How to Choose the Right Shocks

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Conservation Laws

Introduction – Analytic Theories

$$\boxed{\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)}$$

$t \in \mathbb{R}_+$ time
$x \in \mathbb{R}^N$ space
$u \in \mathbb{R}^n$ unknown

f smooth flux
g smooth source

Scalar MultiD $\quad n = 1,\ N \geq 1$

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

Existence

(Kružkov: Mat.Sb., 1970)

Uniqueness

(Kružkov: Mat.Sb., 1970)

Dependence on data

(Kružkov: Mat.Sb., 1970)

Dependence on f, g

(Colombo, Mercier, Rosini: CMS, 2009)

Systems in 1D $\quad n \geq 1,\ N = 1$

$$\partial_t u + \partial_x f(u) = 0$$

Existence

(Glimm: CPAM, 1965)

Uniqueness

(Bressan & c.: 1999, 2000)

Dependence on data

(Bressan & c.: 1995, 2000)

Dependence on f

(Bianchini, Colombo: PAMS, 2002)

$$\boxed{n \geq 1,\ N \geq 1?}$$

Introduction – Key Features

1. Evolution
2. Irreversible
3. Finite Speed
4. Conservation
5. Singularities

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Simplest Case

$n = 1, N = 1$,

$f = f(u)$, $g \equiv 0$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_o(x) \end{cases}$$

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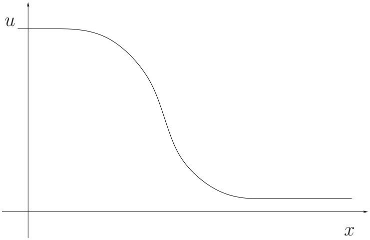
$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_o(x) \end{cases}$$

$f(u) = \lambda\, u \qquad \partial_t u + \lambda\, \partial_x u = 0 \qquad u(t,x) = u_o(x - \lambda\, t)$



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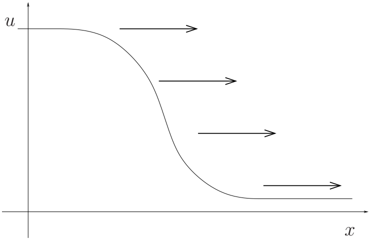

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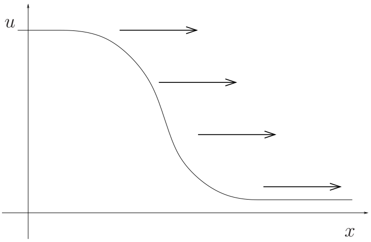

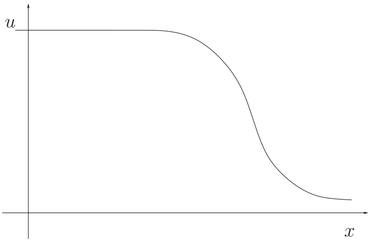

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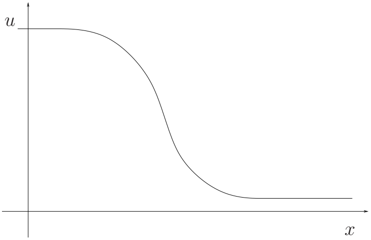
f non linear $\qquad \partial_t u + f'(u)\,\partial_x u = 0 \qquad u(t,x) = ?$



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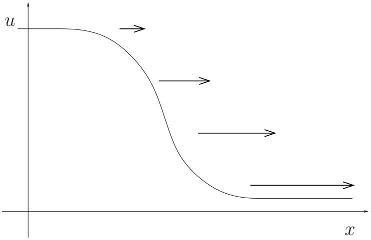

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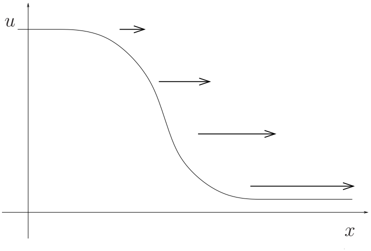

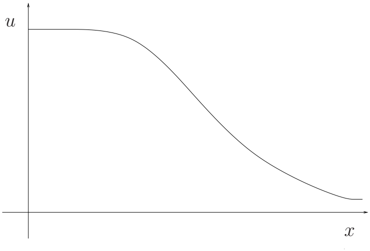

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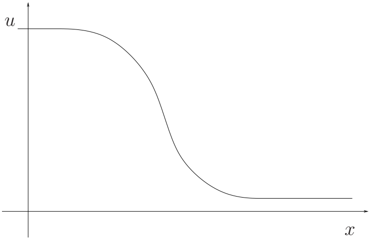

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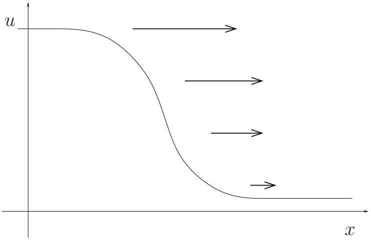

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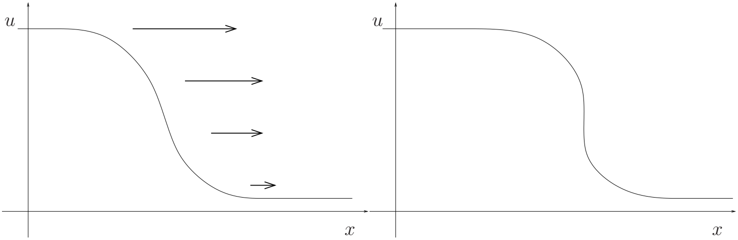

Discontinuities

Rankine–Hugoniot Conditions

Riemann Problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = \begin{cases} u^\ell & x < 0 \\ u^r & x > 0 \end{cases} \end{cases}$$

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Solution

$$u(t,x) = \begin{cases} u^\ell & x < \Lambda\, t \\ u^r & x > \Lambda\, t \end{cases}$$

$\Lambda = ?$

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Conservation:

$$\int_{x_1}^{x_2} u(t_2,x)\,\mathrm{d}x - \int_{x_1}^{x_2} u(t_1,x)\,\mathrm{d}x = \int_{t_1}^{t_2} f\left(u(t,x_1)\right)\mathrm{d}t - \int_{t_1}^{t_2} f\left(u(t,x_2)\right)\mathrm{d}t$$

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$$(u^\ell - u^r)(x_2 - x_1) = \Big(f(u^\ell) - f(u^r)\Big)(t_2 - t_1)$$

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$$(u^\ell - u^r)(x_2 - x_1) = \Big(f(u^\ell) - f(u^r)\Big)\,(t_2 - t_1)$$

$$\boxed{f(u^\ell) - f(u^r) = \Lambda(u^\ell - u^r)}$$

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$$n \geq 1,\ N = 1$$

Selecting the Right Discontinuities

Many choices!

Selecting the Right Discontinuities

Many choices!

Lack of Uniqueness = Lack of Information

Selecting the Right Discontinuities

Many choices!
Lack of Uniqueness = Lack of Information

Vanishing viscosity:
$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon\, \partial^2_{xx} u^\varepsilon$ and then $\varepsilon \to 0$

Selecting the Right Discontinuities

Many choices!

Lack of Uniqueness = Lack of Information

Vanishing viscosity:

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon\, \partial^2_{xx} u^\varepsilon \text{ and then } \varepsilon \to 0$$

Entropy inequality:

$$q \text{ entropy with } \eta \text{ entropy flux} \quad \rightleftharpoons \quad D\eta(u)\, Df(u) = Dq(u)$$

$$u \text{ entropy solution} \quad \rightleftharpoons \quad \partial_t \eta(u) + \partial_x q(u) \leq 0$$

for all convex entropies

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Linear Stability

Selecting the Right Discontinuities

Many choices!

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Linear Stability

Lax (Liu) Inequalities

Selecting the Right Discontinuities

Many choices!

STABILITY

Selecting the Right Discontinuities

Many choices!

STABILITY

Shocks & Rarefactions

Selecting the Right Discontinuities

Many choices!

STABILITY

Shocks & Rarefactions

(But other choices can also be necessary...)

Linearizing?

Linearizing?

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2}\, u^2\right) = 0 \\ u^\vartheta(0,x) = \vartheta\, x\, \chi_{[0,1]}(x) \end{cases}$$

$$u^\vartheta(t,x) = \frac{\vartheta\, x}{1 + \vartheta\, t}\, \chi_{[0,\sqrt{1+\eta\, t}]}(x)$$

$$\lim_{\Delta\vartheta \to 0} \frac{u^{\vartheta+\Delta\vartheta}(t) - u^\vartheta(t)}{\vartheta} \text{ not in } \mathbf{L}^1!$$

(Bressan, Guerra: DCDS, 1997)

More Information:

A. Bressan: *Hyperbolic systems of conservation laws*
Oxford University Press, 2000

C.M. Dafermos: *Hyperbolic conservation laws in continuum physics*
Springer, 2016

H. Holden & N.H. Risebro: *Front tracking for hyp. conservation laws*
Springer, 2015

P.G. LeFloch: *Hyperbolic systems of conservation laws*
Birkhäuser, 2002

Y. Lu: *Hyp. cons. laws and the compensated compactness method*
Chapman & Hall/CRC, 2003

D. Serre: *Systems of conservation laws 1 & 2*
Cambridge University Press, 1999