Existence and uniqueness for Mean Field Games with state constraints[*]

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Abstract

In this paper, we study deterministic mean field games for agents who operate in a bounded domain. In this case, the existence and uniqueness of Nash equilibria cannot be deduced as for unrestricted state space because, for a large set of initial conditions, the uniqueness of the solution to the associated minimization problem is no longer guaranteed. We attack the problem by interpreting equilibria as measures in a space of arcs. In such a relaxed environment the existence of solutions follows by set-valued fixed point arguments. Then, we give a uniqueness result for such equilibria under a classical monotonicity assumption.

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1 Introduction

Mean field games (MFG) theory has been introduced simultaneously by Lasry and Lions ([11], [12], [13]) and by Huang, Malhamé and Caine ([9], [10]) in order to study large population differential games. The main idea of such a theory is to borrow from statistical physics the general principle of a mean-field approach to describe equilibria in a system of many interacting particles.

In game theory, for a system with a finite number of players, the natural notion of equilibrium is the one introduced by John Nash. So, the notion of mean-field equilibrium suggested by Lasry-Lions is justified as being the limit, as $N \to \infty$, of the Nash equilibria for N-player games, under the assumption that players are symmetric and rational.

In deterministic settings, the equilibrium found in the mean field limit turns out to be a solution of the forward-backward system of PDEs

$$(MFG) \begin{cases} -\partial_t u + H(x, Du) = F(x, m) & \text{in } [0, T] \times \Omega, \\ \partial_t m - div(m D_p H(x, Du)) = 0 & \text{in } [0, T] \times \Omega, \\ m(0) = m_0 \quad u(x, T) = G(x, m(T)) \end{cases} \tag{1.1}$$

which couples a Hamilton-Jacobi-Bellman equation (for the value function u of the generic player) with a continuity equation (for the density m of players). Here $\Omega \subset \mathbb{R}^n$ represents the domain in the state space in which agents are supposed to operate.

The well-posedness of system (1.1) was developed for special geometries of the domain Ω, namely when Ω equals the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, or the whole space $\mathbb{R}^n$ (see, e.g., [6], [12], [13]). The goal of the present paper is to study the well-posedness of the MFG problem subject to state constraints, that is, when players are confined into a compact domain $\overline{\Omega} \subseteq \mathbb{R}^n$.

In the above references, the solution of (1.1) on $[0,T] \times \mathbb{T}^n$ is obtained by a fixed point argument which uses in an essential way the fact that viscosity solutions of the Hamilton-Jacobi equation

$$-\partial_t u + H(x, Du) = F(x, m) \quad \text{in } [0,T] \times \mathbb{T}^n$$

are smooth on a sufficiently large set to allow the continuity equation

$$\partial_t m - div(m D_p H(x, Du)) = 0 \quad \text{in } [0,T] \times \mathbb{T}^n$$

to be solvable. Specifically, it is known that u is of class $C_{loc}^{1,1}$ outside a closed singular set of zero Lebesgue measure. In this way, the coefficient $D_p H(x, Du)$ in the continuity equation turns out to be locally Lipschitz continuous on a "sufficiently large" open set. Such an "almost smooth" structure is lost in the presence of state constraints [7, Example 1.1]. Therefore, in order to prove the existence of solutions to (1.1) a complete change of paradigm is necessary.

In this paper, following the Lagrangian formulation of the unconstrained MFG problem proposed in [4], we define a "relaxed" notion of constrained MFG equilibria and solutions, for which we give existence and uniqueness results. Such a formulation consists of replacing probability measures on $\overline{\Omega}$ with measures on arcs in $\overline{\Omega}$. More precisely, on the metric space

$$\Gamma = \left\{ \gamma \in AC(0,T; \mathbb{R}^n) : \gamma(t) \in \overline{\Omega}, \quad \forall t \in [0,T] \right\}$$

with the uniform metric, for any $t \in [0,T]$ we consider the evaluation map $e_t : \Gamma \to \overline{\Omega}$ defined by

$$e_t(\gamma) = \gamma(t) \qquad (\gamma \in \Gamma).$$

Given any probability measure m_0 on $\overline{\Omega}$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0\sharp\eta = m_0$ and we consider, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, the functional

$$J_\eta[\gamma] = \int_0^T \left[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), e_t\sharp\eta) \right] dt + G(\gamma(T), e_T\sharp\eta) \qquad (\gamma \in \Gamma). \qquad (1.2)$$

Then, we call a measure $\eta \in \mathcal{P}_{m_0}(\Gamma)$ a *constrained MFG equilibrium* for m_0 if η is supported on the set of all curves $\overline{\gamma} \in \Gamma$ such that

$$J_\eta[\overline{\gamma}] \leq J_\eta[\gamma] \ \ \forall \gamma \in \Gamma, \ \gamma(0) = \overline{\gamma}(0).$$

Thus, we obtain the existence of constrained MFG equilibria for m_0 (Theorem 3.1) by applying the Kakutani fixed point theorem [14]. At this point, it is natural to define a *mild solution of the constrained MFG problem* in $\overline{\Omega}$ as a pair $(u,m) \in C([0,T] \times \overline{\Omega}) \times C([0,T]; \mathcal{P}(\overline{\Omega}))$, where m is given by $m(t) = e_t\sharp\eta$ for some constrained MFG equilibrium η for m_0 and

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t) = x}} \left\{ \int_t^T \left[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \right] ds + G(\gamma(T), m(T)) \right\}.$$

In this way, the existence of mild solutions of the constrained MFG problem in $\overline{\Omega}$ (Corollary 4.1) becomes an easy corollary of the existence of equilibria for m_0 (Theorem 3.1), whereas the uniqueness issue for such a problem remains a more challenging question. As observed by Lasry and Lions, in absence of state constraints uniqueness can be addressed by imposing suitable monotonicity assumptions on the data. We show that the same general strategy can be adopted even for constrained problems (Theorem 4.1). However, we have to interpret the Lasry-Lions method differently because, as recalled above, solutions are highly nonsmooth in our case.

The results of this paper can be regarded as an initial step of the study of deterministic MFG systems with state constraints. The natural sequel of our analysis would be to show that mild solutions to the constrained MFG problem in $\overline{\Omega}$ satisfy the MFG system in a suitable point-wise sense and, possibly, derive the uniqueness of solutions from such a system.

This paper is organised as follows. In Section 2, we introduce the notation and recall preliminary results. In Section 3, we define constrained MFG equilibria and we prove their existence. Section 4 is devoted to the study of mild solutions of the constrained MFG problem, in particular to the uniqueness issue.

2 Preliminaries

2.1 Notation

Throughout this paper we denote by $|\cdot|$, $\langle \cdot \rangle$, respectively, the Euclidean norm and scalar product in $\mathbb{R}^n$. For any subset $S \subset \mathbb{R}^n$, $\overline{S}$ stands for its closure, ∂S for its boundary and $S^c = \mathbb{R}^n \setminus S$ for the complement of S. We denote by $\mathbf{1}_S : \mathbb{R}^n \to \{0, 1\}$ the characteristic function of S, i.e.,

$$\mathbf{1}_S(x) = \begin{cases} 1 & x \in S, \\ 0 & x \in S^c. \end{cases}$$

We write $AC(0, T; \mathbb{R}^n)$ for the space of all absolutely continuous $\mathbb{R}^n$-valued functions on $[0, T]$, equipped with the uniform metric. We observe that such a space is not complete.
For any measurable function $f : [0, T] \to \mathbb{R}^n$, we set

$$||f||_2 = \left(\int_0^T |f|^2 \, dt \right)^{\frac{1}{2}}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^2$ boundary. The distance function from $\overline{\Omega}$ is the function $d_\Omega : \mathbb{R}^n \to [0, +\infty[$ defined by

$$d_\Omega(x) := \inf_{y \in \overline{\Omega}} |x - y| \quad (x \in \mathbb{R}^n).$$

We define the oriented boundary distance from $\partial\Omega$ by

$$b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x) \quad (x \in \mathbb{R}^n). \tag{2.1}$$

We recall that, since the boundary of Ω is of class $\mathcal{C}^2$, there exists $\rho_0 > 0$ such that

$$b_\Omega(\cdot) \in C_b^2 \ \text{ on } \ \partial\Omega + B_{\rho_0} = \left\{ y \in B(x, \rho_0) : x \in \partial\Omega \right\}, \tag{2.2}$$

where C_b^2 is the set of all functions with bounded derivates of first and second order. Throughout the paper, we suppose that ρ_0 is fixed so that (2.2) holds.

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2.2 Results from Measure Theory

In this section we introduce, without proof, some basic tools needed in the paper (see, e.g., [1]).
Let X be a separable metric space, we denote by $\mathscr{B}(X)$ the family of the Borel subset of X and by $\mathcal{P}(X)$ the family of all Borel probability measures on X. The support of $\mu \in \mathcal{P}(X)$, $supp(\mu)$, is the closed set defined by

$$supp(\mu) := \Big\{ x \in X : \mu(V) > 0 \text{ for each neighborhood V of } x \Big\}. \tag{2.3}$$

We say that a sequence $(\mu_n) \subset \mathcal{P}(X)$ is narrowly convergent to $\mu \in \mathcal{P}(X)$ if

$$\lim_{n \to \infty} \int_X f(x) \, d\mu_n(x) = \int_X f(x) \, d\mu(x) \quad \forall f \in C_b^0(X),$$

where $C_b^0(X)$ is the set of all bounded continuous functions on X.
We recall an interesting link between narrow convergence of probability measures and Kuratowski convergence of their supports.

Proposition 2.1. *If $(\mu_n) \subset \mathcal{P}(X)$ is a sequence narrowly converging to $\mu \in \mathcal{P}(X)$ then $supp(\mu) \subset K - \liminf_{n \to \infty} supp(\mu_n)$, i.e.*

$$\forall x \in supp(\mu) \, \exists \, x_n \in supp(\mu_n) : \lim_{n \to \infty} x_n = x.$$

The following theorem is a useful characterization of relatively compact sets with respect to narrow topology.

Theorem 2.1. *(Prokhorov's Theorem) If a set $\mathcal{K} \subset \mathcal{P}(X)$ is tight, i.e.*

$$\forall \epsilon > 0 \, \exists \, K_\epsilon \text{ compact in } X \text{ such that } \widehat{\eta}(K_\epsilon) \geq 1 - \epsilon \, \forall \widehat{\eta} \in \mathcal{K},$$

then $\mathcal{K}$ is relatively compact in $\mathcal{P}(X)$ with respect to narrow topology. Conversely, if X is a separable complete metric space then every relatively compact subset of $\mathcal{P}(X)$ is tight.

Let X be a separable metric space. We recall that X is a Radon space if every Borel probability measure $\mu \in \mathcal{P}(X)$ satisfies

$$\forall B \in \mathscr{B}(X), \forall \epsilon > 0, \, \exists K_\epsilon \text{ compact with } K_\epsilon \Subset B \text{ such that } \mu(B \setminus K_\epsilon) \leq \epsilon.$$

Let us denote by d the distance on X and, for $p \in [1, +\infty)$, by $\mathcal{P}_p(X)$ the set of probability measures m on X such that

$$\int_X d^p(x_0, x) \, dm(x) < +\infty, \quad \forall x_0 \in X.$$

The *Monge-Kantorowich distance* on $\mathcal{P}_p(X)$ is given by

$$d_p(m, m') = \inf_{\lambda \in \Pi(m,m')} \left[\int_{X^2} d(x,y)^p \, d\lambda(x,y) \right]^{1/p}, \tag{2.4}$$

where $\Pi(m, m')$ is the set of Borel probability measures on $X \times X$ such that $\lambda(A \times \mathbb{R}^n) = m(A)$ and $\lambda(\mathbb{R}^n \times A) = m'(A)$ for any Borel set $A \subset X$. In the particular case when $p = 1$, the distance d_p takes the name of Kantorovich-Rubinstein distance and the following formula holds

$$d_1(m, m') = sup\Big\{ \int_X f(x) \, dm(x) - \int_X f(x) \, dm'(x) \mid f : X \to \mathbb{R} \text{ is 1-Lipschitz}\Big\}, \tag{2.5}$$

for all $m, m' \in \mathcal{P}_1(X)$. In the next result, we recall the relationship between the weak-$*$ convergence of measures and convergence with respect to d_p.

Proposition 2.2. *If a sequence of measures $\{\mu_n\}_{n\geq 1} \subset \mathcal{P}_p(X)$ converges to μ for d_p, then $\{\mu_n\}_{n\geq 1}$ weakly converges to μ. "Conversely", if μ_n is concentrated on a fixed compact subset of X for all $n \geq 1$ and $\{\mu_n\}_{n\geq 1}$ weakly converges to μ, then the $\{\mu_n\}_{n\geq 1}$ converges to μ in d_p.*

Given separable metric spaces X_1 and X_2 and a Borel map $f : X_1 \to X_2$, we recall that the push-forward of a measure $\mu \in \mathcal{P}(X_1)$ through f is defined by

$$f\sharp\mu(B) := \mu(f^{-1}(B)) \quad \forall B \in \mathscr{B}(X_2). \tag{2.6}$$

The push-forward is characterized by the fact that

$$\int_{X_1} g(f(x)) \, d\mu(x) = \int_{X_2} g(y) \, df\sharp\mu(y) \tag{2.7}$$

for every Borel function $g : X_2 \to \mathbb{R}$.
We conclude this preliminary session by recalling the disintegration theorem.

Theorem 2.2. *Let X, Y be Radon separable metric spaces, $\mu \in \mathcal{P}(X)$, let $\pi : X \to Y$ be a Borel map and let $\eta = \pi\sharp\mu \in \mathcal{P}(Y)$. Then there exists an η-a.e. uniquely determined Borel measurable family* [1] *of probabilities $\{\mu_y\}_{y\in Y} \subset \mathcal{P}(X)$ such that*

$$\mu_y(X \setminus \pi^{-1}(y)) = 0 \text{ for } \eta\text{-a.e. } y \in Y \tag{2.8}$$

and

$$\int_X f(x) \, d\mu(x) = \int_Y \Big(\int_{\pi^{-1}(y)} f(x) \, d\mu_y(x) \Big) \, d\eta(y) \tag{2.9}$$

for every Borel map $f : X \to [0, +\infty]$.

3 Constrained MFG equilibria

3.1 Aproximation of constrained trajectories

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Let Γ be the metric subspace of $AC(0, T; \mathbb{R}^n)$ defined by

$$\Gamma = \Big\{ \gamma \in AC(0, T; \mathbb{R}^n) : \gamma(t) \in \overline{\Omega}, \quad \forall t \in [0, T] \Big\}.$$

For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{ \gamma \in \Gamma : \gamma(0) = x \} . \tag{3.1}$$

Lemma 3.1. *Let $\gamma \in AC(0, T; \mathbb{R}^n)$ and suppose that $d_\Omega(\gamma(t)) < \rho_0$ for all $t \in [0, T]$. Then $d_\Omega \circ \gamma \in AC(0, T)$ and*

$$\frac{d}{dt}(d_\Omega \circ \gamma)(t) = \big\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t) \big\rangle \mathbf{1}_{\Omega^c}(\gamma(t)) \text{ a.e. } t \in [0, T]. \tag{3.2}$$

Moreover,

$$N_\gamma := \{ t \in [0, T] : \gamma(t) \in \partial\Omega, \exists \, \dot{\gamma}(t), \langle Db_\Omega(\gamma(t)), \dot{\gamma}(t) \rangle \neq 0 \} \tag{3.3}$$

is a discrete set.

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[1] We say that $\{\mu_y\}_{y\in Y}$ is a Borel family (of probability measures) if $y \in Y \longmapsto \mu_y(B) \in \mathbb{R}$ is Borel for any Borel set $B \subset X$.

Proof. First we prove that N_γ is a discrete set. Let $t \in N_\gamma$, then there exists $\epsilon > 0$ such that $\gamma(s) \notin \partial\Omega$ for any $s \in (]t - \epsilon, t + \epsilon[\setminus \{t\}) \cap [0, T]$. Therefore, N_γ is composed of isolated points and so it is a discrete set.

Let us now set $\phi(t) = (d_\Omega \circ \gamma)(t)$ for all $t \in [0, T]$. We note that $\phi \in AC(0, T)$ because it is the composition of $\gamma \in AC(0, T; \mathbb{R}^n)$ with the Lipschitz continuous function $d_\Omega(\cdot)$. Denote by D the set of $t \in [0, T]$ such that there exists the first order derivative of γ in t, i.e.,

$$D = \{t \in [0, T] : \exists \dot{\gamma}(t)\}.$$

We observe that D has full Lebesgue measure and we decompose D in the following way:

$$D = \underbrace{\{t \in D : \gamma(t) \notin \partial\Omega\}}_{D_0} \cup \underbrace{\{t \in D : \gamma(t) \in \partial\Omega\}}_{D_1}.$$

By [8, Theorem 4, pg 129], for all $t \in D_0$ the first order derivative of ϕ is equal to

$$\dot{\phi}(t) = \begin{cases} 0 & \gamma(t) \in \Omega \\ \langle Db_\Omega(\gamma(t)), \dot{\gamma}(t) \rangle & \gamma(t) \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Now, consider $t \in D_1 \setminus N_\gamma$. Since $\gamma(t) \in \partial\Omega$, one has that

$$\frac{\phi(t + h) - \phi(t)}{h} = \frac{d_\Omega(\gamma(t + h))}{h},$$

for all $h > 0$. Since $\gamma(t + h) = \gamma(t) + h\dot{\gamma}(t) + o(h)$ and d_Ω is Lipschitz continuous, we obtain

$$0 \le \frac{d_\Omega(\gamma(t + h))}{h} \le \frac{o(h)}{h} + \frac{d_\Omega(\gamma(t) + h\dot{\gamma}(t))}{h}.$$

Hence, one has that

$$0 \le \liminf_{h \to 0} \frac{d_\Omega(\gamma(t + h))}{h} \le \limsup_{h \to 0} \frac{d_\Omega(\gamma(t + h))}{h} \le \limsup_{h \to 0} \frac{d_\Omega(\gamma(t) + h\dot{\gamma}(t))}{h}. \qquad (3.4)$$

Moreover, by the regularity of b_Ω, we obtain

$$d_\Omega(\gamma(t) + h\dot{\gamma}(t)) \le |b_\Omega(\gamma(t) + h\dot{\gamma}(t))| \le |h| \, |\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t) \rangle| + o(h). \qquad (3.5)$$

Thus, since $t \in D \setminus N_\gamma$, we conclude that

$$\limsup_{h \to 0} \frac{d_\Omega(\gamma(t) + h\dot{\gamma}(t))}{h} \le |Db_\Omega(\gamma(t)), \dot{\gamma}(t) \rangle| = 0. \qquad (3.6)$$

So $\dot{\phi}(t) = 0$ and the proof is complete. $\qquad \square$

Proposition 3.1. *Let $x_i \in \overline{\Omega}$ be such that $x_i \to x$ and let $\gamma \in \Gamma[x]$. Then there exists $\gamma_i \in \Gamma[x_i]$ such that:*

   (i) $\gamma_i \to \gamma$ uniformly on $[0, T]$;

   (ii) $\dot{\gamma}_i \to \dot{\gamma}$ a.e. on $[0, T]$;

  (iii) $|\dot{\gamma}_i(t)| \le C|\dot{\gamma}(t)|$ for any $i \ge 1$, a.e. $t \in [0, T]$, and some constant $C \ge 0$.

Proof. Let $\widehat{\gamma}_i$ be the trajectory defined by

$$\widehat{\gamma}_i(t) = \gamma(t) + x_i - x. \tag{3.7}$$

We observe that $d_\Omega(\widehat{\gamma}_i(t)) \le \rho_0$ for all $t \in [0, T]$ and all sufficiently large i, say $i \ge i_0$. Indeed,

$$d_\Omega(\widehat{\gamma}_i(t)) \le |\widehat{\gamma}_i(t) - \gamma(t)| = |x_i - x|.$$

Since $x_i \to x$, we have that $d_\Omega(\widehat{\gamma}_i(t)) \le \rho_0$ for all $t \in [0, T]$ and $i \ge i_0$. We denote by γ_i the projection of $\widehat{\gamma}_i$ on $\overline{\Omega}$, i.e.,

$$\gamma_i(t) = \widehat{\gamma}_i(t) - d_\Omega(\widehat{\gamma}_i(t))Db_\Omega(\widehat{\gamma}_i(t)) \quad \forall t \in [0, T]. \tag{3.8}$$

We note that $\gamma_i \in \Gamma[x_i]$. Moreover, γ_i converges uniformly to γ on $[0, T]$. Indeed,

$$|\gamma_i(t) - \gamma(t)| = |x_i - x - d_\Omega(\widehat{\gamma}_i(t))Db_\Omega(\widehat{\gamma}_i(t))| \le 2|x_i - x|, \quad \forall\, t \in [0, T].$$

By Lemma 3.1, $d_\Omega(\widehat{\gamma}_i(\cdot)) \in AC(0, T)$ and $\frac{d}{dt}\left(d_\Omega(\widehat{\gamma}_i(t))\right) = \langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\widehat{\gamma}}_i(t)\rangle \mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))$ a.e. $t \in [0, T]$. Using the regularity of b_Ω, we obtain

$$\dot{\gamma}_i(t) = \dot{\gamma}(t) - \langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t)) - d_\Omega(\widehat{\gamma}_i(t))D^2b_\Omega(\widehat{\gamma}_i(t))\dot{\gamma}(t), \text{ a.e. } t \in [0, T].$$

Therefore, there exists a constant $C \ge 0$ such that $|\dot{\gamma}_i(t)| \le C|\dot{\gamma}(t)|$ for any $i \ge i_0$, a.e. $t \in [0, T]$. Finally, we have to show that $\dot{\gamma}_i \to \dot{\gamma}$ almost everywhere on $[0, T]$. Since $\widehat{\gamma}_i \to \gamma$ and $\gamma \in \Gamma[x]$, one has that

$$d_\Omega(\widehat{\gamma}_i(t))D^2b_\Omega(\widehat{\gamma}_i(t))\dot{\gamma}(t) \xrightarrow{i\to\infty} 0, \quad \forall t \in [0, T].$$

So, we have to prove that

$$-\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t)) \xrightarrow{i\to\infty} 0, \text{ a.e. } t \in [0, T]. \tag{3.9}$$

We note that

$$\left|\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))\right| \le \left|\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle\right|, \text{ a.e. } t \in [0, T]. \tag{3.10}$$

Fix $t \in [0, T]$ such that (3.10) holds. If $\gamma(t) \in \Omega$ then $\widehat{\gamma}_i(t) \in \Omega$ for i large enough and (3.9) holds. On the other hand, if $\gamma(t) \in \partial\Omega$, then passing to the limit in (3.10), we have that

$$\limsup_{i\to\infty}\left|\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))\right| \le \limsup_{i\to\infty}\left|\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle\right|.$$

Since $\gamma_i \to \gamma$ uniformly on $[0, T]$, one has that

$$\limsup_{i\to\infty}\left|\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\rangle\right| = \left|\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t)\rangle\right|. \tag{3.11}$$

By Lemma 3.1, we have that $\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t)\rangle = 0$ for $t \in [0, T] \setminus N_\gamma$, where N_γ is the discret set defined in (3.3). So (3.9) holds for a.e. $t \in [0, T]$. Thus, $\dot{\gamma}_i$ converges almost everywhere to $\dot{\gamma}$ on $[0, T]$. This completes the proof. $\qquad\square$

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3.2 Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let $\mathcal{P}(\overline{\Omega})$ be the set of all Borel probability measures on $\overline{\Omega}$ endowed with the Kantorovich-Rubinstein distance d_1 defined in (2.4). We suppose throughout that $F, G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $L : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ are given continuous functions. Moreover, we assume the following conditions.

(L1) $L \in C^1(\overline{\Omega} \times \mathbb{R}^n)$ and for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$,

$$|D_x L(x, v)| \leq C(1 + |v|^2), \tag{3.12}$$

$$|D_v L(x, v)| \leq C(1 + |v|), \tag{3.13}$$

for some constant $C > 0$.

(L2) There exist constants $c_1, c_0 > 0$ such that

$$L(x, v) \geq c_1 |v|^2 - c_0, \quad \forall (x, v) \in \overline{\Omega} \times \mathbb{R}^n. \tag{3.14}$$

(L3) $v \longmapsto L(x, v)$ is convex for all $x \in \overline{\Omega}$.

Remark 3.1. (i) As $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$ is a compact set, the continuity of F and G implies that they are bounded and uniformly continuous on $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$.
(ii) In (L1), L is assumed to be of class $C^1(\overline{\Omega} \times \mathbb{R}^n)$ just for simplicity. All the results of this paper hold true if L is locally Lipschitz—hence, a.e. differentiable—in $\overline{\Omega} \times \mathbb{R}^n$ and satisfies the growth conditions (3.12) and (3.13) a.e. on $\overline{\Omega} \times \mathbb{R}^n$, see Remark 3.3 below.

3.3 Existence of constrained MFG equilibria

For any $t \in [0, T]$, we denote by $e_t : \Gamma \to \overline{\Omega}$ the evaluation map defined by

$$e_t(\gamma) = \gamma(t), \quad \forall \gamma \in \Gamma.$$

For any $\eta \in \mathcal{P}(\Gamma)$, we define

$$m^\eta(t) = e_t \sharp \eta, \quad \forall t \in [0, T]. \tag{3.15}$$

Lemma 3.2. *The following holds true.*

(i) $m^\eta \in C([0, T]; \mathcal{P}(\overline{\Omega}))$ *for any* $\eta \in \mathcal{P}(\Gamma)$.

(ii) *Let* $\eta_i, \eta \in \mathcal{P}(\Gamma)$, $i \geq 1$, *be such that* η_i *is narrowly convergent to* η. *Then* $m^{\eta_i}(t)$ *is narrowly convergent to* $m^\eta(t)$ *for all* $t \in [0, T]$.

Proof. First, we prove point (i). By definition (3.15), it is obvious that $m^\eta(t)$ is a Borel probability measure on $\overline{\Omega}$ for any $t \in [0, T]$. Let $\{t_k\} \subset [0, T]$ be a sequence such that $t_k \to \bar{t}$. We want to show that

$$\lim_{t_k \to \bar{t}} \int_{\overline{\Omega}} f(x) m^\eta(t_k, dx) = \int_{\overline{\Omega}} f(x) m^\eta(\bar{t}, dx), \tag{3.16}$$

for any $f \in C(\overline{\Omega})$. Since $m^\eta(t_k) = e_{t_k} \sharp \eta$ and $e_{t_k}(\gamma) = \gamma(t_k)$, we have that

$$\lim_{t_k \to \bar{t}} \int_{\overline{\Omega}} f(x) m^\eta(t_k, dx) = \lim_{t_k \to \bar{t}} \int_\Gamma f(e_{t_k}(\gamma)) \, d\eta(\gamma) = \lim_{t_k \to \bar{t}} \int_\Gamma f(\gamma(t_k)) \, d\eta(\gamma).$$

Since $f \in C(\overline{\Omega})$ and $\gamma \in \Gamma$, then $f(\gamma(t_k)) \to f(\gamma(\overline{t}))$ and $|f(\gamma(t_k))| \leq ||f||_\infty$. Therefore, by Lebesgue's dominated convergence theorem, we have that

$$\lim_{t_k \to \overline{t}} \int_\Gamma f(\gamma(t_k)) \, d\eta(\gamma) = \int_\Gamma f(\gamma(\overline{t})) \, d\eta(\gamma). \tag{3.17}$$

Thus, recalling the definition of m^η, we obtain (3.16). Moreover, by Proposition 2.2, we conclude that $d_1(m^\eta(t_k), m^\eta(\overline{t})) \to 0$. This completes the proof of point (i).

In order to prove point (ii), we suppose that η_i is narrowly convergent to η. Then, for all $f \in C(\overline{\Omega})$ we have that

$$\lim_{i \to \infty} \int_{\overline{\Omega}} f(x) m^{\eta_i}(t, \, dx) = \lim_{i \to \infty} \int_\Gamma f(\gamma(t)) \, d\eta_i(\gamma) = \int_\Gamma f(\gamma(t)) \, d\eta(\gamma) = \int_{\overline{\Omega}} f(x) m^\eta(t, \, dx).$$

Hence, $m^{\eta_i}(t)$ is narrowly convergent to $m^\eta(t)$ for all $t \in [0, T]$. $\qquad\square$

For any fixed $m_0 \in \mathcal{P}(\overline{\Omega})$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0 \sharp \eta = m_0$. For all $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$J_\eta[\gamma] = \int_0^T \left[L(\gamma(t), \dot\gamma(t)) + F(\gamma(t), m^\eta(t)) \right] dt + G(\gamma(T), m^\eta(T)), \quad \gamma \in \Gamma. \tag{3.18}$$

Remark 3.2. We note that $\mathcal{P}_{m_0}(\Gamma)$ is nonempty. Indeed, let $j : \overline{\Omega} \to \Gamma$ be the continuous map defined by

$$j(x)(t) = x \quad \forall t \in [0, T].$$

Then,

$$\eta := j \sharp m_0$$

is a Borel probability measure on Γ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$.

For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$\Gamma^\eta[x] = \left\{ \gamma \in \Gamma[x] : J_\eta[\gamma] = \min_{\Gamma[x]} J_\eta \right\}. \tag{3.19}$$

Definition 3.1. *Let $m_0 \in \mathcal{P}(\overline{\Omega})$. We say that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if*

$$supp(\eta) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^\eta[x]. \tag{3.20}$$

In other words, $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if for η-a.e. $\overline{\gamma} \in \Gamma$ we have that

$$J_\eta[\overline{\gamma}] \leq J_\eta[\gamma], \quad \forall \gamma \in \Gamma[\overline{\gamma}(0)].$$

The main result of this section is the existence of constrained MFG equilibria for m_0.

Theorem 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L1)-(L3) hold true. Let $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ be continuous. Then, there exists at least one constrained MFG equilibrium for m_0.*

Theorem 3.1 will be proved in Section 3.4. Now, we will show some properties of $\Gamma^\eta[x]$ that we will use in what follows.

Lemma 3.3. *For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$ the following holds true.*

(i) $\Gamma^\eta[x]$ *is a nonempty set.*

(ii) *All* $\gamma \in \Gamma^\eta[x]$ *satisfy*

$$||\dot{\gamma}||_2 \leq K, \tag{3.21}$$

where

$$K = \frac{1}{\sqrt{c_1}} \left[T \max_{\overline{\Omega}} L(x, 0) + 2T \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |F| + 2 \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |G| + Tc_0 \right]^{\frac{1}{2}} \tag{3.22}$$

and c_0, c_1 *are the constants in* (3.14). *Consequently, all minimizers* $\gamma \in \Gamma^\eta[x]$ *are* $\frac{1}{2}$*-Hölder continuous of constant* K.

In addition, if $\eta \in \mathcal{P}_{m_0}(\Gamma)$ *is a constrained MFG equilibrium for* m_0, *then* $m^\eta(t) = e_t \sharp \eta$ *is* $\frac{1}{2}$*-Hölder continuous of constant* K.

Proof. By classical results in the calculus of variation (see, e.g., [5, Theorem 6.1.2]), there exists at least one mimimizer of $J_\eta[\cdot]$ on Γ for any fixed initial point $x \in \overline{\Omega}$. So $\Gamma^\eta[x]$ is a nonempty set.

Let $x \in \overline{\Omega}$ and let $\gamma \in \Gamma^\eta[x]$. By comparing the cost of γ with the cost of the constant trajectory $\gamma(0) \equiv x$, one has that

$$\int_0^T \left[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^\eta(t)) \right] dt + G(\gamma(T), m^\eta(T)) \tag{3.23}$$

$$\leq \int_0^T \left[L(x, 0) + F(x, m^\eta(t)) \right] dt + G(x, m^\eta(T))$$

$$\leq \left[T \max_{\overline{\Omega}} L(x, 0) + T \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |F| + \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |G| \right].$$

Using (3.14) in (3.23), one has that

$$||\dot{\gamma}||_2 \leq \frac{1}{\sqrt{c_1}} \left[T \max_{\overline{\Omega}} L(x, 0) + 2T \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |F| + 2 \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |G| + Tc_0 \right]^{\frac{1}{2}} = K, \tag{3.24}$$

where c_0, c_1 are the constants in (3.14). This completes the proof of point (ii) since the Hölder regularity of γ is a direct consequence of the estimate (3.24).

Finally, we claim that, if η is a constrained MFG equilibrium for m_0, then the map $t \to m^\eta(t)$ is $\frac{1}{2}$-Hölder continuous with constant K. Indeed, for any $t_1, t_2 \in [0, T]$, we have that

$$d_1(m^\eta(t_2), m^\eta(t_1)) = \sup_\phi \int_{\overline{\Omega}} \phi(x) \left(m^\eta(t_2, dx) - m^\eta(t_1, dx) \right), \tag{3.25}$$

where the supremum is taken over the set of all 1-Lipschitz continuous maps $\phi : \overline{\Omega} \to \mathbb{R}$. Since $m^\eta(t) = e_t \sharp \eta$ and the map ϕ is 1-Lipschitz continuous, one has that

$$\int_{\overline{\Omega}} \phi(x) \left(m^\eta(t_2, dx) - m^\eta(t_1, dx) \right) = \int_\Gamma \left[\phi(e_{t_2}(\gamma)) - \phi(e_{t_1}(\gamma)) \right] d\eta(\gamma)$$

$$= \int_\Gamma \left[\phi(\gamma(t_2)) - \phi(\gamma(t_1)) \right] d\eta(\gamma) \leq \int_\Gamma |\gamma(t_2) - \gamma(t_1)| \, d\eta(\gamma).$$

Since η is a constrained MFG equilibrium for m_0, property (ii) yields

$$\int_\Gamma |\gamma(t_2) - \gamma(t_1)| \, d\eta(\gamma) \leq K \int_\Gamma |t_2 - t_1|^{\frac{1}{2}} \, d\eta(\gamma) = K |t_2 - t_1|^{\frac{1}{2}}.$$

Hence, we conclude that

$$d_1(m^\eta(t_2), m^\eta(t_1)) \leq K |t_2 - t_1|^{\frac{1}{2}}, \quad \forall \, t_1, t_2 \in [0, T]$$

and the map $t \longmapsto m^\eta(t)$ is 1/2-Hölder continuous. $\qquad\square$

Lemma 3.4. *Let η_i, $\eta \in \mathcal{P}_{m_0}(\Gamma)$ be such that η_i narrowly converges to η. Let $x_i \in \overline{\Omega}$ be such that $x_i \to x$ and let $\gamma_i \in \Gamma^{\eta_i}[x_i]$ be such that $\gamma_i \to \overline{\gamma}$. Then $\overline{\gamma} \in \Gamma^{\eta}[x]$. Consequently, $\Gamma^{\eta}[\cdot]$ has closed graph.*

Proof. We want to prove that

$$J_\eta[\overline{\gamma}] \le J_\eta[\gamma], \quad \forall \gamma \in \Gamma[x] \text{ such that } \int_0^T |\dot{\gamma}|^2 \, dt < \infty. \tag{3.26}$$

We observe that the above request is not restrictive because, by assuption (L2), if $\int_0^T |\dot{\gamma}|^2 \, dt = \infty$ then the above inequality is trivial.

Fix $\gamma \in \Gamma[x]$ with $\int_0^T |\dot{\gamma}|^2 \, dt < \infty$, by Proposition 3.1, we have that there exists $\widehat{\gamma}_i \in \Gamma[x_i]$ such that $\widehat{\gamma}_i \to \gamma$ uniformly on $[0, T]$, $\dot{\widehat{\gamma}}_i \to \dot{\gamma}$ a.e. on $[0, T]$ and $|\dot{\widehat{\gamma}}_i(t)| \le C|\dot{\gamma}(t)|$ for any $i \ge 1$, a.e. $t \in [0, T]$, and some constant $C \ge 0$. Since $\gamma_i \in \Gamma^{\eta_i}[x_i]$, one has that

$$J_{\eta_i}[\gamma_i] \le J_{\eta_i}[\widehat{\gamma}_i], \quad \forall i \ge 1. \tag{3.27}$$

So, in order to prove (3.26), we have to check that

(a) $J_\eta[\overline{\gamma}] \le \liminf_{i \to \infty} J_{\eta_i}[\gamma_i]$;

(b) $\lim_{i \to +\infty} J_{\eta_i}[\widehat{\gamma}_i] = J_\eta[\gamma]$.

First we show that (a) holds, that is,

$$\liminf_{i \to \infty} \left\{ \int_0^T \left[L\big(\gamma_i(t), \dot{\gamma}_i(t))\big) + F\big(\gamma_i(t), m^{\eta_i}(t)\big) \right] dt + G(\gamma_i(T), m^{\eta_i}(T)) \right\}$$
$$\ge \int_0^T \left[L\big(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))\big) + F\big(\overline{\gamma}(t), m^{\eta}(t)\big) \right] dt + G(\overline{\gamma}(T), m^{\eta}(T)). \tag{3.28}$$

First of all, we recall that, by Lemma 3.2, $m^{\eta_i}(t)$ narrowly converges to $m^{\eta}(t)$ for all $t \in [0, T]$. Owing to the convergence of γ_i to $\overline{\gamma}$, the narrow convergence of $m^{\eta_i}(t)$ to $m^{\eta}(t)$, and our assumption on F and G, we conclude that

$$\int_0^T F(\gamma_i(t), m^{\eta_i}(t)) \, dt \xrightarrow{i \to \infty} \int_0^T F(\overline{\gamma}(t), m^{\eta}(t)) \, dt,$$
$$G(\gamma_i(T), m^{\eta_i}(T)) \xrightarrow{i \to \infty} G(\overline{\gamma}(T), m^{\eta}(T)).$$

Up to taking a subsequence of γ_i, we can assume that $\dot{\gamma}_i \rightharpoonup \dot{\overline{\gamma}}$ in $L^2(0, T; \mathbb{R}^n)$ without loss of generality. By assumption (L3), one has that

$$\int_0^T L(\gamma_i(t), \dot{\gamma}_i(t)) = \int_0^T L(\overline{\gamma}(t), \dot{\gamma}_i(t)) \, dt + \int_0^T \left[L(\gamma_i(t), \dot{\gamma}_i(t)) - L(\overline{\gamma}(t), \dot{\gamma}_i(t)) \right] dt$$
$$\ge \int_0^T \left[L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)) + \langle D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)), \dot{\gamma}_i - \dot{\overline{\gamma}} \rangle \right] dt + \int_0^T \left[L(\gamma_i(t), \dot{\gamma}_i(t)) - L(\overline{\gamma}(t), \dot{\gamma}_i(t)) \right] dt.$$

Since $\gamma_i \in \Gamma^{\eta_i}[x_i]$ and $\gamma_i \to \overline{\gamma}$, by (L1), we obtain

$$\int_0^T \left[L(\gamma_i(t), \dot{\gamma}_i(t)) - L(\overline{\gamma}(t), \dot{\gamma}_i(t)) \right] dt \xrightarrow{i \to \infty} 0.$$

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Moreover, since $\dot{\gamma}_i \rightharpoonup \dot{\overline{\gamma}}$ in $L^2(0, T; \mathbb{R}^n)$, one has that

$$\int_0^T \left\langle D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)), \dot{\gamma}_i - \dot{\overline{\gamma}} \right\rangle dt \xrightarrow{i \to \infty} 0.$$

Thus, (3.28) holds.

Finally, we prove (b), i.e.,

$$\lim_{i \to \infty} \left\{ \int_0^T L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) + F(\widehat{\gamma}_i(t), m^{\eta_i}(t)) \, dt + G(\widehat{\gamma}_i(T), m^{\eta_i}(T)) \right\}$$
$$= \int_0^T L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^{\eta}(t)) \, dt + G(\gamma(T), m^{\eta}(T)).$$

Owing to the convergence of $\widehat{\gamma}_i$ to γ, the narrow convergence of $m^{\eta_i}(t)$ to $m^{\eta}(t)$ for all $t \in [0, T]$, and our assumption on F and G, one has

$$\int_0^T F(\widehat{\gamma}_i(t), m^{\eta_i}(t)) \, dt \xrightarrow{i \to \infty} \int_0^T F(\gamma(t), m^{\eta}(t)) \, dt,$$
$$G(\widehat{\gamma}_i(T), m^{\eta_i}(T)) \xrightarrow{i \to \infty} G(\gamma(T), m^{\eta}(T)).$$

Hence, we only need to prove that

$$\liminf_{i \to \infty} \int_0^T L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) \, dt = \int_0^T L(\gamma(t), \dot{\gamma}(t)) \, dt. \tag{3.29}$$

Owing to (L1), one has that

$$\left| \int_0^T \left[L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\gamma}(t)) \right] dt \right|$$
$$\leq \int_0^T \left| L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\widehat{\gamma}}_i(t)) \right| dt + \int_0^T \left| L(\gamma(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\gamma}(t)) \right| dt$$
$$\leq \|\widehat{\gamma}_i - \gamma\|_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2 \right) dt + \int_0^T \left| \int_0^1 \left\langle D_v L(\gamma(t), \lambda \dot{\widehat{\gamma}}_i + (1 - \lambda)\dot{\gamma}), \dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t) \right\rangle d\lambda \right| dt$$
$$\leq \|\widehat{\gamma}_i - \gamma\|_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2 \right) dt + C \int_0^T \int_0^1 \left(1 + |\dot{\widehat{\gamma}}_i| + |\dot{\gamma}| \right) |\dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t)| \, dt.$$

Since $\widehat{\gamma}_i \to \gamma$ uniformly on $[0, T]$ and $|\dot{\widehat{\gamma}}_i(t)| \leq C|\dot{\gamma}(t)|$ for any $i \geq 1$ and for any $t \in [0, T]$, we have that

$$\|\widehat{\gamma}_i - \gamma\|_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2 \right) dt \xrightarrow{i \to \infty} 0.$$

In addition, since $\dot{\widehat{\gamma}}_i \to \dot{\gamma}$ a.e. on $[0, T]$, by Lebesgue's dominated convergence theorem we obtain

$$C \int_0^T \int_0^1 \left(1 + |\dot{\widehat{\gamma}}_i| + |\dot{\gamma}| \right) \left| \dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t) \right| dt \xrightarrow{i \to \infty} 0. \tag{3.30}$$

This gives (b) and the proof is complete. $\qquad \square$

Remark 3.3. The above proof can be adapted to treat the case of a locally Lipschitz Lagrangian L as was mentioned in Remark 3.1. Indeed, it suffices to replace the gradient $D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))$ with a measurable selection of the subdifferential $\partial_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))$.

3.4 Proof of Theorem 3.1

In this section we prove Theorem 3.1 using a fixed point argument. First of all, we recall that, by Theorem 2.2, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, there exists a unique Borel measurable family of probabilities $\{\eta_x\}_{x \in \overline{\Omega}}$ on Γ which disintegrates η in the sense that

$$
\begin{cases}
\eta(d\gamma) = \int_{\overline{\Omega}} \eta_x(d\gamma) \, dm_0(x), \\
supp(\eta_x) \subset \Gamma[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega}.
\end{cases}
\tag{3.31}
$$

We introduce the set-valued map $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$ by defining, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$,

$$
E(\eta) = \left\{ \widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : supp(\widehat{\eta}_x) \subseteq \Gamma^\eta[x] \; m_0 - \text{a.e. } x \in \overline{\Omega} \right\}.
\tag{3.32}
$$

Then, it is immediate to realize that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if and only if $\eta \in E(\eta)$. We will therefore show that the set-valued map E has a fixed point. For this purpose, we will apply Kakutani's Theorem [14]. The following lemmas are intended to check that the assumptions of such a theorem are satisfied by E.

Lemma 3.5. *For any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, $E(\eta)$ is a nonempty convex set.*

Proof. First, we note that $E(\eta)$ is a nonempty set. Indeed, by (i) of Lemma 3.3, Lemma 3.4, and [2, Theorem 8.1.4] we have that $x \longmapsto \Gamma^\eta[x]$ is measurable. Moreover, by [2, Theorem 8.1.3], $x \longmapsto \Gamma^\eta[x]$ has a Borel measurable selection $x \longmapsto \gamma_x^\eta$. Thus, the measure $\widehat{\eta}$, defined by $\widehat{\eta}(\mathcal{B}) = \int_{\overline{\Omega}} \delta_{\{\gamma_x^\eta\}}(\mathcal{B}) \, dm_0(x)$ for any $\mathcal{B} \in \mathscr{B}(\Gamma)$, belongs to $E(\eta)$.

Now we want to check that $E(\eta)$ is a convex set. Let $\{\eta_i\}_{i=1,2} \in E(\eta)$ and let $\lambda_1, \lambda_2 \geq 0$ be such that $\lambda_1 + \lambda_2 = 1$. Since η_i are Borel probability measures, $\widehat{\eta} := \lambda_1 \eta_1 + \lambda_2 \eta_2$ is a Borel probability measure as well. Moreover, for any Borel set $B \in \mathscr{B}(\overline{\Omega})$ we have that

$$
e_0 \sharp \widehat{\eta}(B) = \widehat{\eta}(e_0^{-1}(B)) = \sum_{i=1}^2 \lambda_i \eta_i(e_0^{-1}(B)) = \sum_{i=1}^2 \lambda_i e_0 \sharp \eta_i(B) = \sum_{i=1}^2 \lambda_i m_0(B) = m_0(B).
\tag{3.33}
$$

So, $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma)$. Since $\eta_1 \in E(\eta)$, there exists a unique Borel measurable family of probabilities $\{\eta_{1,x}\}_{x \in \overline{\Omega}}$ on Γ which disintegrates η_1 as in (3.31) and there exists $A_1 \subset \overline{\Omega}$, with $m_0(A_1) = 0$, such that

$$
supp(\eta_{1,x}) \subset \Gamma^\eta[x], \quad x \in \overline{\Omega} \setminus A_1.
\tag{3.34}
$$

In the same way, $\eta_2(d\gamma) = \int_{\overline{\Omega}} \eta_{2,x}(d\gamma) \, dm_0(x)$ can be disintegrated and one has that

$$
supp(\eta_{2,x}) \subset \Gamma^\eta[x] \quad x \in \overline{\Omega} \setminus A_2,
\tag{3.35}
$$

where A_2 is such that $m_0(A_2) = 0$. Hence, $\widehat{\eta}$ can be disintegrated in the following way:

$$
\begin{cases}
\widehat{\eta}(d\gamma) = \int_{\overline{\Omega}} \left(\lambda_1 \eta_{1,x} + \lambda_2 \eta_{2,x} \right)(d\gamma) dm_0(x), \\
supp(\lambda_1 \eta_{1,x} + \lambda_2 \eta_{2,x}) \subset \Gamma^\eta[x] \quad x \in \overline{\Omega} \setminus (A_1 \cup A_2),
\end{cases}
\tag{3.36}
$$

where $m_0(A_1 \cup A_2) = 0$. This completes the proof. $\qquad\square$

Lemma 3.6. *The map $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$ has closed graph.*

Proof. Let $\eta_i, \eta \in \mathcal{P}_{m_0}(\Gamma)$ be such that η_i is narrowly convergent to η. Let $\widehat{\eta}_i \in E(\eta_i)$ be such that $\widehat{\eta}_i$ is narrowly convergent to $\widehat{\eta}$. We have to prove that $\widehat{\eta} \in E(\eta)$. Since $\widehat{\eta}_i$ narrowly converges to $\widehat{\eta}$, we have that $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma)$ and there exists a unique Borel measurable family of probabilities $\{\widehat{\eta}_x\}_{x \in \overline{\Omega}}$ on Γ such that $\widehat{\eta}(d\gamma) = \int_{\overline{\Omega}} \widehat{\eta}_x(d\gamma)\, dm_0(x)$ and $supp(\widehat{\eta}_x) \subset \Gamma[x]$ for m_0-a.e. $x \in \overline{\Omega}$. Hence, $\widehat{\eta} \in E(\eta)$ if and only if $supp(\widehat{\eta}_x) \subseteq \Gamma^\eta[x]$ for m_0-a.e $x \in \overline{\Omega}$. Let $\Omega_0 \subset \overline{\Omega}$ be an m_0-null set such that

$$supp(\widehat{\eta}_x) \subset \Gamma[x] \quad \forall x \in \overline{\Omega} \setminus \Omega_0.$$

Let $x \in \overline{\Omega} \setminus \Omega_0$ and let $\widehat{\gamma} \in supp(\widehat{\eta}_x)$. Since $\widehat{\eta}_i$ narrowly converges to $\widehat{\eta}$, then, by Proposition 2.1, one has that

$$\exists\, \widehat{\gamma}_i \in\ supp(\widehat{\eta}_i) \text{ such that } \lim_{i \to \infty} \widehat{\gamma}_i = \widehat{\gamma}.$$

Let $\widehat{\gamma}_i(0) = x_i$, with $x_i \in \overline{\Omega}$. Since $\widehat{\eta}_i \in E(\eta_i)$ and $\widehat{\gamma}_i \in supp(\widehat{\eta}_i)$, we have that $\widehat{\gamma}_i$ is an optimal trajectory for $J_{\eta_i}[\cdot]$, i.e.,

$$J_{\eta_i}[\widehat{\gamma}_i] \leq J_{\eta_i}[\gamma] \quad \forall \gamma \in \Gamma[x_i]. \tag{3.37}$$

As η_i narrowly converges to η, applying Lemma 3.4, we conclude that $\widehat{\gamma} \in \Gamma^\eta[x]$. Since x is any point in $\overline{\Omega} \setminus \Omega_0$, we have shown that $\widehat{\eta} \in E(\eta)$. $\qquad \square$

We denote by Γ_K the set of trajectories $\gamma \in \Gamma$ such that γ satisfies (3.21), i.e.,

$$\Gamma_K = \{\gamma \in \Gamma : ||\dot{\gamma}||_2 \leq K\} \tag{3.38}$$

where K is the constant given by (3.22). By the definition of $E(\eta)$ in (3.32) and Lemma 3.3, we deduce that

$$E(\eta) \subseteq \mathcal{P}_{m_0}(\Gamma_K) \quad \forall \eta \in \mathcal{P}_{m_0}(\Gamma). \tag{3.39}$$

Remark 3.4. In general Γ fails to be complete as a metric space. Then, by Theorem 2.1, $\mathcal{P}_{m_0}(\Gamma)$ is not a compact set. On the other hand, if Γ is replaced by Γ_K then $\mathcal{P}_{m_0}(\Gamma_K)$ is a compact convex subset of $\mathcal{P}_{m_0}(\Gamma)$. Indeed, the convexity of $\mathcal{P}_{m_0}(\Gamma_K)$ follows by the same argument used in the proof of Lemma 3.5. As for compactness, let $\{\eta_k\} \subset \mathcal{P}_{m_0}(\Gamma_K)$. Since Γ_K is a compact set, $\{\eta_k\}$ is tight. So, by Theorem 2.1, one finds a subsequence, that we denote again by η_k, which narrowly converges to some probability measure $\eta \in \mathcal{P}_{m_0}(\Gamma_K)$.

We will restrict domain of interest to $\mathcal{P}_{m_0}(\Gamma_K)$, where Γ_K is given by (3.38). Hereafter, we denote by E the restriction $E_{|\mathcal{P}_{m_0}(\Gamma_K)}$.

Conclusion

By Remark 3.4 and Remark 3.2, $\mathcal{P}_{m_0}(\Gamma_K)$ is a nonempty compact convex set. Moreover, by Lemma 3.5, $E(\eta)$ is nonempty convex set for any $\eta \in \mathcal{P}_{m_0}(\Gamma_K)$ and, by Lemma 3.6, the set-valued map E has closed graph. Then, the assumptions of Kakutani's Theorem are satisfied and so there exists $\overline{\eta} \in \mathcal{P}_{m_0}(\Gamma_K)$ such that $\overline{\eta} \in E(\overline{\eta})$.

4 Mild solution of the constrained MFG problem

In this section we define mild solutions of the constrained MFG problem in $\overline{\Omega}$. Moreover, under the assumptions of Section 3.2, we prove the existence of such solutions. Then, we give a uniqueness result under a classical monotonicity assumption on F and G.

Definition 4.1. *We say that $(u, m) \in C([0, T] \times \overline{\Omega}) \times C([0, T]; \mathcal{P}(\overline{\Omega}))$ is a mild solution of the constrained MFG problem in $\overline{\Omega}$ if there exists a constrained MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that*

(i) $m(t) = e_t \sharp \eta$ for all $t \in [0, T]$;

(ii) u *is given by*

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t) = x}} \left\{ \int_t^T [L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(s))] \, ds + G(\gamma(T), m(T)) \right\}, \quad (4.1)$$

for $(t,x) \in [0,T] \times \overline{\Omega}$.

A direct consequence of Theorem 3.1 is the following result.

Corollary 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L1)-(L3) hold true. Let $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ be continuous. There exists at least one mild solution (u,m) of the constrained MFG problem in $\overline{\Omega}$.*

Before proving our uniqueness result, we recall the following definitions.

Definition 4.2. *We say that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is monotone if*

$$\int_{\overline{\Omega}} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0, \quad (4.2)$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$.
We say that F is strictly monotone if

$$\int_{\overline{\Omega}} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0, \quad (4.3)$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$ and $\int_{\overline{\Omega}}(F(x, m_1) - F(x, m_2))d(m_1 - m_2)(x) = 0$ if and only if $F(x, m_1) = F(x, m_2)$ for all $x \in \overline{\Omega}$.

Example 4.1. Assume that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is of the form

$$F(x, m) = \int_{\overline{\Omega}} f(y, (\phi \star m)(y)) \phi(x - y) \, dy, \quad (4.4)$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth even kernel with compact support and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that $z \to f(x, z)$ is strictly increasing for all $x \in \overline{\Omega}$. Then F satisfies condition (4.3). Indeed, for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$, we have that

$$\int_{\overline{\Omega}} [F(x, m_1) - F(x, m_2)] \, d(m_1 - m_2)(x)$$
$$= \int_{\overline{\Omega}} \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] \phi(x - y) \, dy \, d(m_1 - m_2)(x)$$
$$= \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] \int_{\overline{\Omega}} \phi(x - y) \, d(m_1 - m_2)(x) \, dy$$
$$= \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] [(\phi \star m_1)(y) - (\phi \star m_2)(y)] \, dy.$$

Since $z \to f(x, z)$ is increasing, then one has that

$$\int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] [(\phi \star m_1)(y) - (\phi \star m_2)(y)] \, dy \geq 0.$$

Moreover, if the term on the left-side above is equal to zero, then we obtain

$$[f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] [(\phi \star m_1)(y) - (\phi \star m_2)(y)] = 0 \quad \text{a.e. } y \in \overline{\Omega}.$$

As $z \to f(x, z)$ is strictly increasing, we deduce that $\phi \star m_1(y) = \phi \star m_2(y)$ for any $y \in \overline{\Omega}$ and so $F(\cdot, m_1) = F(\cdot, m_2)$.

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Theorem 4.1. *Suppose that F and G satisfy (4.3). Let η_1, $\eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria and let J_{η_1} and J_{η_2} be the associated functionals. Then J_{η_1} is equal to J_{η_2}. Consequently, if (u_1, m_1), (u_2, m_2) are mild solutions of the constrained MFG problem in $\overline{\Omega}$, then $u_1 = u_2$.*

Proof. Let η_1, $\eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria, such that $m_1(t) = e_t \sharp \eta_1$, $m_2(t) = e_t \sharp \eta_2$ and let u_1, u_2 be the value functions of J_{η_1} and J_{η_2}, respectively. Let $x_0 \in \overline{\Omega}$ and let γ be an optimal trajectory for u_1 at $(0, x_0)$. We get

$$u_1(0, x_0) = \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_1(s)) \Big] \, ds + G(\gamma(T), m_1(T)),$$

$$u_2(0, x_0) \leq \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_2(s)) \Big] \, ds + G(\gamma(T), m_2(T)).$$

Therefore,

$$G(\gamma(T), m_1(T)) - G(\gamma(T), m_2(T)) \leq u_1(0, x_0) - u_2(0, x_0)$$

$$- \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_1(s)) \Big] \, ds + \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_2(s)) \Big] \, ds$$

$$= u_1(0, x_0)) - u_2(0, x_0) + \int_0^T F(\gamma(s), m_2(s)) - F(\gamma(s), m_1(s)) \, ds.$$

Recalling that $supp(\eta_1) \subset \Gamma^{\eta_1}[x_0]$, we integrate on Γ respect to $d\eta_1$ to obtain

$$\int_\Gamma \Big[G(\gamma(T), m_1(T)) - G(T, m_2(T)) \Big] d\eta_1(\gamma) \leq$$

$$\leq \int_\Gamma \Big[u_1(0, \gamma(0)) - u_2(0, \gamma(0)) \Big] d\eta_1(\gamma) + \int_\Gamma \int_0^T \Big[F(\gamma(s), m_2(s)) - F(\gamma(s), m_1(s)) \Big] \, ds \, d\eta_1(\gamma).$$

Recalling the definition of e_t and using the change of variables $e_t(\gamma) = x$ in the above inequality, we get

$$\int_\Gamma \Big[G(\overbrace{\gamma(T)}^{e_T(\gamma)}, m_1(T)) - G(\overbrace{\gamma(T)}^{e_T(\gamma)}, m_2(T)) \Big] d\eta_1(\gamma) = \int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T)) \Big] m_1(T, dx),$$

$$\int_\Gamma \Big[u_1(0, \overbrace{\gamma(0)}^{e_0(\gamma)}) - u_2(0, \overbrace{\gamma(0)}^{e_0(\gamma)}) \Big] d\eta_1(\gamma) = \int_{\overline{\Omega}} \Big[u_1(0, x) - u_2(0, x) \Big] m_1(0, dx),$$

$$\int_0^T \int_\Gamma \Big[F(\overbrace{\gamma(s)}^{e_s(\gamma)}, m_2(s)) - F(\overbrace{\gamma(s)}^{e_s(\gamma)}, m_1(s)) \Big] d\eta_1(\gamma) \, ds = \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s)) \Big] m_1(s, dx) \, ds.$$

So, we deduce that

$$\int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T)) \Big] m_1(T, dx) \tag{4.5}$$

$$\leq \int_{\overline{\Omega}} \Big[u_1(0, x) - u_2(0, x) \Big] m_1(0, dx) + \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s)) \, m_1(s, dx) \Big] \, ds.$$

In a similar way, we obtain

$$\int_{\overline{\Omega}} \Big[G(x, m_2(T)) - G(x, m_1(T)) \Big] m_2(T, dx) \tag{4.6}$$

$$\leq \int_{\overline{\Omega}} \Big[u_2(0, x) - u_1(0, x) \Big] m_2(0, dx) + \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_1(s)) - F(x, m_2(s)) \Big] m_2(s, dx) \, ds.$$

Summing the inequalities (4.5) and (4.6), we deduce that

$$\int_{\overline{\Omega}} \left[G(x, m_1(T)) - G(x, m_2(T)) \right] (m_1(T, dx) - m_2(T, dx))$$

$$\leq \int_{\overline{\Omega}} \left[u_1(0, x) - u_2(0, x) \right] (m_1(0, dx) - m_2(0, dx))$$

$$+ \int_0^T \int_{\overline{\Omega}} \left[F(x, m_2(s)) - F(x, m_1(s)) \right] (m_1(s, dx) - m_2(s, dx)) \, ds$$

Since $m_1(0, dx) = m_2(0, dx) = m_0$, by using the monotonicity assumption on G and F, we obtain that

$$0 \geq \int_0^T \int_{\overline{\Omega}} \left[F(x, m_2(s)) - F(x, m_1(s)) \right] (m_1(s, dx) - m_2(s, dx)) \, ds \geq$$

$$\int_{\overline{\Omega}} \left[G(x, m_1(T)) - G(x, m_2(T)) \right] (m_1(T, dx) - m_2(T, dx)) \geq 0.$$

Therefore,

$$\int_{\overline{\Omega}} \left[F(x, m_2(s)) - F(x, m_1(s)) \right] (m_1(s, dx) - m_2(s, dx)) = 0 \quad \text{a.e. } s \in [0, T],$$

and also

$$\int_{\overline{\Omega}} \left[G(x, m_1(T)) - G(x, m_2(T)) \right] (m_1(T, dx) - m_2(T, dx)) = 0.$$

Thus, by the strict monotonicity of F and G, we conclude that $F(x, m_1) = F(x, m_2)$ for all $x \in \overline{\Omega}$ and $G(x, m_1) = G(x, m_2)$ for all $x \in \overline{\Omega}$. Consequently, we have that J_{η_1} is equal to J_{η_2}. $\qquad \square$

Remark 4.1. Suppose that G satisfies (4.2) and F satisfies the following condition

$$\int_{\overline{\Omega}} \left[F(x, m_1) - F(x, m_2) \right] d(m_1 - m_2)(x) > 0,$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$ with $m_1 \neq m_2$. Then, proceeding as in the proof of Theorem 4.1, one can show that the mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$ is unique.

References

[1] Ambrosio, L., Gigli, N., Savare, G., *Gradient flows in metric spaces and in the space of probability measures. Second edition*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008.

[2] Aubin, J. P., Frankowska, H., *Set-Valued Analysis*, Birkhäuser Boston, Basel, Berlin, 1990

[3] Bardi, M., and Capuzzo Dolcetta, I., *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, 1996.

[4] Benamou, J. D., Carlier, G., Santambrogio, F., *Variational Mean Field Games*, 141-171, 2017.

[5] Cannarsa, P., and Sinestari, C., *Semiconcave functions, Hamilton-Jacobi Equations and optimal control*, Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[6] Cardaliaguet, P., Notes on mean feld games from P.-L. Lions' lectures at Collège de France, 2012, https://www.ceremade.dauphine.fr/∼ cardalia/MFG100629.pdf.

[7] Cardaliaguet, P., Marchi, C., *Regularity of the Eikonal Equation with Neumann Boundary Conditions in the Plane: Application to Fronts with Nonlocal Terms*, SIAM J. Control and Optimization, Vol 45, 1017-1038, 2006.

[8] Evans, L.C., Gariepy, R.F., *Measure Theory and Fine Properties of Functions*, Studies in Adavance Mathematics, CRC Press, Ann Arbor, 1992.

[9] Huang, M., Caines, P.E. and Malhamé, R.P., *Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ϵ-Nash equilibria*, Automatic Control, IEEE Transactions on 52, no. 9, 1560-1571, 2007

[10] Huang, M., Malhamé, R.P., Caines, P.E., *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainly equivalence principle.*, Communication in information and systems Vol 6, no.3, pp. 221-252, 2006.

[11] Lasry, J.-M., Lions, P.-L., *Jeux àÂ champ moyen. I. Le cas stationnaire.*, C. R. Math. Acad. Sci. Paris 343,no. 9, 619-625, 2006.

[12] Lasry, J.-M., Lions, P.-L., *Jeux à champ moyen. II. Horizon fini et contrôle optimal.*, C. R. Math. Acad. Sci. Paris 343, no. 9, 679-684, 2006.

[13] Lasry, J.-M.,Lions, P.-L., *Mean field games* Jpn. J. Math. 2, no. 1, 229-260, 2007.

[14] Kakutani, S., *A generalization of Brouwer's fixed point theorem*, Duke Math J., Vol 8, no. 3, 457-459, 1941