Attainable sets for conservation laws on networks

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Why studing conservation laws on network?

The recent great interest to conservation laws on network is motivated by application in:

- vehicular traffic in roads
- data networks
- supply chains
- air traffic management
- gas pipelines
- irrigation channels
- biomedical

Conservation laws with discontinuous flux

As first simple example we take the network made by two arcs and a junction point, this brings to the following Cauchy problem for scalar conservation law in one space dimension:

$$u_t + f(x,u)_x = 0, \qquad x \in \mathbb{R}, \quad t > 0$$
$$u|_{t=0} = u_0 \qquad x \in \mathbb{R}$$

where the flux function $f(x,u)$ is a discontinuous function given by

$$f(x,u) = f_l(u)1_{x<0} + f_r(u)1_{x>0}$$

We assume that:

- $f_l, f_r \in \mathcal{C}^1(\mathbb{R})$, strictly convex,
- $u_0 \in L^\infty(\mathbb{R})$.

There exist applications of this kind of model also in problems which do not involve network as:

- traffic flow (with roads whose amplitude varies for work in progress or change of number of lanes)
- sedimentation
- two-phase flow in porous media
- Saint Venant model of blood flow

WEAK SOLUTIONS

It is well known that after a finite time, this problem does not possess in general a continuous solution also if u_0 is smooth and $f_l = f_r$. Hence we consider a solution in a weak sense, that is $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that for all $\varphi \in \mathcal{C}_0^\infty(\overline{\mathbb{R} \times \mathbb{R}_+})$

$$\int_{-\infty}^{\infty} \int_0^{\infty} u \frac{\partial \varphi}{\partial t} + f(x,u) \frac{\partial \varphi}{\partial x} dxdt + \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)dx = 0.$$

This condition is satisfied if and only if u is a weak solution of

$$\begin{aligned} u_t + f_l(u)_x &= 0, \qquad x < 0, \quad t > 0, \\ u_t + f_r(u)_x &= 0, \qquad x > 0, \quad t > 0. \end{aligned}$$

and satisfies the Rankine-Hugoniot conditions

$$f_l(u^-(t)) = f_r(u^+(t)).$$

INTERFACE ENTROPY CONDITION

Weak solutions are, in general, not unique, thus additional admissibility criteria are necessary to single out a unique solution; these criteria are called entropy condition and are closely related to the nature of the physical phenomena described by the model. We choose the following entropy condition:

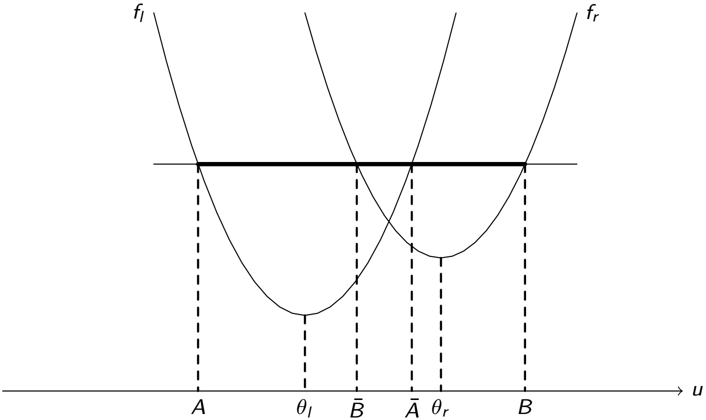
Definition 1

Let $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm = u(0\pm, t)$ exists a.e $t > 0$. Then we define $I_{AB}(t)$, the interface entropy functional, by

$$\begin{aligned} I_{AB}(t) =& (f_l'(u^-(t)) - f_l'(A))sign(u^-(t) - A) \\ & - (f_r'(u^+(t)) - f_r'())sign(u^+(t) - B). \end{aligned}$$

The function u is said to satisfy the interface entropy condition relative to the connection (AB) if for a.e. $t > 0$

$$I_{AB}(t) \geq 0$$


f_l
f_r
u
A
θ_l
$\bar{B}$
$\bar{A}$
θ_r
B

Definition 2

Let (AB) be a connection and $f(x,u) = f_l(u)1_{x<0} + f_r(u)1_{x>0}$. Let $u_0 \in L^\infty(\mathbb{R})$. Then $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be an (AB) entropy solution if:

(i) u is a weak solution of

$$\begin{aligned} u_t + f(x,u)_x = 0, \qquad x \in \mathbb{R}, \quad t > 0 \\ u\big|_{t=0} = u_0 \qquad x \in \mathbb{R}. \end{aligned}$$

(ii) u satisfies Lax-Oleinik-Kruzkov entropy condition away from the interface $\{x = 0\}$.

(iii) At the interface $\{x = 0\}$, u satisfies the (AB) interface entropy condition

Important assumption

We consider all the connections such that

$$A \neq \theta_l \text{ and } B \neq \theta_r$$

since under this hypothesis the solution are all of bounded variation!

We analyze the problem from the point of view of Control Theory following two approaches:

I) The initial data determine a unique (AB) entropy solution (in the sense of Def 2). We shall regard the initial data as controls.

II) The initial data determines many solutions in the sense of Def 1. We consider all such solutions which are selected by assigning a particular incoming flux at the discontinuity interface. In this case we shall regard both the initial data and the inflow flux at the discontinuity interface as controls (this correspond to regard the inflow flux at a junction of a network as controls).

In particular we want to characterize

$$\mathcal{A}(T, U) = \textit{set of attainable profiles}$$
$$\mathcal{F}(T, U) = \textit{set of attainable traces of flux on the discontinuity interface}$$

where $T > 0$ and U is a set of controls.

Our first result regards the exact controllability at time T: given a function $\omega \in \mathcal{A}(T) \subset BV(\mathbb{R})$ we are able to find $u_0 \in L^\infty(\mathbb{R})$ such that $u(T, \cdot) = \omega$.
The main tool for this analysis is given by the theory of generalized characteristics of Dafermos.
This kind of study is inspired to previous works as:

- On the Attainable Set for Scalar Nonlinear Conservation Laws with Boundary Conditions,(F.Ancona,A.Marson 1998);
- the attainable set for a class of triangular systems of conservation laws (B. Andreianov, C. Donadello, S. S. Ghoshal and U. Razafison 2015)
- On the attainable set for a scalar nonconvex conservation law (B. Andreianov, C. Donadello, A. Marson, Preprint)

Theorem

$\mathcal{A}(T)$ is given by the union of the following three sets:

$\mathcal{A}_1(T)$ is the set of all the functions $\omega \in BV(\mathbb{R})$ for which there exists $R > 0$ such that

$$f_r'(\omega(x)) > \frac{x}{T} \quad \forall x \in (0, R),$$
$$f_r'(\omega(x)) \leq \frac{x}{T} \quad \forall x \in (R, +\infty),$$
$$f_l'(\omega(x)) \geq \frac{x}{T} \quad \forall x \in (-\infty, 0),$$
$$\varphi_R : x \mapsto -f_l'(f_l^{-1} f_r(\omega(x))) \left(T - \frac{x}{f_r'(\omega(x))} \right) \textit{ is not decreasing on } (0, R),$$
$$\psi_R^1 : x \mapsto x - f_r'(\omega(x))T \textit{ is not decreasing on } (R, +\infty),$$
$$\psi_R^2 : x \mapsto x - f_l'(\omega(x))T \textit{ is not decreasing on } (-\infty, 0)$$

$\mathcal{A}_2(T)$ is the set of all the functions $\omega \in BV(\mathbb{R})$ for which there exists $L \in (-\infty, 0]$ such that

$$f_l'(\omega(x)) < \frac{x}{T} \quad \forall x \in (L, 0),$$

$$f_l'(\omega(x)) \geq \frac{x}{T} \quad \forall x \in (-\infty, L),$$

$$f_r'(\omega(x)) \leq \frac{x}{T} \quad \forall x \in (0, +\infty),$$

$$\varphi_L : x \mapsto -f_r'(f_r^{-1} f_l(\omega(x)) \left(T - \frac{x}{f_l'(\omega(x))} \right) \textit{ is not decreasing on } (L, 0),$$

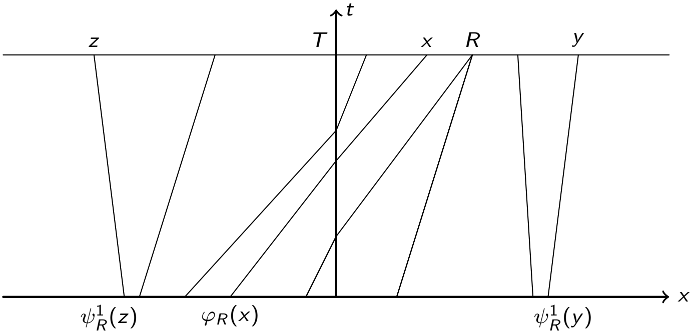
$$\psi_L^1 : x \mapsto x - f_l'(\omega(x))T \textit{ is not decreasing on } (-\infty, L),$$

$$\psi_L^2 : x \mapsto x - f_r'(\omega(x))T \textit{ is not decreasing on } (0, +\infty)$$

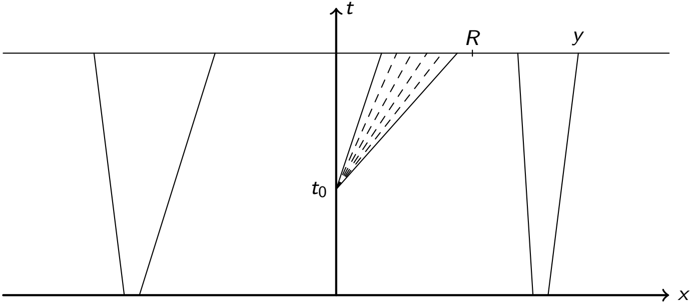
$\mathcal{A}_3(T)$ is the set of all the functions $\omega \in BV(\mathbb{R})$, for which there exists $L \in (-\infty, 0]$ and $R \in [0, \infty)$ such that

$$\begin{aligned}
&\omega(x) = A \quad \forall x \in (L, 0),\\
&\omega(x) = B \quad \forall x \in (0, R),\\
&f_r'(\omega(x)) \leq \frac{x}{T} \quad \forall x \in (R, +\infty),\\
&f_l'(\omega(x)) \geq \frac{x}{T} \quad \forall x \in (-\infty, L),\\
&\psi_l : x \mapsto x - f_l'(\omega(x))T \ \textit{is not decreasing on } (-\infty, L),\\
&\psi_r : x \mapsto x - f_r'(\omega(x))T \ \textit{is not decreasing on } (R, +\infty)
\end{aligned}$$

Characteristics's behavior ($\mathcal{A}_1$)

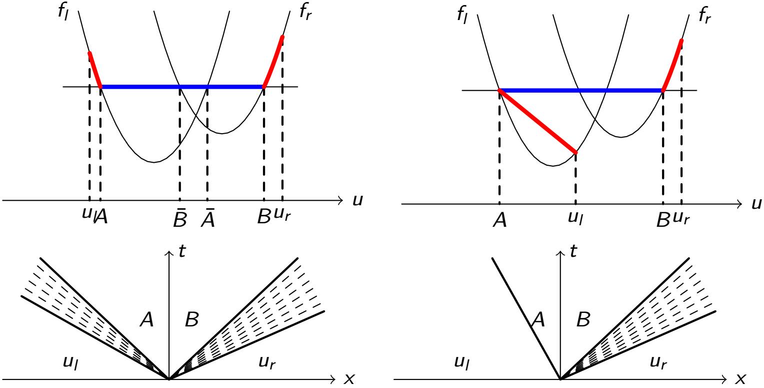


We can prove that no rarefactions can arise on the interface for $t > 0$.



The reason can be understood considering the initial data which generate rarefactions in the origin.

For which Riemann problems does the solution have rarefactions in the origin?



But all the initial data for which the solution has rarefactions in the origin do not satisfy the interface entropy condition. Thus if we have a rarefaction at time $t_0 > 0$, the solution does not satisfy the interface entropy condition at time $t = t_0$, but this is not enough since we want the interface entropy condition to be satisfied for a.a. $t > 0$. What we can prove is that there exists a set of positive measure for which the interface entropy condition is not satisfied.

Thanks for your attention!