Rapid Stabilization of a Schrödinger Equation

Ludovick Gagnon
Work in collaboration with Jean-Michel Coron and Morgan Morancey

Université Côte d'Azur

August 25, 2017

Outline

1. Introduction of the method in the finite dimensional setting
2. Application to the Linearized Schrödinger Equation
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Rapid stabilization

Consider, for $T > 0$,

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1}$$

Goal : Design a feedback law $u(t) = K(x(t))$ such that for every $\lambda > 0$, there exists $C > 0$

$$\|x(t)\| \leq Ce^{-\lambda t}\|x_0\|, \quad \forall x_0.$$

A classical result for finite dimensional system of the form (1) is that the controllability implies the rapid stabilization.

Rapid stabilization through linear transformations

A recent idea is to prove the rapid stabilization in the finite or infinite dimensional setting through linear transformations. This method, often and misleadingly referred as the backstepping method, was brought by Boskovic Balogh and Krstic. Consider

$$\begin{cases} \xi'(t) = (A - \lambda I)\xi(t), & t \in [0, T], \\ \xi(0) = \xi_0. \end{cases}$$

This equation is exponentially stable for $\lambda > \max(\Re(\lambda_i))$

$$\|\xi(t)\| \leq e^{-(\lambda - \max(\Re(\lambda_i)))t}\|\xi_0\|.$$

The idea relies on the mapping

$$\begin{aligned} \xi &= Tx, \\ u &= Kx. \end{aligned}$$

If one is able to find such continuous mappings such that T is invertible, then one obtains the rapid stabilization of (1) :

$$\|x(t)\| \leq \|T^{-1}\xi(t)\| \leq \|T^{-1}\|e^{(\|A\|-\lambda)t}\|\xi_0\| \leq \|T^{-1}\|\|T\|e^{(\|A\|-\lambda)t}\|x_0\|.$$

Rapid stabilization through linear transformations

Using the equation satisfied by x and ξ, (T, K) must satisfy

$$\begin{aligned} \xi_t &= Tx_t \\ \Leftrightarrow (A-\lambda I)\xi &= TAx + TBu \\ \Leftrightarrow (A-\lambda I)Tx &= TAx + TBKx \end{aligned}$$

On the last line, we impose the uniqueness condition $TB = B$ to obtain

$$AT - \lambda T = TA + BK, \tag{2}$$

$$TB = B. \tag{3}$$

In the finite dimensional setting, if (1) is controllable, then one can find such (T, K)

Theorem (J. M. Coron, '16)

Suppose $A \in M_n(\mathbb{R}), B \in \mathbb{R}^n$ and (1) is controllable. Then there exists a unique $(T, K) \in GL_n(\mathbb{R}) \times \mathbb{R}^{1\times n}$.

Proof

We provide a proof adapted to the infinite dimensional setting. The strategy consists to first find a (Riesz) basis to solve (2). This basis is then used together with $TB = B$ to obtain (T, K). The uniqueness is ensured using $TB = B$. Finally the invertiblity is proved using the observability of the adjoint equation.

Proof :

Let $\{\lambda_i, e_i\}_{1 \leq i \leq n}$ be the (simple) eigenvalues and eigenvectors of A. We have

$$(\lambda_i + \lambda - A)Te_i = -BKe_i, \quad \forall i \in \{1, \dots, n\} \tag{4}$$

Since $Ke_i \in \mathbb{R}$ are unknowns, consider

$$(\lambda_i + \lambda - A)\widetilde{T}e_i = -B, \quad \forall i \in \{1, \dots, n\}. \tag{5}$$

We are able to prove that $\{\widetilde{T}e_i\}_{1 \leq i \leq n}$ is a basis of $\mathbb{R}^n$ using the controllability assumption. This part being only technical, we proceed with the rest of the proof.

Construction of T and K

We define Ke_i with the uniqueness condition $TB = B$:

$$\begin{aligned} & B = TB \\ \iff \quad & B = T\sum_{i=1}^{n} b_i e_i \\ \iff \quad & B = \sum_{i=1}^{n} b_i K e_i \widetilde{T} e_i. \end{aligned} \tag{6}$$

Since $\{\widetilde{T} e_i\}_{1 \leq i \leq n}$ is a basis, then there exists $\{a_i\}_{1 \leq i \leq n}$ such that

$$\sum_{i=1}^{n} a_i \widetilde{T} e_i = B.$$

We therefore obtain $Ke_i = a_i / b_i$. We use here the controllability to obtain that $b_i \neq 0, \forall i \in \{1, \ldots, n\}$. We finally obtain Te_i by the relation $Te_i = Ke_i \widetilde{T} e_i$.

Invertibility of T

We prove the invertibility of T by showing $\mathrm{Ker}\ T^* = \{0\}$.

Assume $0 \neq x \in \mathrm{Ker}\ T^*$. Then,

$$T^* A^* x = A^* T^* x + K^* B^* T^* x + \lambda T^* x = 0.$$

Therefore, $A^* x \in \mathrm{Ker}\ T^*$. By iterating this process, we obtain $(A^*)^{k+1} x \in \mathrm{Ker}\ T^*$ for all $k \in \mathbb{N}$. However,

$$B^* (A^*)^k x = B^* T^* (A^*)^k x = 0, \quad \forall k \in \mathbb{N},$$

which, by observability of the adjoint system implies $x = 0$.

Bilinear Schrödinger equation

Let $T > 0$. Consider

$$\begin{cases} i\partial_t \psi = -\Delta\psi - u(t)\mu(x)\psi, & (x,t) \in (0,1) \times (0,T), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1). \end{cases} \tag{NLS}$$

1. $\mu \in H^3((0,1);\mathbb{R})$,
2. $u \in L^2((0,T);\mathbb{R})$,
3. $\psi_0 \in H^3_{(0)}((0,1);\mathbb{C})$

The space $H^3_{(0)}(0,1)$ is defined by

$$H^3_{(0)} = D((-\Delta)^{3/2}) := \left\{ \phi \in H^3 \cap H^1_0 \,|\, \phi''(0) = \phi''(1) = 0 \right\},$$

and is endowed with the norm

$$\|\psi\|^2_{H^3_{(0)}} = \sum_{k \in \mathbb{N}^*} \lambda_k^3 |\langle \psi, \varphi_k \rangle|^2,$$

where $(\lambda_k, \varphi_k) = ((k\pi)^2, \sqrt{2}\sin(\lambda_k x))$. We denote the eigenstates of (NLS) $\Phi_k(x,t) := e^{-i\lambda_k t}\varphi_k(x)$.

Local controllability of NLS

Theorem (Beauchard, Laurent, (Exact controllability) '10)

Let $T > 0$ *and* $\mu \in H^3((0,1);\mathbb{R})$ *such that there exists* $C > 0$ *such that*

$$|\langle \mu\varphi_1, \varphi_k \rangle| \geq \frac{C}{k^3}, \quad \forall k \in \mathbb{N}^*. \qquad \text{(HypCont)}$$

Then (NLS) *is locally exactly controllable* $\Phi_1(x,t)$*.*

Notice that the controllability assumption implies that $\mu\varphi_1$, belonging to $H^3 \cap H^1_0$, does not belong to $H^3_{(0)}$:

$$\|\mu\varphi_1\|^2_{H^3_{(0)}} = \sum_{k\in\mathbb{N}^*} \lambda_k^3 |\langle \mu\varphi_1, \varphi_k \rangle|^2 = +\infty$$

This is a typical result : B must be singular enough to obtain the controllability but regular enough to keep the well-posedness of the equation [Beauchard, Laurent, (Hidden regularity) '10]. We have the following behaviour

$$n^3\langle \psi, \varphi_n \rangle \in \ell^2(\mathbb{N}^*;\mathbb{C}), \quad n^3\langle \mu\varphi_1, \varphi_n \rangle \in \ell^\infty(\mathbb{N}^*;\mathbb{R}).$$

Rapid stabilization of the linearized Schrödinger equation

Applying the method presented earlier to

$$\begin{cases} i\partial_t\psi = -\Delta\psi - u(t)\mu(x)\varphi_1, & (x,t) \in (0,1)\times(0,T), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1), \end{cases} \tag{Lin}$$

we obtain

Theorem (J.-M. Coron, L. G., M. Morancey, '16)

Let $T > 0$. Assume the controllability hypothesis on $\mu\varphi_1$. Then, for every $\lambda > 0$, there exist $C > 0$ and a feedback $v(t) = K(\psi(t,\cdot))$ such that for every $\Psi_0 \in H^3_{(0)}$ the solution of (Lin) satisfies

$$\|\Psi(t,\cdot)\|_{H^3_{(0)}} \leq Ce^{-\lambda t}\|\Psi_0\|_{H^3_{(0)}}.$$

Target equation

Since the feedback has to be real-valued, write $\Psi^1 + i\Psi^2 = \Psi$.

$$\begin{cases} \partial_t \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} = \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} + v(t) \begin{pmatrix} 0 \\ (\mu\varphi_1)(x) \end{pmatrix}, & (t,x) \in (0,1) \times (0,T), \\ \Psi^1(t,0) = \Psi^1(t,1) = 0, \quad \Psi^2(t,0) = \Psi^2(t,1) = 0, & t \in (0,T), \\ \Psi^1(0,x) = \Psi^1_0(x), \quad \Psi^2(0,x) = \Psi^2_0(x), & x \in (0,1). \end{cases}$$

We define $\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} := T \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}$ where ξ is the solution of the target system

$$\begin{cases} \partial_t \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} - \lambda \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, & (t,x) \in (0,1) \times (0,T), \\ \xi^1(t,0) = \xi^1(t,1) = 0, \quad \xi^2(t,0) = \xi^2(t,1) = 0, & t \in \times(0,T), \\ \xi^1(0,x) = \xi^1_0(x), \quad \xi^2(0,x) = \xi^2_0(x), & x \in \times(0,1), \end{cases}$$

with $\begin{pmatrix} \xi^1_0 \\ \xi^2_0 \end{pmatrix} = T \begin{pmatrix} \Psi^1_0 \\ \Psi^2_0 \end{pmatrix}$.

Definition of T and K

To analyze T and K, we consider the kernel transformations

$$
\begin{aligned}
T : H^3_{(0)} \times H^3_{(0)} &\longrightarrow H^3_{(0)} \times H^3_{(0)} \\
\begin{pmatrix} \Psi^1(t,\cdot) \\ \Psi^2(t,\cdot) \end{pmatrix} &\longmapsto \int_0^1 \begin{pmatrix} k_{11}(x,y) & k_{12}(x,y) \\ k_{21}(x,y) & k_{22}(x,y) \end{pmatrix} \begin{pmatrix} \Psi^1(t,y) \\ \Psi^2(t,y) \end{pmatrix} dy, \\
K : H^3_{(0)} \times H^3_{(0)} &\longrightarrow \mathbb{R} \\
\begin{pmatrix} \Psi^1(t,\cdot) \\ \Psi^2(t,\cdot) \end{pmatrix} &\longmapsto \int_0^1 \alpha^1(y)\Psi^1(t,y) + \alpha^2(y)\Psi^2(t,y) dy.
\end{aligned}
$$

As in the finite dimensional setting, set

$$
\begin{aligned}
A : (H^5_{(0)})^2 &\longrightarrow (H^3_{(0)})^2 \\
\begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} &\longmapsto \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix},
\end{aligned}
\qquad
\begin{aligned}
B : \mathbb{R} &\longrightarrow \left(C^\infty \times (H^3 \cap H^1_0)((0,1);\mathbb{R}) \right) \\
a &\longmapsto a \begin{pmatrix} 0 \\ (\mu\varphi_1)(x) \end{pmatrix},
\end{aligned}
$$

PDE on the kernels

Using once again $\xi_t = T\psi_t$ and the uniqueness condition $TB = B$

$$T\begin{pmatrix}0\\(\mu\varphi_1)(x)\end{pmatrix} = \int_0^1 \begin{pmatrix}k_{12}(x,y)(\mu\varphi_1)(y)\\k_{22}(x,y)(\mu\varphi_1)(y)\end{pmatrix} dy = \begin{pmatrix}0\\(\mu\varphi_1)(x)\end{pmatrix}.$$

we obtain a PDE on the kernels

$$\begin{cases}(\Delta_y k_{11} - \Delta_x k_{22} - \lambda k_{12})(x,y) = 0,\\(\Delta_y k_{12} + \Delta_x k_{21} + \lambda k_{11})(x,y) = 0,\\(\Delta_y k_{21} + \Delta_x k_{12} - \lambda k_{22})(x,y) - \alpha^2(y)(\mu\varphi_1)(x) = 0,\\(\Delta_y k_{22} - \Delta_x k_{11} + \lambda k_{21})(x,y) + \alpha^1(y)(\mu\varphi_1)(x) = 0,\\k_{ij}(x,0) = k_{ij}(x,1) = 0,\\k_{ij}(0,y) = k_{ij}(1,y) = 0.\end{cases}$$

Write

$$k^{ij}(x,y) = \sum_{k\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*} c^{ij}_{nk}\varphi_k(x)\varphi_n(y), \quad \alpha^i(y) = \sum_{n\in\mathbb{N}^*}\alpha^i_n\varphi_n(y).$$

Riesz basis

As in the finite dimensional setting,

Step 1 : the controllability implies

$$\left\{ \widetilde{T} \begin{pmatrix} \varphi_n/\lambda_n^{3/2} \\ 0 \end{pmatrix}, \widetilde{T} \begin{pmatrix} 0 \\ \varphi_n/\lambda_n^{3/2} \end{pmatrix} \right\}_{n\in\mathbb{N}^*}$$

is a basis of $(H^3_{(0)})^2$.

Step 2 : we obtain $\alpha_n^2 = K((0, \varphi_n/\lambda_n^{3/2})^{\mathrm{tr}})$ by using, formally, $TB = B$

$$B = \sum_{n\in\mathbb{N}^*} \lambda_n^{3/2} \langle \mu\varphi_1, \varphi_n \rangle \begin{pmatrix} 0 \\ \varphi_n/\lambda_n^{3/2} \end{pmatrix} \simeq \sum_{n\in\mathbb{N}^*} \lambda_n^{3/2} \langle \mu\varphi_1, \varphi_n \rangle \frac{\alpha_n^2}{\lambda_n^{3/2}} T \begin{pmatrix} 0 \\ \varphi_n/\lambda_n^{3/2} \end{pmatrix}.$$

Recall that $B \notin (H^3_{(0)})^2$. We circumvent this issue by writing

$$\alpha_n^2 = \lambda_n^{3/2}(1 + a_n^2),$$

Using the basis to change the $\simeq$ to a $=$, we obtain $(\alpha_n^1/\lambda_n^{3/2}, a_n^2/\lambda_n^{3/2}) \in (\ell^2(\mathbb{N}^*;\mathbb{R}))^2$. Therefore $(\alpha_n^2/\lambda_n^{3/2}) \in \ell^\infty(\mathbb{N}^*;\mathbb{R})$.

Regularity of T and K

The behaviour of α_n^i allows to prove

Proposition

The transformation T is linear continuous from $(H^3_{(0)})^2$ into itself.

However, the linear application K is unbounded in $(H^3_{(0)})^2$. Indeed, if it was true, then

$$\left| K \begin{pmatrix} 0 \\ \psi^2 \end{pmatrix} \right| = \left| \sum_{n \in \mathbb{N}^*} \alpha_n^2 \langle \psi^2, \varphi_n \rangle \right| = \left| \sum_{n \in \mathbb{N}^*} \frac{\alpha_n^2}{n^3} (n^3 \langle \psi^2, \varphi_n \rangle) \right| < +\infty.$$

which would imply $\left(\frac{\alpha_n^2}{n^3} \right)_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{R})$. We remark here that the singular behaviour of α_n^2 is necessary as $\left(\frac{\alpha_n^2}{n^3} \right)_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{R})$ would imply that T is compact.

Invertibility of T

Despite the unboundedness of K, $TA + BK = AT - \lambda T$ has a sense using $TB = B$

$$T(A + BK) = AT - \lambda T$$

in the domain

$$\begin{aligned} D(A+BK) &:= \left\{ \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \in X^3_{(0)} \, ; \, (A+BK) \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \in X^3_{(0)} \right\} \\ &= \left\{ \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \in (H^5 \cap H^3_{(0)}) \times H^5_{(0)} \, ; \, -\Delta \Psi^1 + K \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \mu \varphi_1 \in H^3_{(0)} \right\} \end{aligned}$$

which is dense in $(H^3_{(0)})^2$. We then prove the invertibility of T using the fact that T is a Fredholm operator. It is therefore sufficient to prove $\mathrm{Ker}\ T^* = \{0\}$ using the proof used in the finite dimensional setting.

We finally prove, using classical techniques, that $(A + BK)$ defines a infinitesimal generator of semi-group on $(H^3_{(0)})^2$. Therefore

$$\begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}_t = (A+BK) \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}$$

is well-posed and the invertibility of T implies the rapid stabilizaton.

A word on the local rapid stabilization of NLS (in progress)

We prove

Proposition (Coron, G., Morancey, '17)

There exists $r > 0$ such that the linearized equations (NLS) *around $a \in \mathcal{B}_r(\varphi_1)$, with $\mathcal{B}_r(\varphi_1) := \{a \in H^3_{(0)};\ |a - \varphi_1|_{H^3_{(0)}} < r\}$, are exactly controllable.*

We then use the same techniques to construct invertible transformations $T(a)$ for $a \in \mathcal{B}_r$.
We prove that the scalar product

$$Q(a)(\psi_1, \psi_2) = \langle T(a)\psi_1, T(a)\psi_2 \rangle_{H^3_{(0)}}$$

possesses the following estimates

$$\frac{1}{C}|\psi|^2_{H^3_{(0)}} \leq Q(a)(\psi, \psi) \leq C|\psi|^2_{H^3_{(0)}},\ \forall a \in \mathcal{B}_r,\ \forall \psi \in H^3_{(0)},$$

$$|[Q'(a)b](\psi_1, \psi_2)| \leq C|b|_{H^3_{(0)}}|\psi_1|_{H^3_{(0)}}|\psi_2|_{H^3_{(0)}},\ \forall a \in \mathcal{B}_r,\ \forall b \in H^3_{(0)},\ \forall(\psi_1, \psi_2) \in (H^3_{(0)})^2.$$

Lyapunov argument

We construct

$$V_1(t) := Q(\psi(t))(\psi(t), \psi(t)). \tag{7}$$

Then,

$$\dot{V}_1(t) \leq -\lambda V_1(t) + [Q_a'(\psi(t))\dot{\psi}(t)](\psi(t), \psi(t)). \tag{8}$$

We are not able to bound $[Q_a'(\psi(t))\dot{\psi}(t)](\psi(t), \psi(t))$ in the right space, as it is the case for nonlinear hyperbolic problems. We therefore construct Lyapunov functions for ψ_t and ψ_{tt} which allows to obtain the local rapid stabilization at the cost of having to consider initial conditions in H^5 with compatibility conditions.

Conclusion

Perspective : abstract results.