The cost of controlling strongly degenerate parabolic equations

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Outlines

1. Some examples of degenerate parabolic equations
2. Controllability cost for a boundary control :
   - upper bounds
   - lower bounds
3. Controllability cost for locally distributed control
4. Works in progress and open questions

Motivation

Properties of null controllability (and inverse problems) of

$$\begin{cases} u_t - \text{div } (A(x)\nabla u) = h(x,t)\chi_\omega, \\ \text{boundary conditions,} \\ \text{initial condition} \end{cases}$$

when Ω bounded domain of $\mathbb{R}^n$ $(n = 1, 2)$; ω sub-domain of Ω, and

$$A : \overline{\Omega} \to M_n(\mathbb{R}), \quad A(x) \text{ symmetric and } \geq 0,$$

but non uniformly positive : for example when

$$\forall x \in \partial\Omega, \quad \det(A(x)) = 0$$

(First step to controllability to trajectories, LQR problems...)
When $A(x)$ is uniformly positive : heat equation Lebeau-Robbiano (95), general case : Fursikov-Imanuvilov (95, 96)

Some examples of degenerate parabolic equations in dim 1 : :

- climatology : the Budyko-Sellers model :

$$RT_t - k_0((1-x^2)T_x)_x = R_a(,x,T) - R_e(T), \quad x \in (-1,1);$$

Ghil (1976...), Diaz (1993...),
Roques-Checkroun-Cristofol-Soubeyrand-Ghil (2014)

- economy : the Black-Scholes model :

$$u_t - x^2 u_{xx} + \cdots = \cdots, x \in (0,L);$$

- combustion theory and quantum mechanics : inverse square potential

$$u_t - u_{xx} - \frac{\mu}{x^2} u = 0, x \in (0,1)$$

Baras-Goldstein (1984), Vásquez-Zuazua (2000),
Vancostenoble-Zuazua (2008).

Some examples in dim 2 :

- aeronautics : the Crocco equation (boundary layer model) :

$$u_t + a(y)u_x - (b(y)u_y)_y = \text{localized control}, x \in (0, L), y(0, 1)$$

with $a(1) = 0 = b(1)$; Oleinik-Samokhin (1999), Buchot-Raymond (2002), M-Raymond-Vancostenoble (2003) (in the simple case where $a(y) = 1$)

- Kolmogorov type operators :

$$f_t + vf_x - f_{vv} = \text{loc. control}, (x, v) \in (0, 2\pi) \times (0, 2\pi)$$

Beauchard-Zuazua (2009), Beauchard (2014)

- Grushin type operators :

$$f_t - f_{xx} - x^{2\gamma} f_{yy} = \text{loc. control}, (x, y) \in (-1, 1) \times (0, 1)$$

Beauchard-Cannarsa-Guglielmi (2014), Beauchard-Miller-Morancey (2015)

- An example in biology : the Fleming-Viot model (genetic frequency model) :

$$u_t - \text{Tr}\,(C(x)D^2u)\cdots = f,$$

where

$$C(x) = (x_i(\delta_{ij} - x_j))_{i,j}, \quad x \in \{x_i \in [0,1], \sum_i x_i \leq 1\};$$

example : $N = 2$: $\det C(x) = 0$ along the sides of the triangle. Cerrai-Clément (2004), Campiti-Rasa (2004), Albanese-Mangino (2015)
- invariance sets for diffusion processes : Aubin-Da Prato (1990,98) : naturally, the diffusion matrix is degenerate in the normal direction at the boundary.

(Motivations for Cannarsa-M-Vancostenoble, Memoirs AMS (2016))

The typical strongly degenerate parabolic equation :

Strongly degenerate : $u_t - (au_x)_x = \cdots$, when $\int_0^1 \frac{1}{a} = +\infty$.
Simplest cases : given $\alpha \geq 1$:

- Locally distributed control :

$$\begin{cases} u_t - (x^\alpha u_x)_x = h(x,t)\chi_{(a,b)}(x), \quad x \in (0,1), t > 0 \\ (x^\alpha u_x(x,t))_{/x=0} = 0 = u(1,t) \\ u(x,0) = u_0(x) \end{cases},$$

- Boundary control at the non degeneracy point :

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0, \quad x \in (0,1), t > 0 \\ (x^\alpha u_x(x,t))_{/x=0} = 0 \\ u(1,t) = H(t) \\ u(x,0) = u_0(x) \end{cases}:$$

The (theoretical) problem

Known (Cannarsa-M-Vancostenoble (2008)) :

Null Controllability $\begin{cases} \text{holds if } \alpha \in [1,2) \text{ (Carleman estimates)} \\ \text{does not hold if } \alpha \geq 2 \text{ (\textit{csq} Micu-Zuazua (2001))} \end{cases}$

Goal : understand the behavior when $\alpha \to 2^-$.

Natural quantity to estimate : "Null controllability cost" :

$$C(\alpha, T) := \sup_{\|u_0\|=1} \Big(\inf_{\text{admissible control } : u(T)=0} \{\|\text{control}\|\} \Big).$$

Expected :

$$C(\alpha, T) \to +\infty \quad \text{as } \alpha \to 2^- : \text{ true ? speed ?}$$

$$?? \leq C(\alpha, T) \leq ???$$

Related literature

"Controllability cost" appears also in

- the 'fast control problem' : behavior of $C(T)$ as $T \to 0$, for several types of equations : Seidman (1984, 2000), Guichal (1985), Miller (2004, 2005, 2006), Tenenbaum-Tucsnak (2007, 2011), Lissy (2014), Benabdallah et al (2014),
- for semilinear parabolic equations : linearized model (first step) :

  $$u_t - \Delta u + a(x,t)u = ...$$

  behavior of $C(\|a\|_\infty, T)$ as $\|a\|_\infty \to \infty$: Fernandez-Cara-Zuazua (2000),

- the 'vanishing viscosity limit' :

$$u_t - \varepsilon u_{xx} + M u_x = \ldots$$

  analysis of the importance of the transport term, behavior of $C(\varepsilon, T, M)$ as $\varepsilon \to 0$, taking care of the size of MT : Coron-Guerrero (2005), Guerrero-Lebeau (2007), Glass (2010), Lissy (2015),
- observability cost for 1D wave equation : Haraux-Liard-Privat (2016) ; an equivalent of the observability cost as $T \to \infty$: Humbert-Privat-Trélat (2016) ;
- optimizing the location of control region of given measure for $1D$ wave equation : Privat-Trélat-Zuazua (2013), for $1D$ heat equation : Privat-Trélat-Zuazua (2017).

2 main (complementary) methods

Main methods to study null controllability, and its cost :

- direct method : solution $= \cdots$, control $= \cdots$ (moment method, kernels)
  - Fattorini-Russell (1971, 74)
  - eigenvalues, eigenfunctions, biorthogonal families
  - sharp upper and lower bounds for some equations (1-D, constant coefficients) ;
  - (representation with kernels : Martin-Rosier-Rouchon (2015), Dardé-Ervedoza (2016), Tucsnak et al (2017))
- Carleman estimates : weighted estimates $\rightsquigarrow$ observability
  - Lebeau-Robbiano (1995), Fursikov-Imanuvilov (1996)
  - observability
  - good upper bounds for a large class of equations (N-D, variable coefficients).

Carleman estimates for the degenerate parabolic equation :

$$C(\alpha, T) \leq e^{\frac{C}{(2-\alpha)^4 T}} : \quad \text{bound from below ? (sharp estimate ?)}$$

Lower and upper estimate of the cost : main result

$$e^{-\frac{1}{c}\frac{1}{(2-\alpha)^{4/3}}(\ln\frac{1}{2-\alpha}+\ln\frac{1}{T})}e^{\frac{c}{T(2-\alpha)^2}} \leq C(\alpha,T) \leq e^{\frac{C}{T(2-\alpha)^2}}.$$

Tools :

- moment method,
- spectral problem,
- biorthogonal families :
  - construction of a particular biorthogonal family (complex analysis),
    $\rightsquigarrow$ upper estimate
  - estimate from below of any biorthogonal family (hilbertian analysis),
    $\rightsquigarrow$ lower estimate,
  - main ingredient : gap conditions on eigenvalues.

Moment method for the boundary control problem

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0, \quad x \in (0,1), t > 0, \\ (x^\alpha u_x(x,t))_{/x=0} = 0, \\ u(1,t) = H(t) \end{cases} :$$

- well-posedness in weighted Sobolev spaces ; $(H \in H^1(0,T))$;
- eigenvalues, eigenfunctions :

$$\begin{cases} -(x^\alpha \Phi_x)_x = \lambda \Phi, \quad x \in (0,1), t > 0, \\ (x^\alpha \Phi_x)_{/x=0} = 0, \\ \Phi(1) = 0 \end{cases} :$$

then eigenvalues $(\lambda_{\alpha,n})_{n \geq 1}$ associated to $(\Phi_{\alpha,n})_{n \geq 1}$

- The moment problem : if $u(T) = 0$, then multiplying by $\Phi_{\alpha,n}(x)e^{\lambda_{\alpha,n}(t-T)}$:

$$\forall n \geq 1, \quad \int_0^T H(t)e^{\lambda_{\alpha,n}t}\,dt = \frac{(u_0, \Phi_{\alpha,n})}{r_{\alpha,n}},$$

with

$$r_{\alpha,n} = \Phi'_{\alpha,n}(1).$$

- Biorthogonal families : $(\sigma^+_{\alpha,m})_{m\geq 1}$ is "biorthogonal" to $(e^{\lambda_{\alpha,n}t})_{n\geq 1}$ in $L^2(0,T)$ if :

$$\forall m, n \geq 1, \quad \int_0^T \sigma^+_{\alpha,m}(t)e^{\lambda_{\alpha,n}t}\,dt = \delta_{mn} = \begin{cases} 1 \text{ if } m = n, \\ 0 \text{ if } m \neq n \end{cases}.$$

Null controllability $\equiv$ biorthogonal family

- if there is a control H_m for $u_0 := \Phi_{\alpha,m}$, then

$$\forall n \geq 1, \quad \int_0^T (r_{\alpha,m} H_m(t)) e^{\lambda_{\alpha,n} t}\, dt = \delta_{mn},$$

hence $(r_{\alpha,m} H_m)_m$ is biorthogonal to $(e^{\lambda_{\alpha,n} t})_{n\geq 1}$ in $L^2(0,T)$:

$$\text{null controllability} \quad \Longrightarrow \quad \text{biorthogonal family};$$

- and the converse is formally true : if $(\sigma^+_{\alpha,m})_{m\geq 1}$ is biorthogonal to $(e^{\lambda_{\alpha,n} t})_{n\geq 1}$ in $L^2(0,T)$, then formally :

$$H_{expl}(t) := \sum_{m=1}^{\infty} \frac{(u_0, \Phi_{\alpha,m})}{\Phi'_{\alpha,m}(1)} \sigma^+_{\alpha,m}(t)$$

solves the moment problem and drives the solution to 0 in time T :

$$\text{biorthogonal family} \quad \Longrightarrow \quad \text{null controllability}.$$

Hence :

$$\begin{cases} \text{lower bound } LB_m \text{ for ANY biorthogonal sequence} \\ \text{information on the eigenfunctions} \end{cases}$$

$\Longrightarrow$ lower bound for NC cost, since :

$$\|H_m\| \geq \frac{LB_m}{|r_{\alpha,m}|} = \frac{LB_m}{\Phi'_{\alpha,m}(1)};$$

$$\begin{cases} \text{upper bound } UB_m \text{ for SOME biorthogonal sequence} \\ \text{information on the eigenfunctions} \end{cases}$$

$\Longrightarrow$ upper bound for NC cost, since :

$$\|H_{expl}\| \leq \Big(\sum_{m=1}^{\infty} \frac{\|\sigma^+_{\alpha,m}\|^2}{\Phi'_{\alpha,m}(1)^2}\Big)^{1/2} \leq \Big(\sum_{m=1}^{\infty} \frac{UB_m^2}{\Phi'_{\alpha,m}(1)^2}\Big)^{1/2};$$

Good informations on biorthogonal families and eigenfunctions ?

Eigenvalues of the degenerate problem

related to Bessel functions and their zeros :
For $\alpha \in [1, 2)$, let

$$\kappa_\alpha := \frac{2-\alpha}{2}, \qquad \nu_\alpha := \frac{\alpha - 1}{2 - \alpha}.$$

Then (Kamke (1948), Everitt-Zettl (1978), Gueye (2014) when $\alpha \in [0, 1)$)

- the eigenvalues : $\forall n \geq 1, \lambda_{\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2$,
- the eigenfunctions $\Phi_{\alpha,n}(x) = \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha})$,

where

- J_{ν_α} is the Bessel function of first kind and of order ν_α,
- and $(j_{\nu_\alpha,n})_{n\geq 1}$ is the sequence of the positive zeros of J_{ν_α}.

Argument :

λ eigenvalue, Φ associated eigenfunction : then the new function Ψ

$$\Phi(x) =: x^{\frac{1-\alpha}{2}} \Psi\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right)$$

satisfies the following ODE :

$$y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\alpha-1}{2-\alpha})^2)\Psi(y) = 0, \quad y \in (0, \frac{2\sqrt{\lambda}}{2-\alpha})$$

which is the Bessel's equation of order ν :

$$y^2\Psi''(x) + y\Psi'(x) + (y^2 - \nu^2)\Psi(x) = 0, \quad y > 0,$$

with $\nu = \nu_\alpha := \frac{\alpha-1}{2-\alpha}$.
Using the boundary conditions and well-posedness setting, we find $\lambda_{\alpha,n}$ and $\Phi_{\alpha,n}$.

Remark and connection with other problems :

- the strongly degenerate parabolic equation :

$$\begin{cases} -(x^\alpha \Phi_x)_x = \lambda \Phi \\ (x^\alpha \Phi_x)(0) = 0 = \Phi(1) \end{cases} \implies \lambda_{\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2,$$

and then

$$\sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}} = \kappa_\alpha (j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n}),$$

and (classical for Bessel functions)

$$j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n} \to \pi \quad \text{as } n \to \infty,$$

hence

$$\sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}} \to_{n\to\infty} \frac{\pi}{2}(2-\alpha) \quad \to 0 \text{ as } \alpha \to 2.$$

- the 'vanishing viscosity limit' :

$$\begin{cases} -\varepsilon\Phi_{xx} - M\Phi_x = \lambda\Phi \\ \Phi(0) = 0 = \Phi(1) \end{cases} \implies \lambda_{\varepsilon,n} = \varepsilon\pi^2 n^2 + \frac{M^2}{4\varepsilon} :$$

hence

$$\sqrt{\lambda_{\varepsilon,n+1}} - \sqrt{\lambda_{\varepsilon,n}} \to_{n\to\infty} \pi\sqrt{\varepsilon} \quad \to 0 \text{ as } \varepsilon \to 0;$$

- the 'fast control problem' : with the normalization $v(x,\tau) := u(x,\tau T)$:

$$\begin{cases} v_\tau - Tv_{xx} = \text{ loc. control}, \\ v_{/x=0} = 0 = v_{/x=1}, \\ v_{/t=0} = u_0, v_{/\tau=1} = 0 \end{cases} \implies \lambda_{T,n} = T\pi^2 n^2,$$

and then $\sqrt{\lambda_{T,n+1}} - \sqrt{\lambda_{T,n}} \to_{\to\infty} \pi\sqrt{T} \quad \to 0 \text{ as } T \to 0.$

Existence and bounds of a biorthogonal sequence under 'gap-min' condition

Existence of biorthogonal sequences :

- Fattorini-Russell (1971, 74) : functional analysis and complex analysis ; gap conditions but not explicit as $T \to 0$;
- many applications of their results and methods ;
- for the dependence $T \to 0$:
  - Seidman-Avdonin-Ivanov (2000), Tenenbaum-Tucsnak (2007, 2011), Lissy (2014) : complex analysis (but they work with $\lambda_n = rn^2 + l.o.t.$, and not enough precise (?) if there is a large parameter hidden in l.o.t.),
  - Glass (2010) : precise, but adaptable to general conditions on the eigenvalues ?

We proved the following version (Cannarsa-M-Vancostenoble (2017)) :

Theorem

Assume that $\lambda_1 \geq 0$, and that

$$\forall n \geq 1, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{min} > 0.$$

Then there exists a family $(\sigma_m^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0,T)$, and for which there is some universal constant C_u independent of T, γ_{min} and m such that

$$\forall m, \quad \|\sigma_m^+\|^2_{L^2(0,T)} \leq C_u e^{-2\lambda_m T} e^{C_u \frac{\sqrt{\lambda_m}}{\gamma_{min}}} e^{\frac{C_u}{\gamma_{min}^2 T}} B(T, \gamma_{min}),$$

with $B(T,\gamma_{min}) = \frac{1}{T} \max(T\gamma_{min}^2, \frac{1}{T\gamma_{min}^2})$.

Proof : mainly the construction of Seidman-Avdonin-Ivanov (2000) (complex analysis techniques), combined with an additionnal parameter (Tenenbaum-Tucsnak (2007), Lissy (2015)) :

Argument

- a Weierstrass product :

$$F_m(z) := \prod_{k=1, k\neq m}^{\infty} \left(1 - \left(\frac{iz - \lambda_m}{\lambda_k - \lambda_m}\right)^2\right)$$

  whose growth is estimated using the gap condition (number of eigenvalues),

- a suitable mollifier $M_m(z)$ (slight change w.r.t. litt) :

$$f_m := F_m M_m \text{ satisfies } \begin{cases} \forall m, n \geq 1, & f_m(-i\lambda_n) = \delta_{mn}, \\ \forall z \in \mathbb{C}, & |f_m(-z)e^{-iz\frac{T}{2}}| \leq C_m e^{\frac{T}{2}|z|} \\ \forall m \geq 1, & f_m \in L^2(\mathbb{R}) \end{cases},$$

- the Paley-Wiener theorem : $f_m(-z)e^{-iz\frac{T}{2}}$ is the inverse Fourier transform of some compactly supported function ϕ_m (properties 2 and 3), that will give the biorthogonal sequence (property 1).

Application : upper bound of the cost

When $\nu_\alpha = \frac{\alpha-1}{2-\alpha} \geq \frac{1}{2}$, the sequence $(j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n})_n$ decays to π (Komornik-Loreti (2005)), hence

$$\sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}} \geq \frac{\pi}{2}(2-\alpha) =: \gamma_{min}(\alpha),$$

hence the existence of a biorthogonal family, and the control

$$H^{(u_0)}(t) := \sum_{m=1}^{\infty} \frac{(u_0, \Phi_{\alpha,m})}{r_{\alpha,m}} \sigma^+_{\alpha,m}(t)$$

is well-defined, drives the solution to 0 in time T, hence

$$C(\alpha, T) \leq \sup_{\|u_0\|=1} \|H^{(u_0)}\|$$

$$\leq \Big(\sum_{m=1}^{\infty} \frac{C_u}{r^2_{\alpha,m}} e^{-2\lambda_{\alpha,m} T} e^{C_u \frac{\sqrt{\lambda_{\alpha,m}}}{\gamma_{min}(\alpha)}} e^{\frac{C_u}{\gamma_{min}(\alpha)^2 T}} B(T, \gamma_{min}) \Big)^{1/2} \leq C_u e^{\frac{C_u}{(2-\alpha)^2 T}}.$$

Lower bound under some global 'gap-max' condition

Motivation :

$$H_m \text{ drives } u_0 := \Phi_{\alpha,m} \text{ to } 0 \text{ in time } T$$

$$\Longrightarrow_{\text{(moment method)}} \int_0^T (r_{\alpha,m} H_m) e^{\lambda_{\alpha,n} t}\, dt = \delta_{mn},$$

hence $(r_{\alpha,m} H_m)_m$ is biorthogonal to $(e^{\lambda_{\alpha,n} t})_n$.

And bound from below, by hilbertian techniques :

- in $\mathbb{R}^2$, $\mathbb{R}^3$:

$$(v_1, v_2) \text{ biorthogonal to } (u_1, u_2) \Longrightarrow \|v_2\| \geq \frac{1}{\operatorname{dist}(u_2, \mathbb{R}u_1)};$$

- similarly in a general context :
  - Hansen (1991) : optimality (w.r.t. m) of Fattorini-Russell,
  - Guichal (1985) optimality (for heat eq.) of Seidman (1984).

We extend Guichal (1985), the goal being to have

$\|r_{\alpha,m} H_m\| \geq \cdots$, hence $\|H_m\| \geq \frac{\cdots}{|r_{\alpha,m}|}$, hence $C(\alpha, T) \geq \frac{\cdots}{|r_{\alpha,m}|}$.

Theorem

(CMV (2017)) Assume that $\lambda_1 \geq 0$*, and that there is some* $0 < \gamma_{min} \leq \gamma_{max}$ *such that*

$$\forall n \geq 1, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}.$$

Then there exists some $C(m, \gamma_{max}, \lambda_1) > 0$ *(given explictly in the paper) and* $c_u > 0$ *independent of* T *and* m *such that : any family* $(\sigma_m^+)_{m \geq 1}$ *which is biorthogonal to the family* $(e^{\lambda_n t})_{n \geq 1}$ *in* $L^2(0, T)$ *satisfies :*

$$\|\sigma_m^+\|_{L^2(0,T)}^2 \geq e^{-2\lambda_m T} e^{\frac{1}{2\gamma_{max}^2 T}} b(T, \gamma_{max}, m),$$

with

$$b(T, \gamma_{max}, m) = \frac{c_u^2}{C(m, \gamma_{max}, \lambda_1)^2\, T} \left(\frac{1}{2\gamma_{max}^2 T}\right)^{2m} \frac{1}{(4\gamma_{max}^2 T + 1)^2}.$$

Argument

- As in $\mathbb{R}^2$, $\mathbb{R}^3$: any biorthogonal sequence verifies

$$\|\sigma_m^+\| \geq \frac{1}{d_{T,m}}, \quad \text{where } d_{T,m} := \text{ dist } (e^{\lambda_m t}, \overline{\text{Vect } \{e^{\lambda_k t}, k \neq m\}});$$

- then Guichal (1985) :

$$d_{T,m} \leq \|e^{\lambda_m t} - \sum_{i=1, i\neq m}^{M+1} A_i e^{\lambda_i t}\| = \|\frac{-1}{\tilde{A}_m} \sum_{i=1}^{M+1} -\tilde{A}_i e^{\lambda_i t}\| = \|\frac{-1}{\tilde{A}_m} q(t)\|,$$

with a special choice of the coefficients $\tilde{A}_i$: chosen such that

$$q(0) = \cdots = q^{(M-1)}(0) = 0, \quad q^{(M)}(0) = 1$$

(in order to have q small) ;

- then (Vandermonde determinant)

$$d_{T,m} \leq \Big(\prod_{i=1, i\neq m}^{M+1} |\lambda_i - \lambda_m| \Big) \Big(\int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} \, ds \Big)^{1/2} \leq \cdots$$

Application

As already seen :

$$\pi \le j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n} \le j_{\nu_\alpha,2} - j_{\nu_\alpha,1},$$

hence $\gamma_{max} = \kappa_\alpha \left(j_{\nu_\alpha,2} - j_{\nu_\alpha,1}\right)$. But (classical)

$$j_{\nu,2} - j_{\nu,1} \sim a\nu^{1/3} \quad \text{as } \nu \to +\infty.$$

Hence $\gamma_{max} \sim c\nu_\alpha^{-2/3} \sim c'(2-\alpha)^{2/3} \quad \text{as } \alpha \to 2^-$, hence

$$C(\alpha, T) \ge \|H_1(\text{ driving } \Phi_{\alpha,1})\| \ge \frac{\cdots}{|r_{\alpha,1}|} e^{\frac{1}{2\gamma_{max}^2 T}} = \frac{\cdots}{|r_{\alpha,1}|} e^{\frac{c}{(2-\alpha)^{4/3} T}};$$

hence

$$e^{\frac{c}{(2-\alpha)^{4/3} T}} \le C(\alpha, T) \le e^{\frac{C}{(2-\alpha)^2 T}}.$$

Hence $C(\alpha, T) \to \infty$ as $\alpha \to 2^-$, BUT gap between $\frac{4}{3}$ and 2...

Additionnal property of the zeros of Bessel functions

The gap $(j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n})_{n\geq 1}$ decays

- from $j_{\nu_\alpha,2} - j_{\nu_\alpha,1} \sim a\nu_\alpha^{1/3} = \frac{b}{(2-\alpha)^{1/3}}$ (large)
- to π.

Hence certainly :

$$\exists N_\nu, \forall n \geq N_\nu, \quad j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n} \leq 2\pi$$

(which was not taken into account). The value of N_ν :

Lemma

(CMV (2017)

$$\forall \nu \geq \frac{1}{2}, \forall n > \nu, \quad j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n} \leq 2\pi.$$

Proof : oscillation theorem of Sturm for second order ODE, in the spirit of Komornik-Loreti (2005).

Lower bound under some asymptotic 'gap-max' condition

Theorem

Assume that $\lambda_1 \geq 0$*, and that there are* $0 < \gamma_{min} \leq \gamma^*_{max} \leq \gamma_{max}$ *such that*

$$\forall n \geq 1, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max},$$

and

$$\forall n \geq N_*, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma^*_{max}.$$

Then any family $(\sigma^+_m)_{m\geq 1}$ *which is biorthogonal to the family* $(e^{\lambda_n t})_{n\geq 1}$ *in* $L^2(0,T)$ *satisfies :*

$$\|\sigma^+_m\|^2_{L^2(0,T)} \geq e^{-2\lambda_m T} e^{\frac{2}{T(\gamma^*_{max})^2}} b^*(T, \gamma_{max}, \gamma^*_{max}, N_*, \lambda_1, m)^2$$

(where b^* *is explictly given in our paper).*

(In the spirit of Ingham theorems)

Application

Applying the previous result with the asymptotic gap of the zeros of Bessel functions :

- $\gamma_{max} = \frac{2-\alpha}{2}(j_{\nu_\alpha,2} - j_{\nu_\alpha,1}) \sim a(2-\alpha)^{2/3}$,
- $\gamma^*_{max} = \frac{2-\alpha}{2} 2\pi = \pi(2-\alpha)$,
- $N_* = [\nu_\alpha] + 1$,

we obtain

$$e^{-\frac{1}{c}\frac{1}{(2-\alpha)^{4/3}}(\ln\frac{1}{2-\alpha}+\ln\frac{1}{T})} e^{\frac{c}{T(2-\alpha)^2}} \leq C(\alpha, T) \leq e^{\frac{C}{T(2-\alpha)^2}}.$$

The locally distributed control problem

$$\begin{cases} u_t - (x^\alpha u_x)_x = h(x,t)\chi_{(a,b)}(x), & x \in (0,1), t > 0, \\ (x^\alpha u_x(x,t))_{/x=0} = 0 = u(1,t) \end{cases}$$

Similar :

- NC holds iff $\alpha < 2$;
- formally admissible control :

$$h(x,t) = h(x,t) := \sum_{m \geq 1} -(u_0, \Phi_{\alpha,m})\, \sigma^+_{\alpha,m}(t)\, \frac{\Phi_{\alpha,m}(x)}{\int_a^b \Phi^2_{\alpha,m}};$$

- hence information needed for upper estimate of the NC cost : lower bound of $\int_a^b \Phi^2_{\alpha,m}$;
- classical (Lagnese (1983)) : $\inf_m \int_a^b \Phi^2_{\alpha,m} > 0$, but not sufficient to estimate $\|h\|$ in function of α ;

- we prove the following :

Proposition

There exists $\gamma_0^* > 0$ independent of $\alpha \in [1,2)$ and of $m \geq 1$ s.t.

$$\forall \alpha \in [1,2), \forall m \geq 1, \quad \int_a^b \Phi_{\alpha,m}^2 \geq \gamma_0^*(2-\alpha);$$

Difficulties

- in our case

$$\int_a^b \Phi_{\alpha,m}^2 = \frac{2}{j_{\nu_\alpha,m}^2 |J'_{\nu_\alpha}(j_{\nu_\alpha,m})|^2} \int_{j_{\nu_\alpha,m}\, a^{\kappa_\alpha}}^{j_{\nu_\alpha,m}\, b^{\kappa_\alpha}} y J_{\nu_\alpha}(y)^2 \, dy,$$

- known : $J_\nu(y)$ either as $y \to +\infty$ or as $\nu \to +\infty$,
- but here y and ν go to $+\infty$ simultaneously ;

- and proof :
  - the Bessel ODE,
  - integral representation of solutions (keeping in mind ν_α large)
  - classical properties of sin (Haraux (1978))
- and then similar upper and lower estimates for the NC cost.

(Rq : another estimate (from above) of Bessel function in Privat-Trélat-Zuazua (2015) p. 957)

So to conclude :

- many properties of Bessel functions to use,
- some of them to complete,
- and explicit and practical results for biorthogonal sequences under global/asymptotic gap conditions.

Complement and other applications

- upper bound of the biorthogonal family under asymptotic gap conditions
- degeneracy inside the domain : NC and its cost in function of the degeneracy parameter (joint work with P. Cannarsa and Roberto Ferretti)
- inverse square potential : precise estimates of the cost in function of the parameter μ (joint work with J. Vancostenoble) (related to Biccari-Zuazua (2016))

Some open questions

- degenerate parabolic equation : the reachable space ? (in the spirit of Dardé-Ervedoza (2016), Tucsnak et al (2017))
- Grushin operator : some of this useful to prove the optimality of the minimal time ? ? (several progresses : Beauchard-Dardé-Ervedoza (2017), Allonsius-Morancey)
- and many others related to the motivating models :
  - Crocco equation (aeronautics) : nonlinear and with several degeneracies,
  - Fleming-Viot model (biology) : double degeneracy at the vertices of the triangle,
  - Budyko-Sellers model (climatology) : reconstruction of coefficients for involved problems (p-Laplacian, memory terms)

Thank you for your attention !