Kolmogorov-type operators on Lie groups: an introduction and main results

Alessia Kogoj

Università degli Studi di Urbino

- *Heat-type equations on stratified Lie groups:*

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \sum_{j=1}^{m} X_j^2 - \partial_t \quad \text{in } \mathbb{R}^{N+1}$$

$X_1, \ldots, X_m$ are smooth first order linear PDOs generating the Lie algebra of a stratified Lie group in $\mathbb{R}^N$

prototype: Heat operator on the Heisenberg group

$$\mathcal{L} = \Delta_{\mathbb{H}_1} - \partial_t := (\partial_{x_1} + 2x_2\partial_{x_3})^2 + (\partial_{x_2} - 2x_1\partial_{x_3})^2 - \partial_t \quad \text{in } \mathbb{R}^4$$

- *Kolmogorov-type operators of the type:*

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \mathrm{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t \quad \text{in } \mathbb{R}^{N+1}$$

A and B are $N \times N$ real constant matrices, $A \geq 0$ possibly degenerate

prototype:

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \partial_{x_1}^2 + x_1\partial_{x_2} - \partial_t \quad \text{in } \mathbb{R}^3$$

- *sub-Kolmogorov-type operators:* [1]

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \sum_{j=1}^{m} X_j^2 + \langle Bx, \nabla \rangle - \partial_t \quad \text{in } \mathbb{R}^{N+1}$$

prototype:

$$\mathcal{L} = \Delta_{\mathbb{H}_1} + x_1 \partial_{x_4} - \partial_t := (\partial_{x_1} + 2x_2 \partial_{x_3})^2 + (\partial_{x_2} - 2x_1 \partial_{x_3})^2 + x_1 \partial_{x_4} - \partial_t \quad \text{in } \mathbb{R}^5$$

- *Fokker-Planck equations of Mumford type:*

$$\mathcal{L} = \mathcal{L}_0 - \partial_t := \partial_{x_1}^2 + \sin x_1 \partial_{x_2} + \cos x_1 \partial_{x_3} - \partial_t \quad \text{in } \mathbb{R}^4$$

Some references:

- *Kinetic Fokker-Planck equations* Helffer-Nier (2005)
- *Kolmogorov operators of stochastic equations* Da Prato (2004)
- *PDEs model in finance* Pascucci (2005)
- *Computer and human vision* Mumford (1994)
- *Curvature Brownian motion* Chirikjian, Maslen, Wang and Zhou (2006)
- *Phase-noise Fokker-Planck equations* August and Zucker (2003)

[1] A.E. Kogoj and E. Lanconelli, *Link of groups and homogeneous Hörmander operators*, Proc. Amer. Math. Soc. (2007).

Theorem 1[2] Let $u \in C^\infty(\mathbb{R}^{N+1})$ be a solution to

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^{N+1}.$$

Suppose $u \in L^p(\mathbb{R}^{N+1})$ for a suitable $p \in [1, \infty[$.
Then

$$u \equiv 0.$$

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remark Let $u \in C^\infty(\mathbb{R}^N)$ be a harmonic function $\Delta u = 0$ in $\mathbb{R}^N$. If $u \in L^p(\mathbb{R}^N)$ and $1 \leq p < \infty$, for every x in $\mathbb{R}^N$:

$$|u(x)| = \left| \fint_{B_r(x)} u(y)\, dy \right| \leq \left(\frac{1}{|B_r(x)|} \right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty$$

[2] A.E. Kogoj and E. Lanconelli, *L^p-Liouville theorems for invariant Partial Differential Equations in $\mathbb{R}^N$*, Nonlinear Anal. (2015)

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$\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ *H= right invariant Haar measure on* $\mathbb{G}$

$$L^p(\mathbb{R}^{N+1}, H), \quad L^p\text{-space with respect to } H$$

Theorem 2[3] Let $u \in C^\infty(\mathbb{R}^{N+1}, \mathbb{R})$ be a solution to

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^n.$$

Then $u \equiv 0$ if one of the following conditions is satisfied

(i) $u \in L^p(\mathbb{R}^{N+1}, H)$ for a suitable $p \in [1, \infty[$;

(ii) $u \geq 0$ and $u^p \in L^1(\mathbb{R}^{N+1}, H)$ for a suitable $p \in]0, 1[$.

Theorem 3 Let $u \in C^2(\mathbb{R}^{N+1}, \mathbb{R})$ be a solution to

$$\mathcal{L}u \geq 0 \text{ in } \mathbb{R}^{N+1}.$$

If $u \in L^p(\mathbb{R}^{N+1}, H)$ for a suitable $p \in [1, \infty[$, then $u \leq 0$.

[3] A. Bonfiglioli and A.E. Kogoj, *Weighted L^p-Liouville Theorems for Hypoelliptic Partial Differential Operators on Lie Groups*, J. Evol. Equ. (2016)

Uniqueness for the Cauchy Problem [4]

$$\begin{cases} \partial_t u = \mathcal{L}_0 u & \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times]0,\infty[\\ u\big|_{t=0} = 0 \end{cases} \qquad \text{(PC)}$$

if u is a solution of (PC) and for some $p \in [1,\infty)$

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^p \, e^{t\, trace(B)} \, dxdt < \infty,$$

then $u \equiv 0$.

[4] A. Bonfiglioli and A.E. Kogoj, *Weighted L^p-Liouville Theorems for Hypoelliptic Partial Differential Operators on Lie Groups*, J. Evol. Equ. (2016)

Uniqueness for the Positive Cauchy Problem [5]

$$\begin{cases} \partial_t u = \mathcal{L}_0 u & \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times]0,\infty[\\ u\big|_{t=0} = u_0 \geq 0 \end{cases} \qquad \text{(PPC)}$$

(PPC) admits at most one nonnegative solution u.

Liouville type theorem

$$\begin{cases} \sum_{i=1}^m X_i^2 u - \partial_t u = 0 & \text{in } \mathbb{R}^{N+1} \\ u \geq 0 & \text{in } \mathbb{R}^{N+1} \end{cases},$$

Assume

$$u(0,t) = O(e^{\varepsilon t}) \qquad \text{as } t \longrightarrow \infty, \quad \forall \varepsilon > 0$$

Then, $u \equiv const.$

[5] A.E. Kogoj, Y. Pinchover and S. Polidoro, *On Liouville-type theorems and the uniqueness of the positive Cauchy problem for a class of hypoelliptic operators*, J. Evol. Equ. (2016)

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Polynomial Liouville Theorem for $\mathcal{L}$ [6]

Let $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a smooth function s. t.

$$\begin{cases} \mathcal{L}u = p & \text{in } \mathbb{R}^{N+1} \\ u \geq q & \text{in } \mathbb{R}^{N+1} \end{cases},$$

where p and q are polynomial functions.
Assume

$$u(0,t) = O(t^m) \qquad \text{as } t \longrightarrow \infty$$

Then, u is a polynomial function.

Moreover, if p and q are identically zero,

$$u \equiv \text{constant}$$

[6] A.E. Kogoj and E. Lanconelli, *An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations*, Mediterr. J. Math. **1** (2004), no.1.
A.E. Kogoj and E. Lanconelli, *Liouville theorems in halfspaces for parabolic hypoelliptic equations*, Ric. Mat. **55** (2006)

Polynomial Liouville Theorem for $\mathcal{L}_0$

Let $P, Q : \mathbb{R}^N \longrightarrow \mathbb{R}$ be polynomial functions
and
let $U : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a smooth function s. t.

$$\mathcal{L}_0 U = P \quad \text{and } U \geq Q, \quad \text{in } \mathbb{R}^N.$$

Then U is a polynomial function.

In particular if P and Q are identically zero

$$U \equiv \text{constant}.$$

Liouville at $t = -\infty$

Let u be a nonnegative solution to the equation

$$\mathcal{L}u = 0$$

in the halfspace $S = \mathbb{R}^N \times] - \infty, 0[$.

Then

$$\lim_{t \longrightarrow -\infty} u(x,t) = \inf_S u \qquad \forall x \in \mathbb{R}^N$$

The evolution PDOs

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i x_j} + \sum_{i=1}^{N} b_i(z)\partial_{x_i} - \partial_t$$

The coefficients $a_{ij} = a_{ji}$ and b_i are of class C^∞ in the strip

$$S = \{z = (x,t) \in \mathbb{R}^{N+1} \mid x \in \mathbb{R}^N,\ T_1 < t < T_2\}$$

(with $-\infty \leq T_1 < T_2 \leq +\infty$) Moreover:

$$q_{\mathcal{L}}(z,\xi) := \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \geq 0, \qquad q_{\mathcal{L}}(z,\cdot) \not\equiv 0 \text{ for every } z \in S$$

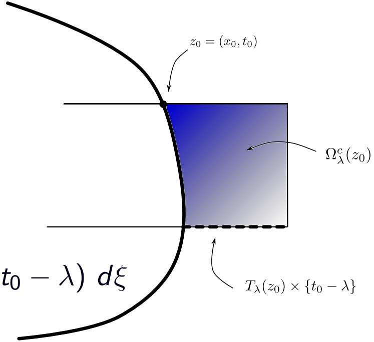
As the operator $\mathcal{L}$ endows the strip S with a structure of Doob β-harmonic space, from the Wiener resolutivity theorem we have the existence[7] of a *generalized solution in the sense of Perron-Wiener* for every $\varphi \in C(\partial\Omega)$ to the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, & \Omega \text{ bounded open set, } \overline{\Omega} \subseteq S \\ u|_{\partial\Omega} = \varphi \end{cases} \tag{DP}$$

$$H_\varphi^\Omega \text{ is } C^\infty(\Omega) \text{ and satisfies } \mathcal{L}u = 0 \text{ in } \Omega$$

[7] A.E. Kogoj, *On the Dirichlet problem for evolution equations : Perron–Wiener solution and a cone-type criterion*, J. Differential Equations (2016)

Ω bounded open set with $\overline{\Omega} \subset S$ and z_0 a point of $\partial\Omega$
$(B_\lambda)_{0<\lambda<1}$ basis of closed neighborhood of x_0 (in $\mathbb{R}^N$)
such that $B_\lambda \subseteq B_\mu$ if $0 < \lambda < \mu \leq 1$
We set $\Omega^c_\lambda(z_0) := (B_\lambda \times [t_0 - \lambda, t_0])\backslash\Omega$
and
$T_\lambda(z_0) = \{x \in \mathbb{R}^N \ : \ (x, t_0 - \lambda) \in \Omega^c_\lambda(z_0)\}$



Finally, we define

$$\gamma_\lambda(z_0) = \int_{T_\lambda(z_0)} \Gamma(z_0; \xi, t_0 - \lambda) \, d\xi$$

and we can state **our criterion**
the point $z_0 \in \partial\Omega$ is $\mathcal{L}$-regular if

$$\limsup_{\lambda \searrow 0} \gamma_\lambda(z_0) > 0$$