Analysis Motivated by Vehicular Traffic and Crowd Dynamics

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Benasque – August 22nd, 2017

Conservation Laws

Introduction – Analytic Theory

$$\boxed{\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u)}$$

$t \in \mathbb{R}_+$ time
$x \in \mathbb{R}^N$ space
$u \in \mathbb{R}^n$ unknown

f smooth flux
g smooth source

Euler	Statement	1755
Riemann	Regular Solutions	1860

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Scalar MultiD $\quad n = 1,\ N \geq 1$

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Existence

(Kružkov: Mat.Sb., 1970)

Uniqueness

(Kružkov: Mat.Sb., 1970)

Dependence on data

(Kružkov: Mat.Sb., 1970)

Dependence on f, g

(Colombo, Mercier, Rosini: CMS, 2009)

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Scalar MultiD $n = 1,\ N \geq 1$	Systems in 1D $n \geq 1,\ N = 1$
$\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u)$	$\partial_t u + \partial_x f(u) = 0$
Existence (Kružkov: Mat.Sb., 1970)	Existence (Glimm: CPAM, 1965)
Uniqueness (Kružkov: Mat.Sb., 1970)	Uniqueness (Bressan & c.: 1999, 2000)
Dependence on data (Kružkov: Mat.Sb., 1970)	Dependence on data (Bressan & c.: 1995, 2000)
Dependence on f, g (Colombo, Mercier, Rosini: CMS, 2009)	Dependence on f (Bianchini, Colombo: PAMS, 2002)

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Systems in 1D $n \geq 1,\ N = 1$

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$$\boxed{n \geq 1,\ N \geq 1?}$$

Introduction – Key Features

1. Evolution
2. Irreversible
3. Finite Speed
4. Conservation
5. Singularities

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Simplest Case

$n = 1, N = 1$,

$f = f(u)$, $g \equiv 0$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0, x) = u_o(x) \end{cases}$$

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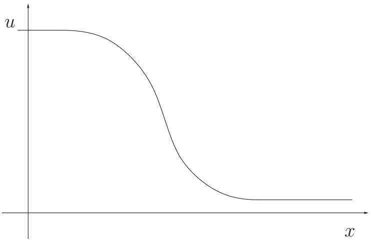
$f(u) = \lambda\, u \qquad \partial_t u + \lambda\, \partial_x u = 0 \qquad u(t,x) = u_o(x - \lambda\, t)$



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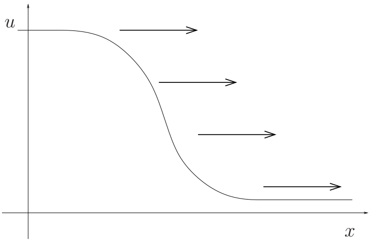

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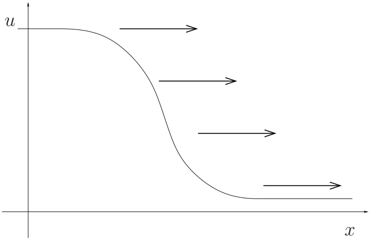

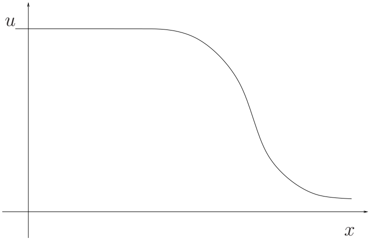

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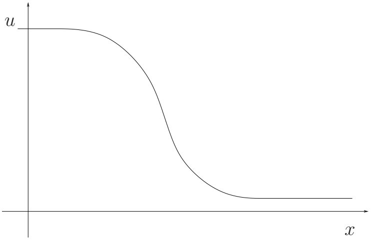
f non linear $\quad \partial_t u + f'(u)\,\partial_x u = 0 \quad u(t,x) = ?$



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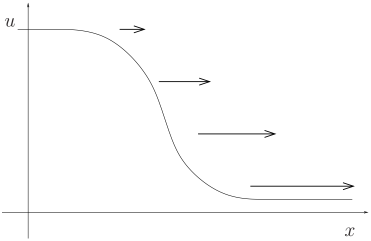

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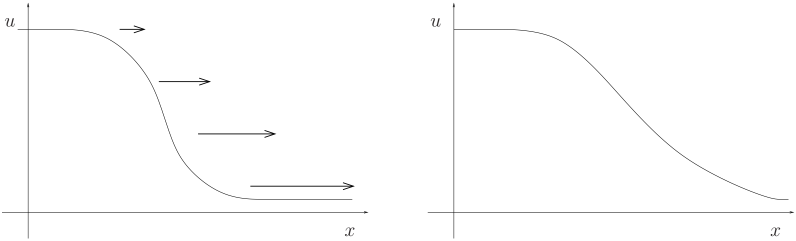

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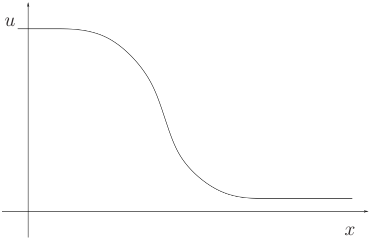

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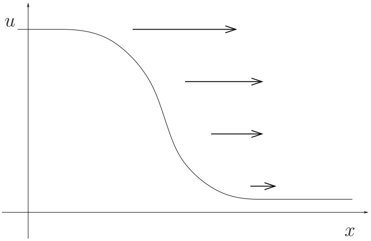

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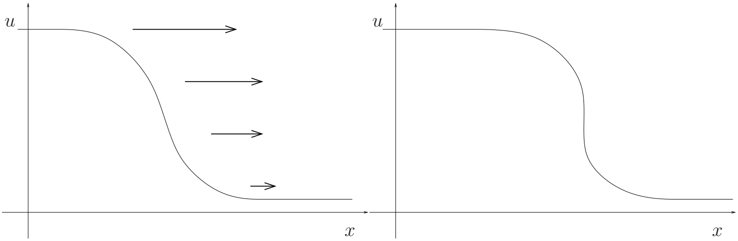

Introduction

Discontinuities!

Discontinuities!

Viscosity Entropy Stability

Discontinuities!

Viscosity Entropy Stability

They all agree!

Vehicular Traffic

Vehicular Traffic – Macroscopic Models

$$t = \text{time} \qquad x = \text{space}$$

$$\rho = \begin{cases} \text{(density)} \\ \text{occupancy} \end{cases}$$

cars are conserved

Vehicular Traffic – Macroscopic Models

$$t = \text{time} \qquad x = \text{space}$$

$$\rho = \begin{cases} \text{(density)} \\ \text{occupancy} \end{cases}$$

$$\partial_t \rho + \partial_x (\rho\, v) = 0$$

$$\boxed{\boxed{v = ?}}$$

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LWR (Lighthill, Whitham: Proc. London. A., 1955)
(Richards: Operations Res., 1956)

v decreasing

$$v(0) = v_{\max}$$
$$v(R) = 0$$

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- Second Order Models
  - *Requiem* (Daganzo: Transp. Research B, 1995)
  - *Resurrection* (Aw, Rascle: SIAM Appl. Math., 2000)
  - (Zhang: Transp. Research B, 2002)

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- Second Order Models
- 2–Phase Models
  - (Colombo: SIAM Appl. Math., 2002)
  - (Colombo, Marcellni, Rascle: SIAM Appl. Math., 2010)
  - (Blandin, Work, Goatin, Piccoli, Bayen: SIAM Appl. Math., 2010)

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- Second Order Models
- 2–Phase Models
- Multi–Population Models
  - (Zhang, Jin: Transp. Research Rec., 2002)
  - (Benzoni–Gavage, Colombo: EJAM, 2003)

Vehicular Traffic – Macroscopic Models

$$\begin{array}{rcl} t & = & \text{time} \\ x & = & \text{space} \end{array} \qquad \rho = \begin{cases} \text{(density)} \\ \text{occupancy} \end{cases} \qquad \partial_t \rho + \partial_x (\rho\, v) = 0$$

$$\boxed{\boxed{v = ?}}$$

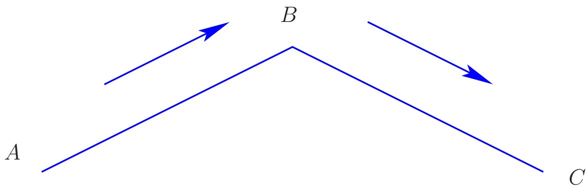
LWR (Lighthill, Whitham: Proc. London. A., 1955)
(Richards: Operations Res., 1956)

v decreasing

$$\begin{array}{rcl} v(0) & = & v_{\max} \\ v(R) & = & 0 \end{array}$$

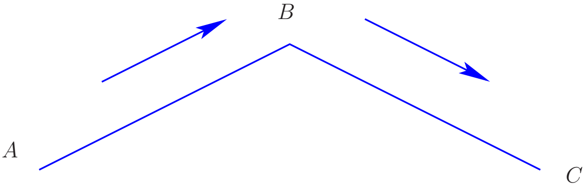
- Second Order Models
- 2–Phase Models
- Multi–Population Models
- Networks
  - (Garavello, Kahn, Piccoli: Book, 2016)

Vehicular Traffic



From	To	Time
A	B	$\frac{\sharp\,\text{cars}}{100}$
B	C	45

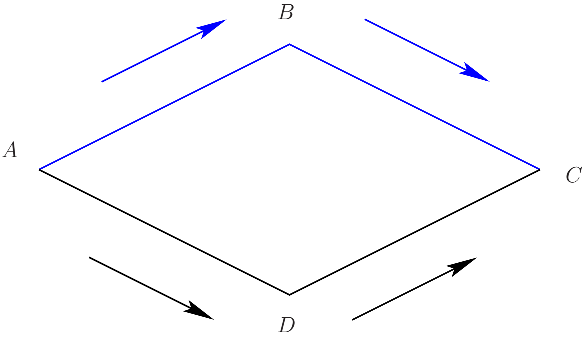
Vehicular Traffic



From	To	Time
A	B	$\frac{\sharp \text{ cars}}{100}$
B	C	45

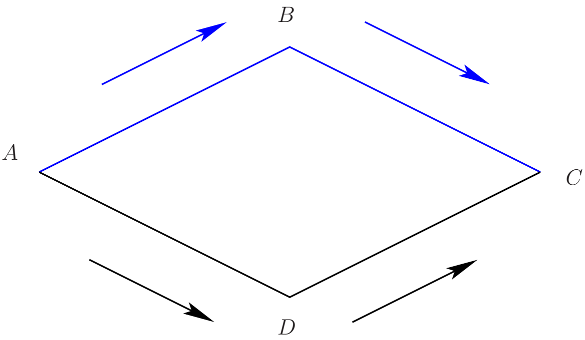
Route ABC: 4000 cars need $\frac{4000}{100} + 45 = 85$

Vehicular Traffic



From	To	Time
A	B	$\frac{\sharp\,\text{cars}}{100}$
B	C	45
A	D	45
D	B	$\frac{\sharp\,\text{cars}}{100}$

Vehicular Traffic

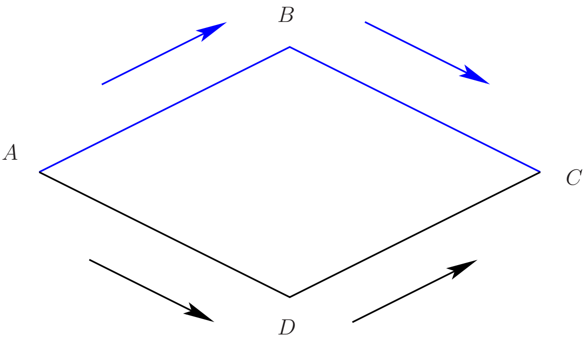


From	To	Time
A	B	$\frac{\sharp\,\text{cars}}{100}$
B	C	45
A	D	45
D	B	$\frac{\sharp\,\text{cars}}{100}$

Route ABC: $\frac{\sharp\,\text{cars}}{100} + 45$

Route ADC: $45 + \frac{\sharp\,\text{cars}}{100}$

Vehicular Traffic

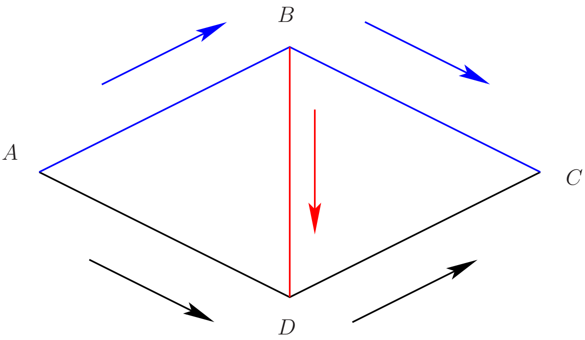


From	To	Time
A	B	$\frac{\sharp\,\text{cars}}{100}$
B	C	45
A	D	45
D	B	$\frac{\sharp\,\text{cars}}{100}$

Route ABC: $\frac{\sharp\,\text{cars}}{100} + 45 \quad \Rightarrow \quad \frac{2000}{100} + 45 = 65$

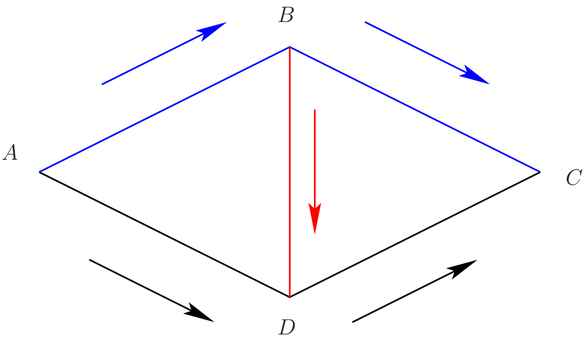
Route ADC: $45 + \frac{\sharp\,\text{cars}}{100} \quad \Rightarrow \quad \frac{2000}{100} + 45 = 65$

Vehicular Traffic



From	To	Time
A	B	$\frac{\sharp\,\text{cars}}{100}$
B	C	45
A	D	45
D	B	$\frac{\sharp\,\text{cars}}{100}$
B	D	0

Vehicular Traffic



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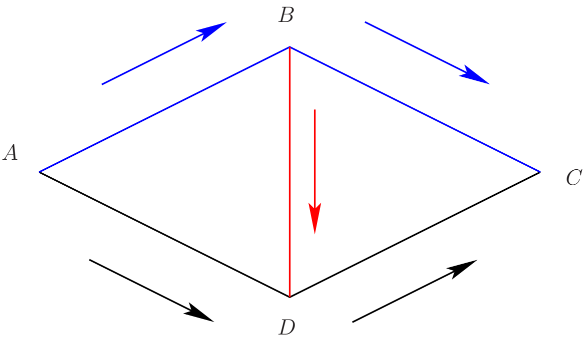
Route ABC: $\frac{\sharp \text{cars}}{100} + 45$

Route ADC: $45 + \frac{\sharp \text{cars}}{100}$

Route $ABDC$: $\frac{\sharp \text{cars}}{100} + 0 + \frac{\sharp \text{cars}}{100}$

Vehicular Traffic



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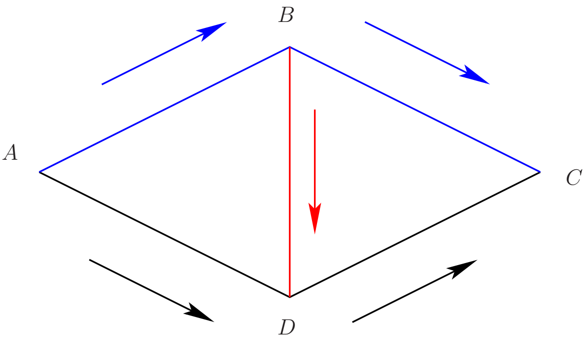
B	D	0

Route ABC: $\frac{\sharp\text{ cars}}{100} + 45$

Route ADC: $45 + \frac{\sharp\text{ cars}}{100}$

Route $ABDC$: $\frac{\sharp\text{ cars}}{100} + 0 + \frac{\sharp\text{ cars}}{100} \quad \Rightarrow \quad \frac{4000}{100} + \frac{4000}{100} = 80$

Vehicular Traffic – Braess Paradox



From	To	Time
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B	C	45
A	D	45
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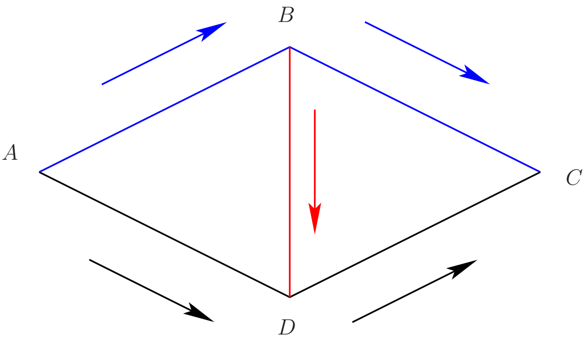
B	D	0

Only ABC	80
ABC + ADC	65
ABC + ADC + ABDC	80

Vehicular Traffic – Braess Paradox



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Nash equilibrium vs. optimality

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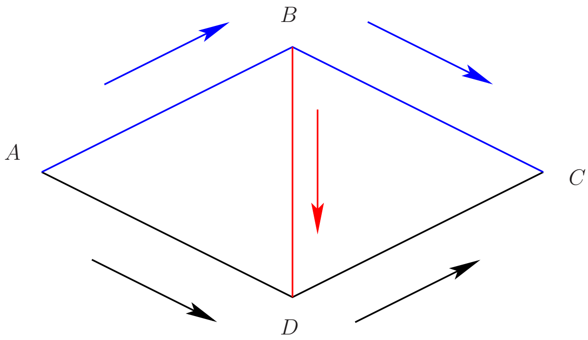

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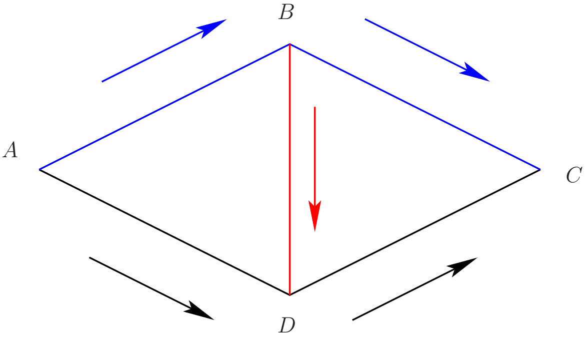
B	D	0

Real!

Stuttgart	Highway segment closed	1968
New York	42^{nd} street closed	22.04.1990
Seoul	6 lanes highway substituted by a park	2008

Vehicular Traffic – Braess Paradox



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A	D	45
D	B	$\frac{\sharp \text{ cars}}{100}$
B	D	0

(Colombo, Holden: JOTA, 2016)

Characterization?
Dynamics?

Crowd Dynamics

Crowd Dynamics

$$\partial_t \rho + \operatorname{div}_x \big(\rho\, v(\rho) \vec v(x)\big) = 0 \qquad \left\{ \begin{array}{rcl} v & = & \text{speed modulus} \\ \vec v & = & \text{velocity direction} \end{array} \right.$$

(Colombo, Facchi, Maternini: HYP2008 Proceedings, 2009)

Crowd Dynamics

$$\partial_t \rho + \mathrm{div}_x \left(\rho \, \vec{V}(\rho, x)\right) = 0$$

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$$\partial_t \rho + \operatorname{div}_x \left(\rho \, \vec{V}(\rho, x) \right) = 0$$

$$\partial_t \rho + \operatorname{div}_x \left(\rho \, v(\rho) \left(\vec{v}(x) \quad + \quad \begin{array}{c} \text{avoid} \\ \text{high} \\ \text{density} \end{array} \quad \right) \right) = 0$$

Crowd Dynamics

$$\partial_t \rho + \mathrm{div}_x \left(\rho \, \vec V(\rho, x)\right) = 0$$

$$\partial_t \rho + \mathrm{div}_x \left(\rho \, v(\rho) \left(\vec v(x) - \frac{\kappa \ \mathrm{grad}_x (\rho * \eta)}{\sqrt{1 + \left\|\mathrm{grad}_x (\rho * \eta)\right\|^2}} \right) \right) = 0$$

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Theorem: If: v is smooth, decreasing, $v(0) = V$, $v(\rho) = 0$;
$\vec{v}$ is smooth;
η is smooth with compact support;

Crowd Dynamics

$$\partial_t \rho + \mathrm{div}_x \left(\rho \, \vec{V}(\rho, x)\right) = 0$$

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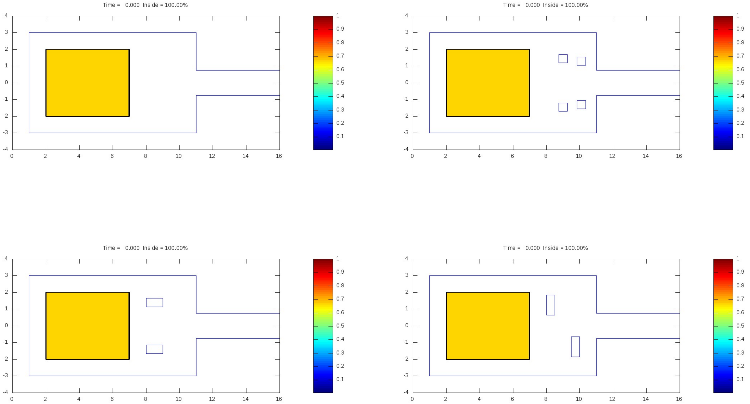
$\vec{v}$ is smooth;
η is smooth with compact support;

Then: Existence & Uniqueness in $\mathbf{L^1}$
Lipschitz Continuity from Data and Equation
Viability (discomfort)

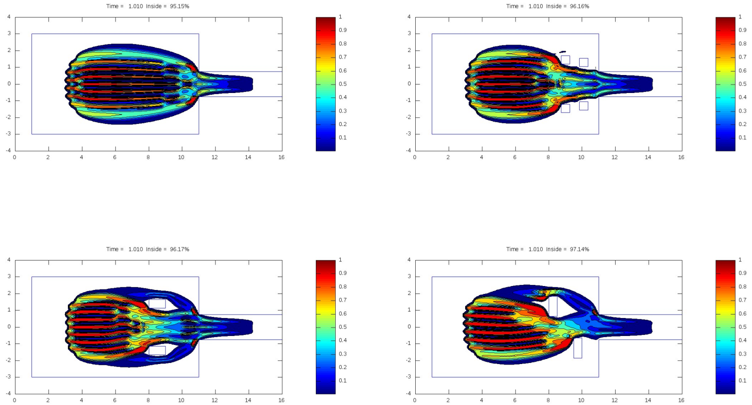
(Colombo, Garavello, Lécureux–Mercier: M3AS, 2012)
(Colombo, Lécureux–Mercier: Acta Math. Sc., 2012)
(Göttlich, Hoher, Schindler, Schleper: Appl.Mat.Mod., 2014)

Crowd Dynamics – Time to Exit



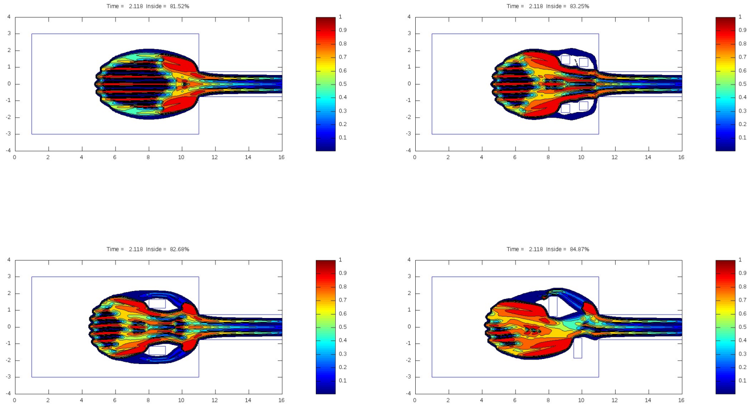
$$t = 0.000$$

Crowd Dynamics – Time to Exit



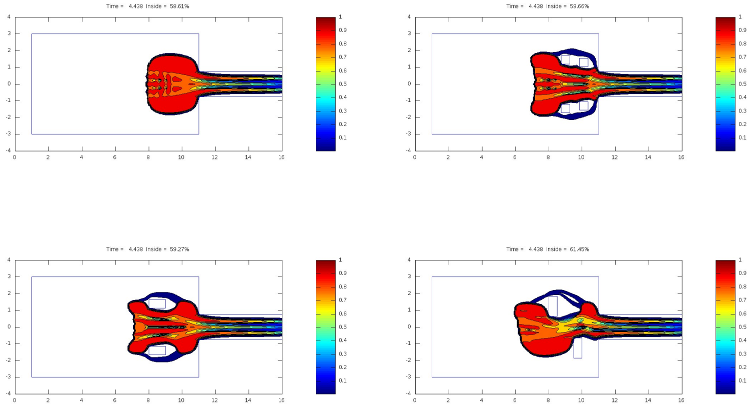
$$t = 1.010$$

Crowd Dynamics – Time to Exit



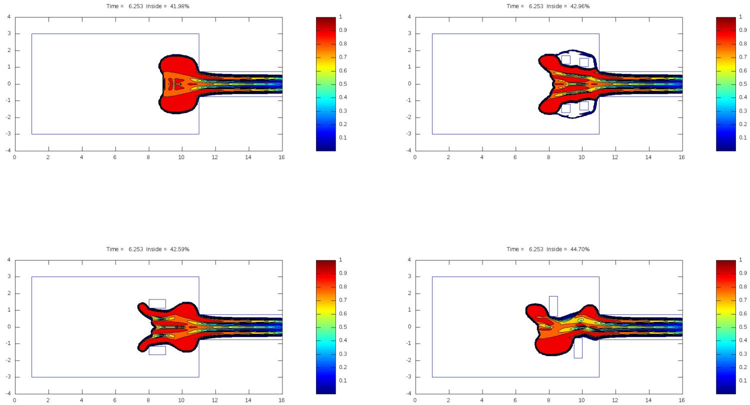
$$t = 2.118$$

Crowd Dynamics – Time to Exit



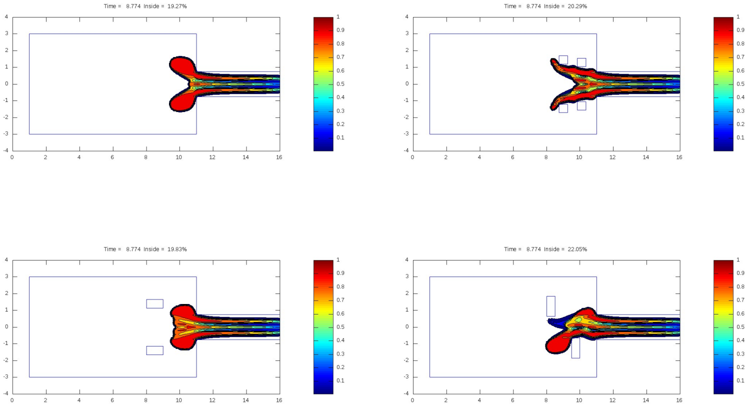
$$t = 4.438$$

Crowd Dynamics – Time to Exit



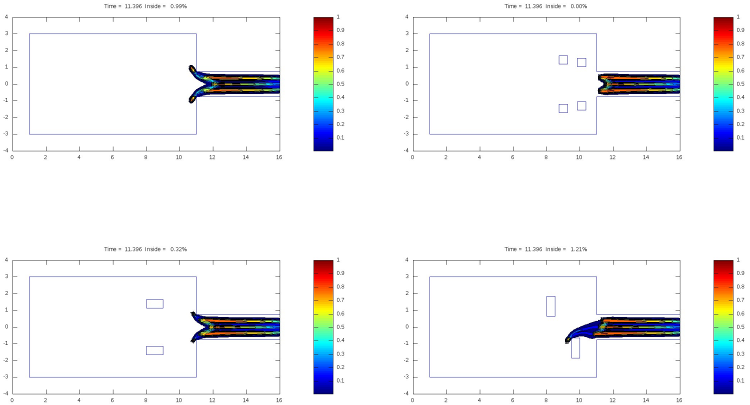
$$t = 6.253$$

Crowd Dynamics – Time to Exit



$$t = 8.774$$

Crowd Dynamics – Time to Exit



$$t = 11.396$$

Crowd Dynamics – Lane Formation

Two populations moving in opposite directions

Crowd Dynamics – Lane Formation

$$\partial_t \rho^1 + \mathrm{div}_x \left[\rho^1\, v(\rho^1) \left(\vec v^1(x) - \frac{\varepsilon_{11}\, \nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1 * \eta)\right\|^2}} - \frac{\varepsilon_{12}\, \nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2 * \eta)\right\|^2}}\right)\right] = 0$$

$$\partial_t \rho^2 + \mathrm{div}_x \left[\rho^2\, v(\rho^2) \left(\vec v^2(x) - \frac{\varepsilon_{21}\, \nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1 * \eta)\right\|^2}} - \frac{\varepsilon_{22}\, \nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2 * \eta)\right\|^2}}\right)\right] = 0$$

$$\vec v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \delta \qquad \eta(x,y) = \left[1-(2x)^2\right]^3 \left[1-(2y)^2\right]^3 \chi_{[-0.5,0.5]^2}(x,y)$$

$$\vec v^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \delta \qquad v(\rho) = 4\,(1-\rho) \qquad \begin{array}{ccccc} \varepsilon_{11} & = & 0.3 & \varepsilon_{12} & = & 0.7 \\ \varepsilon_{21} & = & 0.7 & \varepsilon_{22} & = & 0.3 \end{array}$$

Crowd Dynamics – Shepherd Dog (Consensus)

Given: $$\left\{\begin{array}{ll} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x,\rho,p)\right) = 0 & \text{HCL} \\ \dot p = \varphi\left(t, p, \left(\rho(t) * \eta\right)(p)\right) & \text{ODE} \end{array}\right.$$

Crowd Dynamics – Shepherd Dog (Consensus)

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IF: $v \in \mathbf{C^2}([0,R] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N)$ is such that ...

$\eta \in \mathbf{C^1_c}(\mathbb{R}^N; \mathbb{R})$

φ Caratheodory, Locally Lipschitz, Sublinear

Crowd Dynamics – Shepherd Dog (Consensus)

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IF: $v \in \mathbf{C^2}([0,R] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N)$ is such that ...
$\eta \in \mathbf{C_c^1}(\mathbb{R}^N; \mathbb{R})$
φ Caratheodory, Locally Lipschitz, Sublinear

Then: There exists a solution (u, w), with
- ✓ $\rho = \rho(t,x)$ weak entropy solution to HCL
- ✓ $p = p(t)$ Caratheodory solution to ODE
- ✓ stability estimates

$$\begin{array}{rl} & \left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L^1}} + \left\|(p_1 - p_2)(t)\right\| \\ \le & C(t) \cdot \Big(\left\|\partial_\rho (v_1 - v_2)\right\|_{\mathbf{L^\infty}} + \left\|\operatorname{div}_x (v_1 - v_2)\right\|_{\mathbf{L^1}} \\ & \qquad + \|\varphi_1 - \varphi_2\|_{\mathbf{L^\infty}} + \|\eta_1 - \eta_2\|_{\mathbf{L^1}} \\ & \qquad + \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L^1}} + \|\bar p_1 - \bar p_2\| \Big) \end{array}$$

Crowd Dynamics – Shepherd Dog (Consensus)

(Colombo, Mercier: JNLS, 2012)

Crowd Dynamics – Shepherd Dog (Consensus)

(Colombo, Mercier: JNLS, 2012)

Crowd Dynamics – Policemen vs. Hooligans

$$\begin{cases} \partial_t \rho_i + \mathrm{div}_x \left[\rho^i(1-\rho^i)\left(-w^i(x,p) + \mathcal{A}^i(\rho)\right)\right] = 0 & i = 1,2 \\ \dot{p}^k = I_k(p) + \mathcal{B}_k(\rho) & k = 1,\ldots,4 \end{cases}$$

$$\begin{aligned} \mathcal{A}^1(\rho) &= \frac{\varepsilon_{11}\ \eta*(\rho^1-\bar\rho)\ \nabla_x(\rho^1*\eta)}{\sqrt{1+\|\eta*(\rho^1-\bar\rho)\nabla_x(\rho^1*\eta)\|^2}} + \frac{\varepsilon_{12}\ \eta*(\rho^2-\rho^1)\nabla_x(\rho^2*\eta)}{\sqrt{1+\|\eta*(\rho^2-\rho^1)\nabla_x(\rho^2*\eta)\|^2}}, \\ \mathcal{A}^2(\rho) &= \frac{\varepsilon_{22}\ \eta*(\rho^2-\bar\rho)\ \nabla_x(\rho^2*\eta)}{\sqrt{1+\|\eta*(\rho^1-\bar\rho)\nabla_x(\rho^2*\eta)\|^2}} + \frac{\varepsilon_{21}\ \eta*(\rho^1-\rho^2)\nabla_x(\rho^1*\eta)}{\sqrt{1+\|\eta*(\rho^1-\rho^2)\nabla_x(\rho^1*\eta)\|^2}}, \\ \mathcal{B}_k(\rho)(p) &= \varepsilon_1 \frac{\nabla_x((\bar\eta*\rho^1)(\bar\eta*\rho^2))(p^k)}{\sqrt{1+\|\nabla_x((\bar\eta*\rho^1)(\bar\eta*\rho^2))(p^k)\|^2}} \end{aligned}$$

(Borsche, Colombo, Garavello, Meurer: JNLS, 2015)

Crowd Dynamics – Policemen vs. Hooligans

(Borsche, Colombo, Garavello, Meurer: JNLS, 2015)

Crowd Dynamics – 3D!

Crowd Dynamics – 3D!

Film

Predators – Prey

Hyperbolic		Parabolic
	vs.	
Predators		Prey

Hyperbolic Predators vs. Parabolic Prey

Predators: $u = u(t)$

Prey: $w = w(t)$

$$\begin{cases} \partial_t u = (\alpha\, w - \beta)\, u \\ \partial_t w = (\gamma - \delta\, u)\, w \end{cases}$$

$\alpha =$ predators birth rate due to prey

$\beta =$ predators mortality rate

$\gamma =$ prey birth rate

$\delta =$ prey mortality rate due to predators

Hyperbolic Predators vs. Parabolic Prey

Predators: $u = u(t,x)$
Prey: $w = w(t,x)$

$$\begin{cases} \partial_t u & = (\alpha\, w - \beta)\, u \\ \partial_t w - \mu\, \Delta w & = (\gamma - \delta\, u)\, w \end{cases}$$

α = predators birth rate due to prey

β = predators mortality rate

γ = prey birth rate

δ = prey mortality rate due to predators

Prey

diffuse

Hyperbolic Predators vs. Parabolic Prey

Predators: $u = u(t,x)$

Prey: $w = w(t,x)$

$$\begin{cases} \partial_t u + \operatorname{div}_x \big(u\, v(w)\big) = (\alpha\, w - \beta)\, u \\ \partial_t w - \mu\, \Delta w = (\gamma - \delta\, u)\, w \end{cases}$$

$\alpha =$ predators birth rate due to prey

$\beta =$ predators mortality rate

$\gamma =$ prey birth rate

$\delta =$ prey mortality rate due to predators

Predators	Prey
$v(w) = \kappa \dfrac{\operatorname{grad}(w * \eta)}{\sqrt{1 + \left\Vert \operatorname{grad}(w * \eta) \right\Vert^2}}$	diffuse

Hyperbolic Predator vs. Parabolic Prey

There exists $\mathcal{R} \colon \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ with the properties:

Hyperbolic Predator vs. Parabolic Prey

There exists $\mathcal{R}\colon \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ with the properties:

1. $\mathcal{X}_+ = (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R}) \times (\mathbf{L^1} \cap \mathbf{L^\infty})(\mathbb{R}^N;\mathbb{R})$
2. $\mathcal{R}$ is a semigroup
3. $t \to \mathcal{R}_t(u_o, w_o)$ solves the system
4. $t \to \mathcal{R}_t(u_o, w_o)$ is continuous in time
5. $(u_o, w_o) \to \mathcal{R}_t(u_o, w_o)$ is locally Lipschitz continuous
6. Growth estimates
7. Propagation speed

(Colombo, Rossi: Comm.Math.Sc., 2015)

(Colombo, Marcellini, Rossi: NHM, 2016)

(Rossi, Schleper: M2AN, 2016)

Hyperbolic Predator vs. Parabolic Prey