Multiplicative controllability for semilinear degenerate reaction-diffusion equations

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(joint work with C. Nitsch and C. Trombetti)

VII PARTIAL DIFFERENTIAL EQUATIONS, OPTIMAL DESIGN AND NUMERICS

ORGANIZED BY G. BUTTAZZO, O. GLASS, G. LEUGERING, AND E. ZUAZUA

THEMATIC SESSION ON

"Bilinear and fractional control of partial differential equations"

Benasque, 25 August 2017

Outline

1. Introduction
   - Problem formulation
   - Motivations: Energy balance models in climatology
   - Additive vs multiplicative controllability
2. Main results: multiplicative controllability for sign changing states
   - State of art
   - Main ideas for the proof of the main result
3. Open problems

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Reference

P. Cannarsa, G. F., A. Y. Khapalov, Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign , Journal de Mathématiques Pures et Appliquées, (2017), DOI: 10.1016/j.matpur.2017.07.002 ArXiv: 1510.04203.

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\end{cases} & (\text{for } WDP) \\
a(x)u_x(x,t)|_{x=\pm 1} = 0 \quad t \in (0,T) & (\text{for } SDP)
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u(0,x) = u_0(x) & x \in (-1,1)\,.
\end{cases} \tag{1}
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$\alpha \in L^\infty(Q_T)$, (bilinear control), $u_0 \in L^2(-1,1)$;
$f : Q_T \times \mathbb{R} \to \mathbb{R}$ is such that

- $(x,t,u) \longmapsto f(x,t,u)$ is a Carathéodory function; $u \longmapsto f(x,t,u)$ is differentiable at $u=0$; $t \longmapsto f(x,t,u)$ is locally absolutely continuous;
- $\exists \gamma_* \geq 0, \vartheta \geq 1$ and $\nu \geq 0$ such that, for a.e. $(x,t) \in Q_T, \forall u, v \in \mathbb{R}$, we have
$$|f(x,t,u)| \leq \gamma_*\, |u|^\vartheta,$$
$$-\nu\left(1+|u|^{\vartheta-1}+|v|^{\vartheta-1}\right)(u-v)^2 \leq \left(f(x,t,u)-f(x,t,v)\right)(u-v) \leq \nu(u-v)^2,$$
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Semilinear degenerate problems

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$$\mathbf{a} \in \mathbf{C^0}([-\mathbf{1},\mathbf{1}]) \cap \mathbf{C^1}(]-\mathbf{1},\mathbf{1}[) \,:\, \mathbf{a(x)} > \mathbf{0}\ \forall\, \mathbf{x} \in (-\mathbf{1},\mathbf{1}),\ \mathbf{a(-1)} = \mathbf{a(1)} = \mathbf{0}.$$

We distinguish two cases:

- $\frac{1}{a} \in L^1(-1,1)$ (WDP), e.g. $a(x) = \sqrt{1-x^2}$, $a \notin C^1([-1,1])$ $(\beta_0\beta_1 \leq 0,\ \gamma_0\gamma_1 \geq 0)$;
- $\frac{1}{a} \notin L^1(-1,1)$ (SDP), e.g. $a(x) = 1 - x^2$ (see later Budyko-Sellers climate model) $a \in C^1([-1,1])$ (assume that $\int_0^x \frac{1}{a(s)}\, ds \in L^{q_\vartheta}(-1,1)$, for same $q_\vartheta \geq 1$).

(SD) $H^1_a(-1,1) := \{u \in L^2(-1,1) : u \text{ loc. abs. continuous in } (-1,1), \sqrt{a}u_x \in L^2\}$;

(WD) $H^1_a(-1,1) := \{u \in L^2(-1,1) : u \text{ absolutely continuous in } [-1,1], \sqrt{a}u_x \in L^2\}$.

$$H^2_a(-1,1) := \{u \in H^1_a(-1,1) |\, au_x \in H^1(-1,1)\}$$

Given $T > 0$, let us define the function spaces

$$\mathcal{B}(Q_T) := C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1))$$

$$\mathcal{H}(Q_T) := L^2(0,T; D(A)) \cap H^1(0,T; L^2(-1,1)) \cap C([0,T]; H^1_a(-1,1))$$

Theorem

For all $u_0 \in H^1_a(-1,1)$ there exists a unique strict solution $u \in \mathcal{H}(Q_T)$ to (1).

Theorem

For all $u_0 \in L^2(-1,1)$ there exists a unique strong solution $u \in \mathcal{B}(Q_T)$ to (1).

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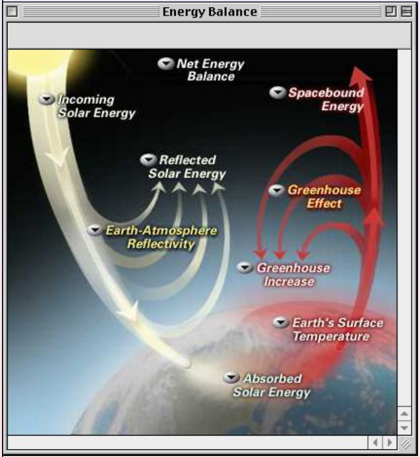
Outline

Energy balance models

The effect of solar radiation on climate



heat variation
$$= R_a - R_e + D$$

- R_a = absorbed energy
- R_e = emitted energy
- D = diffusion

The Budyko-Sellers model (1969)

$\mathcal{M}$ compact surface without boundary (typically S^2)

$$u_t - \Delta_{\mathcal{M}} u = R_a(t,x,u) - R_e(t,x,u)$$

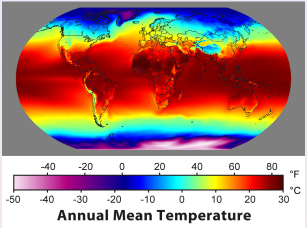
where $u(t,x)$ = temperature distribution

- $R_a(x,u) = Q(t,x)\beta(u)$ $\begin{cases} Q = \text{insolation function} \\ \beta = \text{coalbedo} = 1 - \text{albedo} \end{cases}$
-
  - $R_e(x,u) = A(t,x) + B(t,x)u$ Budyko
  - $R_e(x,u) \backsimeq c\,u^4$ Sellers

One-dimensional BS model

on $\mathcal{M} = \Sigma^2$, $\Delta_{\mathcal{M}} u = \frac{1}{\sin\phi}\left\{\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin\phi}\frac{\partial^2 u}{\partial\lambda^2}\right\}$

ϕ = colatitude λ = longitude



taking average at $x = \cos\phi$ BS model reduces to

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = g(t,x)\,h(u) + f(t,x,u) & x \in]-1,1[\\ (1-x^2)u_x\big|_{x=\pm 1} = 0 \end{cases}$$

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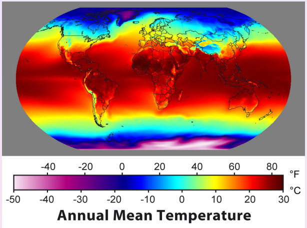

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A prophecy by J. von Neumann

Nature (1955):

Microscopic layers of colored matter spread on an icy surface, or in the atmosphere above one, could inhibit the reflection-radiation process, melt the ice and change the local climate.

$\Rightarrow$ rather than $\downarrow$

$$u_t - \Delta_{\mathcal{M}} u = g(t, \quad x \quad, u) + f(t,x)$$

use $\uparrow$ as control variable

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Other degenerate models: Wright-Fisher models in population genetic and general Kimura diffusion

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Outline

Controllability: heat equation (linear case)

$$\begin{cases} u_t = \Delta u + vu + h(x,t)1\!\!1_\omega \\ u_{|\partial\Omega} = 0 \qquad (\omega \subset \Omega) \\ u_{|t=0} = u_0 \end{cases}$$

Additive controls
(locally distributed source terms)

$$\begin{cases} u_t = \Delta u + vu \\ u_{|\partial\Omega} = g(t) \\ u_{|t=0} = u_0 \end{cases}$$

Boundary controls

$$\begin{cases} u_t = \Delta u + v(x,t)u \\ u_{|\partial\Omega} = 0 \\ u_{|t=0} = u_0 \end{cases}$$

Bilinear controls
(multiplicative controllability)

Definition (Approximate controllability)

$\forall u_0 \in H_0, u^* \in H^*, (H_0, H^* \subseteq L^2(\Omega)), \forall \varepsilon > 0, \exists$ *"a control function"*, $T > 0$ *such that*

$$\|u(\cdot, T) - u^*\|_{L^2(\Omega)} < \varepsilon.$$

Multiplicative controllability and Applied Mathematics

$\Rightarrow$ rather than $\downarrow$

$$u_t - \Delta u = v(x,t) \quad u + h(x,t)$$

use $\uparrow$ as control variable

Remark

$$\Phi : \text{“control”} \longmapsto \text{“solution”}$$

Additive controls vs *Bilinear controls*

$\Phi : h \longmapsto u$ *is a linear map;* $\Phi : v \longmapsto u$ *is a nonlinear map.*

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Additive controls (locally distributed source terms)

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Some references on bilinear control of PDEs

- Ball, Marsden and Slemrod (1982)
  [rod and wave equation]
- Coron, Beauchard, Gagnon, Laurent, Morancey
  [Scrödinger equation, . . .]
- Khapalov (2002–2010)
  [parabolic and hyperbolic equations, swimming models]
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Definition (Approximate controllability)

An evolution system is called globally approximately controllable, if any initial state u_0 in H_0 can be steered into any neighborhood of any target state $u^* \in H^*$ at time T, by a suitable control.

Strong Maximun Principle and obstruction to multiplicative controllability: $H^* \neq H_0^1(\Omega)$

$$u_0(x) = 0 \Longrightarrow u(x,t) = 0$$
$$u_0(x) \geq 0 \Longrightarrow u(x,t) \geq 0$$

If $u_0(x) \geq 0$ in Ω, then the SMP demands that the respective solution to (2) remains nonnegative at any moment of time, regardless of the choice of v. This means that system (2) cannot be steered from any nonnegative u_0 to any target state which is negative on a nonzero measure set in the space domain.

Controllability:

- Nonnegative states
- Sign changing states

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If $u_0(x) \geq 0$ in Ω, then the SMP demands that the respective solution to (2) remains nonnegative at any moment of time, regardless of the choice of v. This means that system (2) cannot be steered from any nonnegative u_0 to any target state which is negative on a nonzero measure set in the space domain.

Controllability:

- Nonnegative states
- Sign changing states

Definition

We say that the system (2) *is nonnegatively globally approximately controllable in* $L^2(\Omega)$, *if for every* $\eta > 0$ *and for any* $u_0,\ u^* \in L^2(\Omega),\ \ u_0,\ u^* \geq 0$, *with* $u_0 \neq 0$ *there are a* $T = T(\eta, u_0, u^*) \geq 0$ *and a bilinear control* $v \in L^\infty(Q_T)$ *such that for the corresponding solution* u *of* (2) *we obtain*

$$\|u(T,\cdot) - u^*\|_{L^2(\Omega)} \leq \eta\,.$$

Reference

P. Cannarsa, G. F., Approximate multiplicative controllability for degenerate parabolic problems with robin boundary conditions, , CAIM, (2011).

Reference

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let us consider (with Carlo Nitsch and Cristina Trombetti)

$$
\begin{cases}
u_t - (a(x)u_x)_x = \alpha(x,t)u + f(x,t,u) & \text{in } Q_T := (-1,1)\times(0,T) \\
\begin{cases}
\begin{cases}
\beta_0 u(-1,t) + \beta_1 a(-1)u_x(-1,t) = 0 & t\in(0,T) \\
& \qquad\qquad (\text{for } WDP) \\
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\end{cases} \\
a(x)u_x(x,t)|_{x=\pm 1} = 0 \qquad t\in(0,T) \quad (\text{for } SDP)
\end{cases} \\
u(0,x) = u_0(x) & x\in(-1,1)\,.
\end{cases}
\tag{3}
$$

We assume that $u_0 \in H^1_a(-1,1)$ has n points of sign change, that is, there exist points $-1 := x^0_0 < x^0_1 < \cdots < x^0_n < x^0_{n+1} := 1$ such that

$$
\begin{gathered}
u_0(x) = 0 \quad \Longleftrightarrow \quad x = x^0_l, \quad l = 1,\ldots,n. \\
u_0(x)u_0(y) < 0, \quad \forall x \in \left(x^0_{l-1}, x^0_l\right), \forall y \in \left(x^0_l, x^0_{l+1}\right).
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Let $u_0 \in H^1_a(-1,1)$ have finitely many points of sign change.

Theorem (G.F., C. Nitsch, C. Trombetti, 2017)

Consider any $u^* \in H^1_a(-1,1)$ which has exactly as many points of sign change in the same order as u_0. Then,

$$\forall \eta > 0\ \exists T > 0,\ \alpha \in L^\infty(Q_T)\ : \| u(\cdot, T) - u^* \|_{L^2(-1,1)} \leq \eta.$$

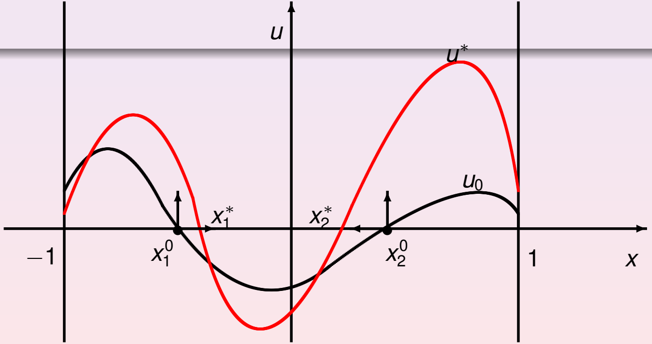


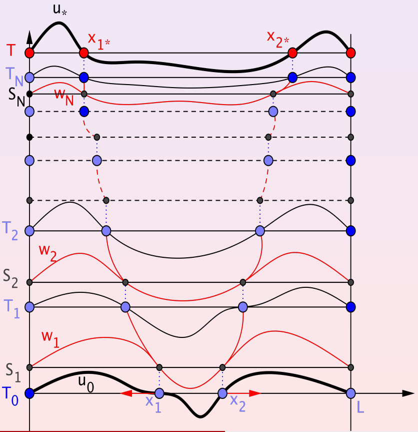
Figure 1. Control of two points of sign change.

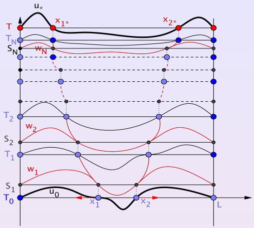
Control strategy: main idea of the proof

Given $N \in \mathbb{N}$ (N will be determined later) we consider the following partition of $[0, T_N]$ in $2N$ intervals:

$$[0, S_1] \cup [S_1, T_1] \cup \cdots \cup [T_{k-1}, S_k] \cup [S_k, T_k] \cup \cdots \cup [T_{N-1}, S_N] \cup [S_N, T_N].$$

$$\alpha_1 \neq 0 \qquad 0 \qquad \cdots \qquad \alpha_k \neq 0 \qquad 0 \qquad \cdots \qquad \alpha_N \neq 0 \qquad 0$$





Construction of the zero curves

- On $[S_k, T_k]$ $(1 \leqslant k \leqslant N)$ we use the Cauchy datum $w_k \in C^{2+\vartheta}([a_0^*, b_0^*]) \cap H_a^1(-1,1)$, $[a_0^*, b_0^*] \subset\subset (-1,1)$ in

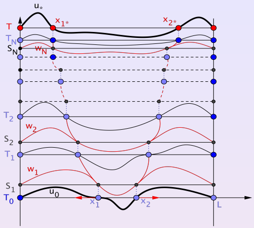
$$\begin{cases} w_t = (a(x)w_x)_x + f(x,t,w), & \text{in} \quad Q_{\mathcal{E}_k} = (-1,1) \times [S_k, T_k], \\ B.C. & t \in [S_k, T_k], \\ w\mid_{t=S_k} = w_k(x), \end{cases}$$

  as a control parameter to be chosen to move the curves of sign change.

- The ℓ-th curve of sign change $(1 \leqslant \ell \leqslant n)$ is given given by solution ξ_ℓ^k

$$\begin{cases} \dot{\xi}_\ell(t) = -\frac{a(\xi_l(t))w_{xx}(\xi_l(t),t)}{w_x(\xi_l(t),t)} - a'(\xi_l(t)), \ t \in [S_k, T_k] \\ \xi_\ell(S_k) = x_\ell^k \end{cases}$$

  where the x_ℓ^k's are the zeros of w_k and so $w(\xi_\ell^k(t), t) = 0$.



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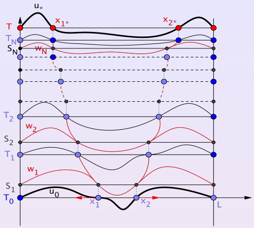

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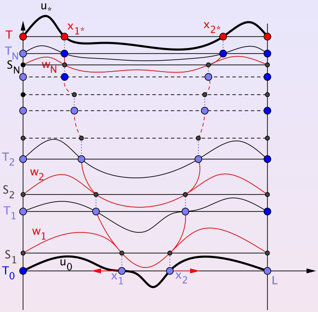

The control parameters w_k's will be chosen to move the curves of sign change in the following way

$$\begin{cases} \dot{\xi}_\ell(t) = -\frac{a(\xi_l(t))w_{xx}(\xi_l(t),t)}{w_x(\xi_l(t),t)} - a'(\xi_l(t)), \ t \in [S_k, T_k] \\ \xi_\ell(S_k) = x_\ell^k \end{cases} \qquad w(\xi_\ell^k(t), t) = 0 \Longrightarrow$$

$$\Longrightarrow \dot{\xi}_\ell(S_k) = -\frac{a(\xi_l(S_k))w_k''(\xi_l(S_k))}{w_k'(\xi_l(S_k))} - a'(\xi_l(S_k)) = \mathrm{sgn}(x_l^* - x_l^0)$$

- To fill the gaps between two successive $[S_k, T_k]$'s, on $[T_{k-1}, S_k]$ we construct the bilinear control α_k that steers the solution of

$$\begin{cases} u_t \;=\; (a(x)u_x)_x \;+\; \alpha_k(x,t)u \;+\; f(x,t,u) & \text{in } (-1,1)\times[T_{k-1}, T_{k-1}+\sigma_k], \\ B.C. & t\in[T_{k-1}, T_{k-1}+\sigma_k], \\ u\;|_{t=T_{k-1}} = u_{k-1} + r_{k-1} \in H^1_a(-1,1), & \end{cases}$$

from $u_{k-1} + r_{k-1}$ to w_k, where u_{k-1} and w_k have the same points of sign change, and $\|r_{k-1}\|_{L^2(0,1)}$ is small. $\alpha_k(x,t)$ piecewise static

Sketch of the proof. In the particular case: $r_{k-1} = 0$ and

$$\exists \delta^* > 0 : \; \delta^* \le \frac{w_k(x)}{u_{k-1}(x)} < 1, \qquad \forall\, x \in (-1,1)\setminus \bigcup_{l=1}^{n}\{x_l\},$$

let us consider $\alpha_k(x,t) \;:=\; \frac{1}{T}\bar{\alpha}_k(x)$, where

$$\bar{\alpha}_k(x) \;=\; \begin{cases} \log\left(\frac{w_k(x)}{u_{k-1}(x)}\right), & \text{for } \; x \neq -1, 1, x_l \;\; (l = 1\ldots, n) \\ 0, & \text{for } \; x = -1, 1, x_l \;\; (l = 1\ldots, n). \end{cases}$$

$$u(x,T) \;=\; e^{\bar{\alpha}_k(x)}u_{k-1}(x) \;+\; \int_0^T e^{\bar{\alpha}_k(x)\frac{(T-\tau)}{T}}\big((a(x)u_x)_x(x,\tau) + f(x,\tau,u(x,\tau))\big)d\tau$$

$$\Rightarrow \|u(\cdot,T) - w_k(\cdot)\|^2_{L^2(-1,1)} \le \; T\,\|\,(a(\cdot)u_x)_x + f(\cdot,\cdot,u)\,\|^2_{L^2(Q_T)}\,.$$

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$$\begin{cases} u_t = (a(x)u_x)_x + \alpha_k(x,t)u + f(x,t,u) & \text{in } (-1,1)\times[T_{k-1}, T_{k-1}+\sigma_k], \\ B.C. & t\in[T_{k-1}, T_{k-1}+\sigma_k], \\ u\ |_{t=T_{k-1}} = u_{k-1} + r_{k-1} \in H^1_a(-1,1), \end{cases}$$

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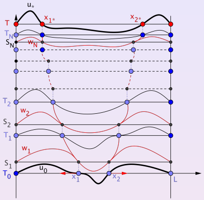
$$\Rightarrow \|u(\cdot,T) - w_k(\cdot)\|^2_{L^2(-1,1)} \le T\,\|\,(a(\cdot)u_x)_x + f(\cdot,\cdot,u)\,\|^2_{L^2(Q_T)}.$$

Closing the loop



- The distance-from-target function satisfies the following estimate, for some $C_1, C_2 > 0$ and $0 < \beta < 1$,

$$0 \leq \sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - x_\ell^*| \leq \sum_{\ell=1}^{n} |x_\ell^0 - x_\ell^*| + C_1 \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\beta}{2}}} - C_2 \sum_{k=n+1}^{N} \frac{1}{k} \xrightarrow{N\to\infty} -\infty$$

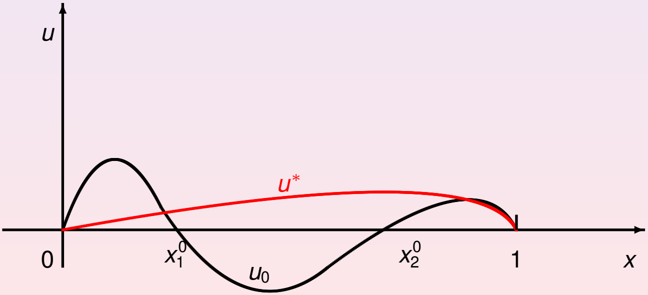
- So the distances of each branch of the null set of the solution from its target points of sign change decreases at a linear-in-time rate while the error caused by the possible displacement of points already near their targets is negligible

- This ensures, by contradiction argument, that $\sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - x_\ell^*| < \epsilon$ within a finite number of steps.

Corollary (G.F., C. Nitsch, C. Trombetti, 2017)

Consider any $u^* \in H^1_a(-1,1)$, *whose amount of points of sign change is less than or equal to the amount of such points for* u_0 *and this points are organized in any order of sign change. Then,*

$$\forall \eta > 0 \ \exists T > 0, \ \alpha \in L^\infty(Q_T) \ : \| \ u(\cdot, T) - u^* \ \|_{L^2(-1,1)} \leq \ \eta.$$



Open problems

- 1-D degenerate reaction-diffusion equations on networks;
- To investigate reaction-diffusion equations in higher space dimensions with initial states that change sign on domains with specific geometries;
- To extend this approach to other nonlinear equations of parabolic type:
  - $\star$ Porous media equation (???) .

Thank you for your attention!