, Seidman) If (A, B) is null controllable in any tyime then $\operatorname{Ran}\Phi_\tau$ does not depend on $\tau > 0$.

Proof. Let $0 < \tau < t$, $\eta \in \operatorname{Ran}\Phi_\tau$ and let $\tilde{u} = \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_\pi \end{bmatrix}$ be a control such that $\tilde{w}(\tau, \cdot) = \eta$. Let u be the control defined by

$$u(\sigma) = \begin{cases} 0 & \text{for} \quad \sigma \in [0, t-\tau], \\ \tilde{u}(\sigma + \tau - t) & \text{for} \quad \sigma \in [t-\tau, t]. \end{cases}$$

Then $w(t, \cdot) = \eta$, thus $\operatorname{Ran}\Phi_\tau \subset \operatorname{Ran}\Phi_t$.

Let now $\eta \in \operatorname{Ran}\Phi_t$ and $\tilde{u}(\sigma) = u(\sigma + t - \tau)$, $\quad \tilde{w}(\sigma) = w(\sigma + t - \tau, \cdot)$. Then $\eta = w(t, \cdot) = \tilde{w}(\tau, \cdot) = \mathbb{T}_\tau \tilde{w}(0, \cdot) + \Phi_\tau \tilde{u}$. Since $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$, we have $\eta \in \operatorname{Ran}\Phi_\tau$, thus $\operatorname{Ran}\Phi_t \subset \operatorname{Ran}\Phi_\tau$.

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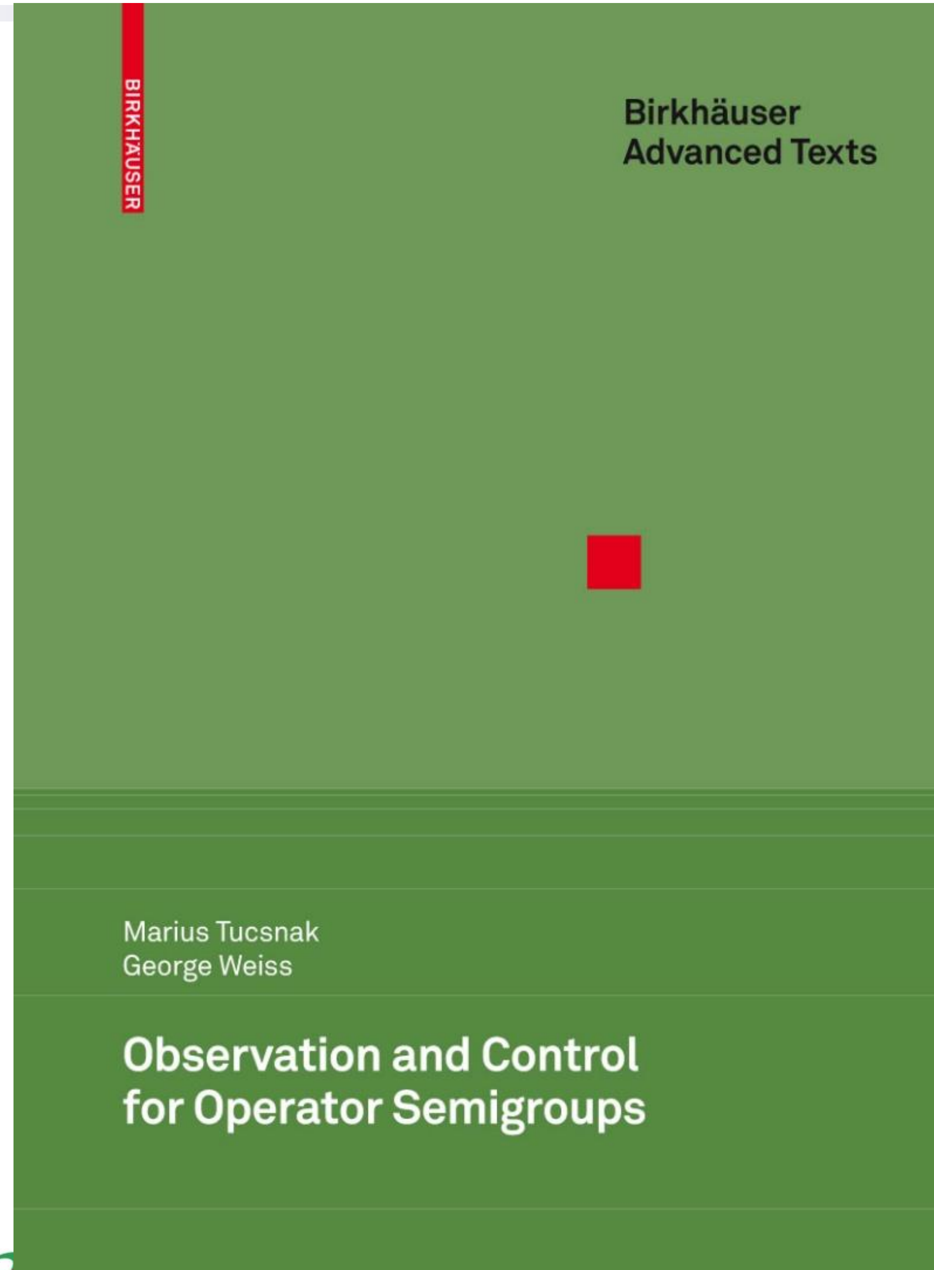
The commercial break:

for the controllability theory of linear time invariant systems one can see



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The 1D heat equation with boundary control:

various representations of the solution

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Fourier series expansion

Writting $\Phi_t u = w(t,\cdot)$ in the form $w(t,x) = \sum_{n\geqslant 1} w_n(t) \sin(n\pi x)$, we obtain the formula

$$(\Phi_\tau u)(x) = \frac{2}{\pi}\sum_{n\geqslant 1} n \left[\int_0^\tau e^{n^2(\sigma-\tau)} u_0(\sigma)\, d\sigma\right] \sin(nx)$$

$$+ \frac{2}{\pi}\sum_{n\geqslant 1} n(-1)^{n+1} \left[\int_0^\tau e^{n^2(\sigma-\tau)} u_\pi(\sigma)\, d\sigma\right] \sin(nx) \qquad (\tau > 0,\ x \in (0,\pi)),$$

The above representation is at the basis of the classical results of Fattorini and Russell who showed, in particular, the null controllability in any time.

The same formula does not seem useful to prove our new results.

Sum of Gaussians

Proposition. $(\Phi_\tau u)(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau-\sigma,x)u_0(\sigma)\,\mathrm{d}\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau-\sigma,x)u_\pi(\sigma)\,\mathrm{d}\sigma$, where

$$K_0(\sigma,x) = -\sqrt{\frac{1}{\pi\sigma}}\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\frac{(x+2m\pi)^2}{4\sigma}} \qquad (\sigma>0,\ x\in\mathbb{R}),$$

$$K_\pi(\sigma,x) = \sqrt{\frac{1}{\pi\sigma}}\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\frac{(x+(2m-1)\pi)^2}{4\sigma}} \qquad (\sigma>0, x\in\mathbb{R}),$$

Proof. Apply the Poisson summation formula to the Fourier series representation or use the method of images.

Proof of the regularity result

A decomposition

$$(\Phi_\tau u)(x) = (\tilde{\Phi}_\tau u)(x) + \left(\tilde{\tilde{\Phi}}_\tau u\right)(x) + \int_0^\tau \frac{\partial \tilde{K}_0}{\partial x}(\tau-\sigma, x) u_0(\sigma)\,\mathrm{d}\sigma$$
$$+ \int_0^\tau \frac{\partial \tilde{K}_\pi}{\partial x}(\tau-\sigma, x) u_\pi(\sigma)\,\mathrm{d}\sigma \qquad (x \in (0,\pi)),$$

where $(\tilde{\Phi}_\tau u)(s) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{\mathrm{e}^{-\frac{s^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{3/2}} s u(\sigma)\,\mathrm{d}\sigma, \quad (\tilde{\tilde{\Phi}}_\tau u)(s) = (\tilde{\Phi}_\tau u)(\pi - s)$

$$\tilde{K}_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}^*} \mathrm{e}^{-\frac{(x+2m\pi)^2}{4\sigma}}, \quad \tilde{K}_\pi(\sigma, x) = \sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}^*} \mathrm{e}^{-\frac{(x+(2m-1)\pi)^2}{4\sigma}}.$$

Since $\left|(s+k\pi)\mathrm{e}^{-\frac{(s+k\pi)^2}{4(\tau-\sigma)}}\right|^2 \leqslant ak^2\mathrm{e}^{\frac{-bk^2}{(\tau-\sigma)}}$ for $k \in \mathbb{Z}\setminus\{-1, 0\}$ and $s \in D$,
it suffices to show that the first two terms in the right-hand side of can be extended to a function holomorphic in $A^2(D)$.

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A result of Aikawa, Hayashi and Saitoh

Theorem. *Let* $\Delta = \left\{ s \in \mathbb{C} \mid -\frac{\pi}{4} < \arg s < \frac{\pi}{4} \right\}$. *For* $s \in \Delta$, $\tau > 0$ *and* $f \in L^2[0,\tau]$ *we set*

$$(P_\tau f)(s) = \int_0^\tau \frac{s e^{-\frac{s^2}{4(\tau-\sigma)}}}{2\sqrt{\pi}(\tau-\sigma)^{\frac{3}{2}}} f(\sigma)\sqrt{\sigma}\, \mathrm{d}\sigma.$$

Then P_τ *defines an isometric isomorphism from* $L^2[0,\tau]$ *onto* $A^2(\Delta, \omega_0)$, *where* $\omega_0(s) = \frac{e^{\frac{\mathrm{Re}\,(s^2)}{2\tau}}}{\tau}$ $\qquad (s \in \Delta)$.

Corollary. $\tilde{\Phi}_\tau \in \mathcal{L}(L^2[0,\tau], A^2(D))$.

This ends the proof of the regularity result.

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Proof of the main result

A result of Levin and Ljubarskii

Theorem. Let $\Lambda = \{(2k+1)(1 \pm i) : k \in \mathbb{Z}\}$ and

$$H(\lambda) = \frac{1}{2}\begin{cases} \pi \mathrm{Im}\,\lambda & \text{for } \lambda \in \Gamma_1, \\ -\pi \mathrm{Re}\,\lambda & \text{for } \lambda \in \Gamma_2, \\ -\pi \mathrm{Im}\,\lambda & \text{for } \lambda \in \Gamma_3, \\ \pi \mathrm{Re}\,\lambda & \text{for } \lambda \in \Gamma_4. \end{cases}$$

Then the family $(g_\lambda)_{\lambda \in \Lambda}$ defined by

$$g_\lambda(s) = \mathrm{e}^{\lambda s}\mathrm{e}^{-\frac{\lambda\pi}{2}}\mathrm{e}^{-H(\lambda)} \qquad (\lambda \in \Lambda,\ s \in \mathbb{C}),$$

is a Riesz basis in $E^2(D)$.

A decomposition result in $E^2(D)$

Lemma 1. *Let $\tau > 0$ and $\varphi \in E^2(D)$. Then there exists $\varphi_1 \in A^2(\Delta, \omega_0)$, $\varphi_2 \in A^2(\pi - \Delta, \omega_\pi)$ such that*

$$\varphi(s) = \varphi_1(s) + \varphi_2(s) \qquad\qquad (s \in D).$$

Proof. Let $\varphi \in E^2(D)$ with $\varphi = \sum_{\lambda \in \Lambda} a_\lambda g_\lambda$. Roughly speaking, we define

$$\varphi_1 = \sum_{\substack{\lambda \in \Lambda \\ \operatorname{Re}\lambda < 0}} a_\lambda g_\lambda \quad \text{and} \quad \varphi_2 = \sum_{\substack{\lambda \in \Lambda \\ \operatorname{Re}\lambda > 0}} a_\lambda g_\lambda,$$

and we use Hilbert's inequality to prove the required estimates.

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Proof of the main result

Let $M_\tau \in \mathcal{L}\left((L^2([0,\pi]))^2, A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi)\right)$ be defined by

$$M_\tau := \begin{bmatrix} P_\tau + R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & Q_\tau + R_{D,\tau}. \end{bmatrix}.$$

Since

$$\left\| \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi))} = 1 \qquad (\tau > 0).$$

$$\lim_{\tau\to 0+} \left\| \begin{bmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau} \end{bmatrix} \right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi))} = 0\,,$$

we have that M_τ is invertible, at least for τ small enough. The result follows by applying Lemma 1.

Extensions and comments

Dirichlet control at one end

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geqslant 0, \ x \in (0,\pi), \\ w(t,0) = u_0(t), \quad w(t,\pi) = 0 & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi)\,. \end{cases}$$

Given $\tau > 0$, we are interested in the range of the *input to state map*:

$$\Phi_\tau^0 u_0 = w(\tau,\cdot) \qquad\qquad (u_0 \in L^2[0,\tau]).$$

Proposition. Denote $G = \{s = x + iy \in \mathbb{C} \ | \ |y| < x \ \text{ and } \ |y| < 2\pi - x\},$,
$\tilde{A}^2(G) = \{\psi \in A^2(G) \ | \ \psi(s) + \psi(2\pi - s) = 0 \ \text{ for all } s \in G\}$,
$\tilde{E}^2(G) = \{\psi \in E^2(G) \ | \ \psi(s) + \psi(2\pi - s) = 0 \ \text{ for all } s \in G\}$.
Then

$$\tilde{E}^2(G) \subset \operatorname{Ran} \Phi_\tau^0 \subset \tilde{A}^2(G).$$

Neumann control at both ends

$$\begin{cases} \dfrac{\partial \theta}{\partial t}(t,x) = \dfrac{\partial^2 \theta}{\partial x^2}(t,x) & t \geqslant 0,\ x \in (0,\pi), \\ \frac{\partial \theta}{\partial x}(t,0) = u_0(t), \quad \frac{\partial \theta}{\partial x}(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ \theta(0,x) = 0 & x \in (0,\pi)\,. \end{cases}$$

The input to state map is $\Phi_\tau^{NN} \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \theta(\tau,\cdot) \qquad (u_0,\ u_\pi \in L^2[0,\tau])$.

Proposition. We have $E^{2,1}(D) \subset \operatorname{Ran} \Phi_\tau^{NN} \subset W^{2,1}(D) \cap \operatorname{Hol}(D)$, where $E^{2,1}(D)$ is the Smirnov-Sobolev space

$$E^{2,1}(D) = \{f \in \operatorname{Hol}(D) \ \mid \ f' \in E^2(D)\}, \tag{1}$$

and $W^{2,1}(D)$ is the usual Sobolev space.

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Some perspectives

- Considering $1D$ parabolic equations with analytic coefficients;
- Several space dimensions?
- Consequences for inverse problems;
- Consequences for time optimal control problems;
- ...