Inverse obstacle problem with partial Cauchy data: a shape optimization approach.

Jérémi Dardé
joint work with Fabien Caubet and Matías Godoy

Institut de Mathématiques de Toulouse, Université Toulouse 3

VII Partial differential equations, optimal design and numerics
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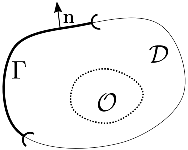
Inverse obstacle problem

- The data:

- $\mathcal{D}$ open set of $\mathbb{R}^d$ ($d \geq 2$), with Lipschitz boundary
- $\Gamma \subset \partial\mathcal{D}$, $|\Gamma| > 0$, $\Gamma_c := \partial\mathcal{D} \setminus \overline{\Gamma}$
- (g_D, g_N): (possibly noisy) Cauchy data, $(g_D, g_N) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.



- Problem:

Find an inclusion $\mathcal{O}$, $\overline{\mathcal{O}} \subset \mathcal{D}$, $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ connected, and $u \in H^1(\Omega)$, s.t.

$$(\mathcal{P}) \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g_D & \text{on } \Gamma \\ \partial_n u = g_N & \text{on } \Gamma \\ u = 0 & \text{on } \partial\mathcal{O} \end{cases}$$

Inverse problems: typical questions

• Classical questions in context of inverse problems:

1) Identifiability - there exists at most one couple $(\mathcal{O}, u)$ solution of $(\mathcal{P})$.

2) Stability - log-type stability (really bad) : *Optimal stability for inverse elliptic boundary value problems with unknown boundaries*, G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze (2000).

3) Reconstruction.

Reconstruction methods (non-exhaustive list)

• 2d problem - methods based on conformal mappings: *Conformal mappings and inverse boundary value problem*, H. Haddar and R. Kress, Inverse Problems **21** (2005).

• Integral equations: *Nonlinear integral equations and the iterative solution for an inverse boundary value problem*, R. Kress and W. Rundell, Inverse Problems (2005).

• Exterior approach, based on the Quasi-reversibility method: *A quasi-reversibility approach to solve the inverse obstacle problem*, L. Bourgeois and J.D., Inverse problems and Imaging (2010).

• Shape optimization methods: *Detecting perfectly insulated obstacles by shape optimization techniques of order two*, L. Afraites, M. Dambrine, K. Eppler, D. Kateb, Discrete Contin. Dyn. Syst. Ser. B (2007).
$\leadsto$ detection of obstacles in fluids:

- *A Kohn-Vogelius formulation to detect an obstacle immersed in a fluid*, F. Caubet, M. Dambrine, D. Kateb, and C. Z. Timimoun, Inverse Probl. Imaging (2013)
- *On the detection of several obstacles in 2d stokes flow: Topological sensitivity and combination with shape derivatives*, F. Caubet, C. Conca, and M. Godoy, Inverse Probl. Imaging (2016).

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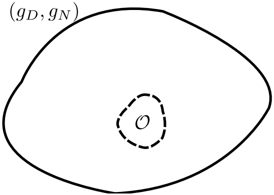
Shape optimization for inverse obstacle problems: general strategy



1. Initial situation
2. choose an arbitrary open set $\omega \Subset \mathcal{D}$
3. compute u solving the direct problem $\begin{cases} \Delta u_\omega = 0 \text{ in } \mathcal{D} \setminus \overline{\omega} \\ \partial_\nu u_\omega = g_N \text{ on } \partial\mathcal{D} \\ u_\omega = 0 \text{ on } \partial\omega \end{cases}$
4. compute $J(\omega) = \int_{\partial D} (g_D - u_\omega)^2 ds \to$ if zero, ok, if not, compute the shape derivative of J w.r.t. $\omega \to$ gradient algorithm.

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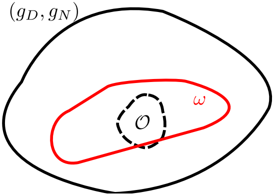

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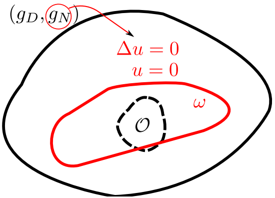

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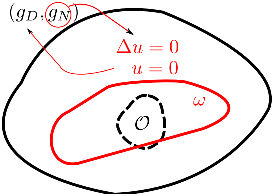

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Kohn-Vogelius functional

- In our computation, we will minimize a Kohn-Vogelius functional:

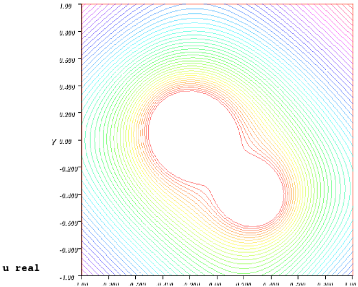
$$\min_{\omega} \ \mathcal{K}(\omega) := \int_{\mathcal{D}\setminus\overline{\omega}} |\nabla(u_\omega^D - u_\omega^N)|^2 \, dx$$

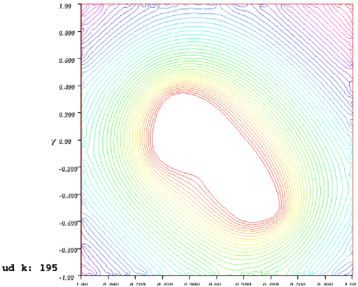
where u_ω^D, u_ω^N solve

$$\begin{cases} \Delta u_\omega^D = 0 \text{ in } \mathcal{D}\setminus\overline{\omega} \\ u_\omega^D = g_D \text{ on } \partial\mathcal{D} \\ u_\omega^D = 0 \text{ on } \partial\omega \end{cases}, \qquad \begin{cases} \Delta u_\omega^N = 0 \text{ in } \mathcal{D}\setminus\overline{\omega} \\ \partial_\nu u_\omega^N = g_N \text{ on } \partial\mathcal{D} \\ u_\omega^N = 0 \text{ on } \partial\omega \end{cases}.$$

- Advantages:
  - (g_D, g_N) are treated symmetrically
  - only volumic quantities
  - numerically: better reconstructions.
- Still a severely ill-posed problem! Regularization?

Example of reconstruction





An example of reconstruction with noisy data.

Incomplete data

• The whole strategy is possible only if the data (g_D, g_N) are available on the whole boundary of the domain $\mathcal{D}$ (at least one of them).

• But in lots of practical applications, some parts of the boundary are unaccessible $\rightarrow$ no measurements on them (particularly true for fluid problems).

• $\Rightarrow$ the whole strategy fails.

• Main objective: propose a *shape optimization* strategy to reconstruct the unknown inclusion when only Cauchy data are available *only on a subpart of the boundary of the domain of study*.

• Clearly, we have to reconstruct both ω and the missing data $\longrightarrow$ *data completion problem*.

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Data completion problem

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- Problem: find $u \in H^1(\Omega)$, s.t. $(\mathcal{P}_c) \begin{cases} \Delta u &= 0 & \text{in } \mathcal{D} \\ u &= g_D & \text{on } \Gamma \\ \partial_n u &= g_N & \text{on } \Gamma \end{cases}$

- This problem is severely ill-posed (exponentially ill-posed), it has *at most* one solution that does not depend continuously on the data.

In particular, the set of data for which the problem has no solution is dense in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \Rightarrow$ high instability $\Rightarrow$ it is mandatory to propose a regularization method to solve the problem numerically.

- In the sequel, we denote by u_{ex} the exact solution corresponding to exact data (g_D, g_N).

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Kohn-Vogelius minimization strategy

• Introduced in *Solving Cauchy problems by minimizing an energy-like functional*, S. Andrieux, T.N. Baranger and A. Ben Abda, Inverse Problems **22**, (2006).

• Main idea: minimize the energy functional

$$\mathcal{K}(\varphi, \psi) := \frac{1}{2} \int_{\mathcal{D}} |\nabla(u_\varphi - u_\psi)|^2 \, dx$$

over all $(\varphi, \psi) \in H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$, where u_φ and u_ψ verify

$$\left\{ \begin{array}{l} \Delta u_\varphi = 0 \text{ in } \mathcal{D} \\ u_\varphi = g_D \text{ on } \Gamma \\ \partial_\nu u_\varphi = \varphi \text{ on } \Gamma_c \end{array} \right. , \qquad \left\{ \begin{array}{l} \Delta u_\psi = 0 \text{ in } \mathcal{D} \\ \partial_\nu u_\psi = g_N \text{ on } \Gamma \\ u_\psi = \psi \text{ on } \Gamma_c. \end{array} \right.$$

• Easy remark: $\mathcal{K}(\varphi, \psi) = 0 \Leftrightarrow u_\varphi = u_{ex} = u_\psi + cte.$

Property

$$\inf_{H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)} \mathcal{K}(\varphi, \psi) = 0$$

Regularization of the K-V functional

• Regularized Kohn-Vogelius functional: for $\varepsilon > 0$, for $(\varphi, \psi) \in H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$,

$$\mathcal{K}_\varepsilon(\varphi, \psi) = \mathcal{K}(\varphi, \psi) + \frac{\varepsilon}{2}\left(\|v_\varphi\|^2_{H^1(\Omega)} + \|v_\psi\|^2_{H^1(\Omega)}\right).$$

with

$$\begin{cases} \Delta v_\varphi = 0 \text{ in } \mathcal{D} \\ v_\varphi = 0 \text{ on } \Gamma \\ \partial_\nu v_\varphi = \varphi \text{ on } \Gamma_c \end{cases}, \qquad \begin{cases} \Delta v_\psi = 0 \text{ in } \mathcal{D} \\ \partial_\nu v_\psi = 0 \text{ on } \Gamma \\ v_\psi = \psi \text{ on } \Gamma_c. \end{cases}$$

Property

There exists a unique $(\varphi_\varepsilon, \psi_\varepsilon) \in H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$ *s.t.*

$$\mathcal{K}_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon) = \underset{(\varphi,\psi)\in H^{-1/2}(\Gamma)}{\text{argmin}} \mathcal{K}_\varepsilon(\varphi, \psi).$$

Convergence results

Property

The sequence $(\varphi_\varepsilon, \psi_\varepsilon)$ *is a minimizing sequence for* $\mathcal{K}$.

Theorem

Suppose $(\mathcal{P}_c)$ *admits a (necessarily unique) solution* u_{ex}. *Then* $(\varphi_\varepsilon, \psi_\varepsilon)$ *converges to* $(\partial_\nu u_{ex}, u_{ex} + cte) \Leftrightarrow u_{\varphi_\varepsilon} \xrightarrow[H^1(\Omega)]{\varepsilon\to 0} u_{ex}$.

Furthermore, the convergence is monotonic: the map
$\varepsilon \mapsto \| u_{\varphi_\varepsilon} - u_{ex}, u_{\psi_\varepsilon} - u_{ex} \|_{H^1(\Omega)\times H^1(\Omega)}$ *is strictly increasing.*

Suppose $(\mathcal{P}_c)$ *does not admit a solution. Then*
$\lim_{\varepsilon\to 0} \| \varphi_\varepsilon, \psi_\varepsilon \|_{H^{-1/2}(\Gamma_c)\times H^{1/2}(\Gamma_c)} = +\infty$.

- It is mandatory to propose a strategy to deal with noisy data.

Derivatives of $\mathcal{K}_\varepsilon$

- We define w_N, $w_D \in H^1(\mathcal{D})$ solutions of

$$\begin{cases} \Delta w_N = \varepsilon v_\psi & \text{in } \mathcal{D} \\ \partial_\nu w_N = \partial_\nu u_\varphi - g_N & \text{on } \Gamma \\ w_N = 0 & \text{on } \Gamma_c \end{cases}, \qquad \begin{cases} \Delta w_D = \varepsilon v_\varphi & \text{in } \mathcal{D} \\ w_D = u_\psi - g_D & \text{on } \Gamma \\ \partial_\nu w_D = 0 & \text{on } \Gamma_c \end{cases}.$$

Property

For all $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})$ in $H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$, we have

$$\frac{\partial \mathcal{K}_\varepsilon}{\partial \varphi}(\varphi, \psi)[\tilde{\varphi}] = \langle \tilde{\varphi}, u_\varphi + \varepsilon v_\varphi + w_D - \psi \rangle_{\Gamma_c}$$

and

$$\frac{\partial \mathcal{K}_\varepsilon}{\partial \psi}(\varphi, \psi)[\tilde{\psi}] = \langle \partial_\nu u_\psi + \varepsilon \partial_\nu v_\psi + \partial_\nu w_N - \varphi, \tilde{\psi} \rangle_{\Gamma_c}.$$

Inverse obstacle problem with partial Cauchy data

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Find an inclusion $\mathcal{O}$, $\overline{\mathcal{O}} \subset \mathcal{D}$, $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ connected, and $u \in H^1(\Omega)$, s.t.

$$(\mathcal{P}) \begin{cases} \Delta u & = 0 & \text{in } \Omega \\ u & = g_D & \text{on } \Gamma \\ \partial_n u & = g_N & \text{on } \Gamma \\ u & = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

• Kohn-Vogelius strategy: minimization of the regularized Kohn-Vogelius functional w.r.t to ω, φ and ψ.

$$\mathcal{K}_\varepsilon(\omega, \varphi, \psi) := \int_{\mathcal{D}\setminus\overline{\omega}} |\nabla(u_\varphi - u_\psi)|^2 \, dx + \frac{\varepsilon}{2} \left(\|v_\varphi\|^2_{H^1(\mathcal{D}\setminus\overline{\omega})} + \|v_\psi\|^2_{H^1(\mathcal{D}\setminus\overline{\omega})} \right).$$

$\rightarrow$ Existence of a minimizer?

Computation of the shape derivative

- As usual, for $\mathbf{V} \in W^{2,\infty}(\mathbb{R}^d)$, compactly supported in $\mathcal{D}$, we note

$$D\mathcal{K}_\varepsilon(\omega) := \lim_{t \to 0} \frac{\mathcal{K}_\varepsilon((\mathbf{I} + t\mathbf{V})\omega) - \mathcal{K}_\varepsilon(\omega)}{t}.$$

Property

We have

$$\begin{aligned} D\mathcal{K}_\varepsilon(\omega) \cdot \mathbf{V} = &- \int_{\partial\omega} (\partial_\nu \rho_N^u \, \partial_\nu u_\varphi + \partial_\nu \rho_N^v \, \partial_\nu v_\varphi)(\mathbf{V} \cdot \nu) \\ &- \int_{\partial\omega} (\partial_\nu \rho_D^u \, \partial_\nu u_\varphi + \partial_\nu \rho_D^v \, \partial_\nu v_\psi)(\mathbf{V} \cdot \nu) \\ &+ \frac{1}{2} \int_{\partial\omega} |\nabla(u_\varphi - u_\psi)|^2 (\mathbf{V} \cdot \nu) \\ &+ \frac{\varepsilon}{2} \int_{\partial\omega} \left(|\nabla v_\varphi|^2 + |\nabla v_\psi|^2 + |v_\varphi|^2 + |v_\psi|^2 \right) (\mathbf{V} \cdot \nu) \end{aligned}$$

Global algorithm

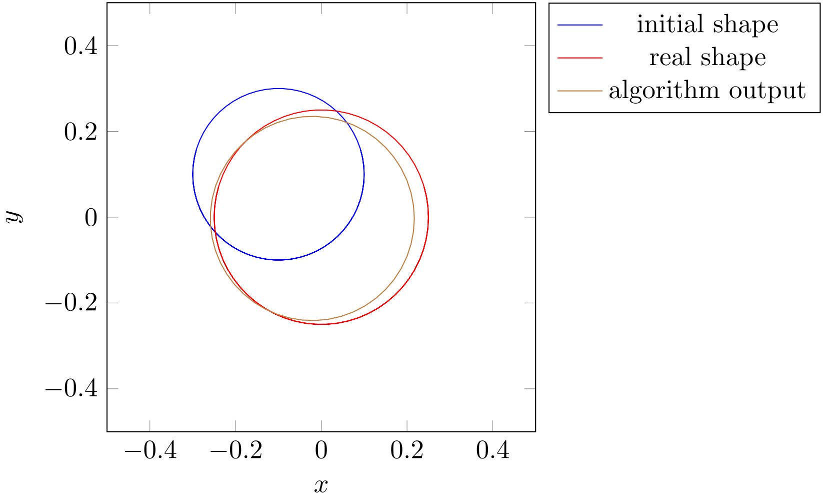
- choose an initial guess $(\omega_0, \varphi_0, \psi_0)$

- at step n,

1. solve 10 (!) elliptic problems in $\mathcal{D} \setminus \overline{\omega_n}$ to obtain u_{φ_n}, u_{ψ_n}, v_{φ_n}, v_{ψ_n}, w_N, w_D, ρ_D^u, ρ_N^u, ρ_D^v and ρ_N^v
2. compute the descent directions $\tilde{\varphi}$ and $\tilde{\psi}$
3. compute the $\nabla \mathcal{K}_\varepsilon(\omega_n)$
4. update φ_n, ψ_n, ω_n (line search) $\rightarrow \varphi_{n+1}$, ψ_{n+1}, ω_{n+1}.

- repeat until stopping criterion is reached.

Reconstructions - easy case



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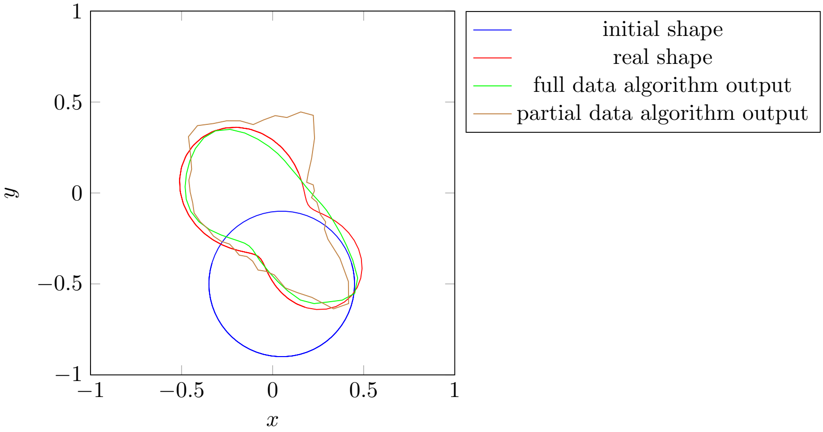
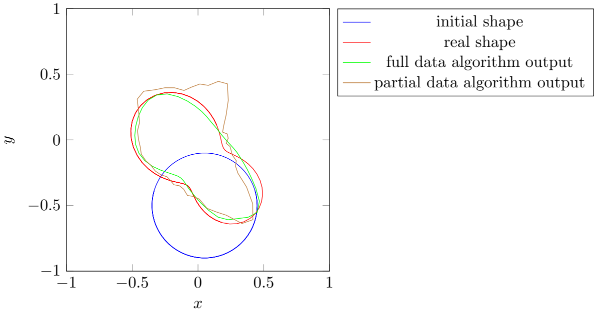


Future works



• In case of noisy data, propose a strategy to set the parameter of regularization w.r.t. noise amplitude
→ for the data completion problem, ok
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• Reconstruction of objects in fluids (Stokes and Navier-Stokes equations)

THANK YOU FOR YOUR ATTENTION

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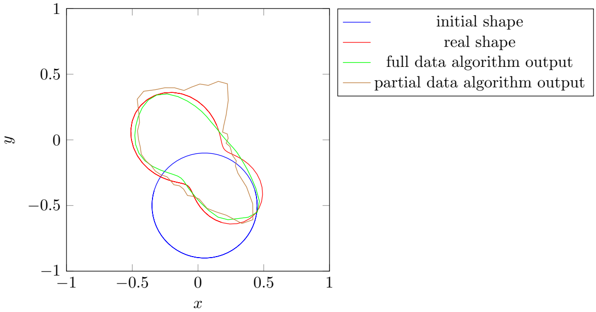

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