Partial Differential Equations, Optimal Design and Numerics

Scalar Nonlocal Balance Laws – Results on Existence, Uniqueness and Regularity[1]

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[1] Joint work with L.Pflug, FAU Erlangen-Nürnberg, Germany

Overview

Relevance of Nonlocal Balance Laws

Possible applications

- Supply chains

D. Armbruster, D. E. Marthaler, C. Ringhofer, K. Kempf, and T.-C. Jo.
A Continuum Model for a Re-entrant Factory.
Operations Research, 54(5):933–950, September 2006.

Jean-Michel Coron, Zhiqiang Wang, and Matthias Kawski.
Analysis of a conservation law modeling a highly re-entrant manufacturing system.
Discrete and Continuous Dynamical Systems-Series B, 2009.

- Chemical ripening processes

Haderlein, M., Segets, D., Gröschel, M., Pflug, L., Leugering, G., and Peukert, W. (2015).
FIMOR: An efficient simulation for ZnO quantum dot ripening applied to the optimization of nanoparticle synthesis.
Chemical Engineering Journal

- Traffic flow

S. Blandin and P. Goatin.
Well-posedness of a conservation law with non-local flux arising in traffic flow modeling.
Numerische Mathematik, 132(2):217–241, 2016.

- Crowd dynamics

Colombo, R. M., Herty, M., and Mercier, M. (2011).
Control of the continuity equation with a non local flow.
ESAIM Control Optim. Calc. Var., 17(2):353–379.

Colombo, R. M. and Lécureux-Mercier, M. (2012).
Nonlocal crowd dynamics models for several populations.
Acta Mathematica Scientia, 32(1):177–196.

In Equations

Suppose $T \in \mathbb{R}_{>0}$ and $(t,x) \in (0,T) \times \mathbb{R}$

Exemplary: Some nonlocal balance laws

- **Supply Chains:** $a, b \in \mathbb{R}$, γ a weight, q the density of products, λ processing velocity

$$q_t(t,x) + \partial_x \left(\lambda\Big(\int_a^b \gamma(t,x,y) q(t,y) \mathrm{d}y \Big) q(t,x) \right) = 0$$

- **Traffic flow:** $\eta \in \mathbb{R}$ (usually small), ω_η a sufficiently smooth kernel, q the density of cars

$$q_t(t,x) + \partial_x \left(q(t,x) \Big(1 - \int_x^{x+\eta} q(t,y) \omega_\eta(y-x) \mathrm{d}y \Big) \right) = 0$$

- **Chemical Ripening Processes:** R ripening velocity, $x_{\min} \in \mathbb{R}_{>0}$ lower bound on the particle size, q_{in} additional particles entering, $n \in \{2,3\}$ the n-th momentum

$$q_t(t,x) + \partial_x \Big(R(W_n[q], t, x) q(t,x) \Big) = q_{\text{in}}(t,x)$$

$$W_n[q](t) = \int_{x_{\min}}^{\infty} y^n q(t,y) \mathrm{d}y$$

All equations are supplemented by an initial condition q_0.

Problem Formulation

Unifying approach for some of the previously mentioned classes of problems: For $T \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$

Considered class of balance laws

$$q_t(t,x) + \partial_x\Big(\lambda\Big[W[q,\gamma,a,b]\Big](t,x)q(t,x)\Big) = h(t,x)$$
$$q(0,x) = q_0(x)$$

supplemented by the nonlocal term W, averaging of the "density" in space

$$W[q,\gamma,a,b](t,x) := \int_{a(x)}^{b(x)} \gamma(t,x,y)q(t,y)\,\mathrm{d}y.$$

λ a sufficiently smooth "flux" function, a, b the boundaries of the nonlocal term, γ a weight in the nonlocal term (e.g. convolution), q_0 initial datum, h space and time dependent source term.

Remark

- No fully local behavior anymore, i.e. solution has to be known between $a(x)$ and $b(x)$ to process in time.
- Still finite propagation, but implicitly dependent on the average density.
- None of the usual existence results (Kružkov, etc.) applicable.

Main Theorem

Theorem: Existence and uniqueness for sufficiently small time

Suppose $q_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $h \in L^\infty((0,T); L^\infty(\mathbb{R}))$, $a, b \in C^1(\mathbb{R})$ with $a', b' \in L^\infty(\mathbb{R})$, γ sufficiently smooth and λ satisfy the "growth conditions" below, there is $T^* \in (0,T]$ s.t. a unique weak solution

$$q \in C([0,T^*]; L^p(\mathbb{R})) \cap L^\infty((0,T^*); L^\infty(\mathbb{R}))$$

exists $(p \in [1,\infty))$.

Growth Condition on λ

For $w \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$ we assume

1. $\lambda[w]$ locally Lipschitz continuous in the spatial variable.
2. $\lambda[w]$ of at most linear growth in the spatial variable.
3. $\operatorname{ess-inf}_{(t,x)\in(0,T)\times\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x}\lambda[w](t,x)$ is finite.

Remark

No Entropy Condition required!

Sketch of the Proof

Part I

Consider for $\lambda \in L^\infty((0,T);W^{1,\infty}(\mathbb{R}))$ the LINEAR balance law

$$\begin{aligned} q_t(t,x) + \partial_x\Big(\tilde{\lambda}(t,x)q(t,x)\Big) &= h(t,x) \\ q(0,x) &= q_0(x). \end{aligned}$$

Thanks to the Lipschitz-continuity of the velocity in the spatial variable, the (unique) weak solution can be globally (!) posed in terms of characteristics

$$q(t,x) = q_0(\xi[t,x](0))\partial_2\xi[t,x](0) + \int_0^t \partial_2\xi[t,x](\tau)h(\tau,\xi[t,x](\tau))\mathrm{d}\tau,$$

where the characteristics for every given $(t,x) \in [0,T] \times \mathbb{R}$ are defined as the (unique!) Carathéodory solution of the integral equality

$$\xi[t,x](\tau) = x + \int_t^\tau \tilde{\lambda}\left(y,\xi[t,x](y)\right)\mathrm{d}y, \quad \tau \in [0,T].$$

Sketch of the Proof

Part II

$$q_t(t,x) + \partial_x\Big(\underbrace{\lambda\Big[W\big[q,\gamma,a,b\big]\Big](t,x)}_{\tilde{\lambda}(t,x)}\, q(t,x)\Big) = h(t,x)$$

Using the solution formula for a linear balance law, we obtain – recalling the nonlocal term –

$$w(t,x) := W[q,\gamma,a,b]\big(t,x\big) := \int_{a(x)}^{b(x)} \gamma(t,x,y)q(t,y)\,\mathrm{d}y$$

for the nonlocal balance law a fixed-point equation in w

$$w(t,x) = \int\limits_{\xi_w[t,a(x)](0)}^{\xi_w[t,b(x)](0)} \gamma(t,x,\xi_w[0,z](t))q_0\big(z\big)\mathrm{d}z + \int\limits_0^t \int\limits_{\xi_w[t,a(x)](\tau)}^{\xi_w[t,b(x)](\tau)} \gamma(t,x,\xi_w[\tau,z](t))h(\tau,z)\mathrm{d}z\,\mathrm{d}\tau,$$

where $\xi_w[t,x]$ is the characteristic curve through $[t,x] \in (0,T) \times \mathbb{R}$ satisfying

$$\xi_w[t,x](\tau) = x + \int_t^\tau \lambda[w]\left(y, \xi_w[t,x](y)\right)\mathrm{d}y, \quad \tau \in [0,T].$$

Sketch of the Proof

Part III

- Applying Banach's fixed-point theorem in the appropriate topology (here $L^\infty((0,T^*);L^\infty(\mathbb{R}))$) and several technical estimates, we obtain for a sufficiently small time horizon the existence of a unique $w^* \in L^\infty((0,T^*);W^{1,\infty}(\mathbb{R}))$.
- With w^* given and Lipschitz-continuous in the spatial variable, we obtain a unique and global solution for the characteristics $\xi_{w^*}[t,x]$, $(t,x) \in [0,T] \times \mathbb{R}$.
- Construct the solution of the nonlocal conservation law for $t \in [0,T^*]$ in terms of characteristics as

$$q(t,x) = q_0(\xi_{w^*}[t,x](0))\partial_2\xi_{w^*}[t,x](0) + \int_0^t \partial_2\xi_{w^*}[t,x](\tau)h(\tau,\xi_{w^*}[t,x](\tau))\mathrm{d}\tau.$$

- Show that the solution is a weak solution and satisfies all required properties.
- Uniqueness of w^* is passed to the uniqueness of the solution (some work).

Extending Time

Clustering a sequence of initial value problems with initial data equal to the previous end data, we can extend time, but not necessarily to any given time horizon.

$$W[q, \gamma, a, b](t, x) = \int_{a(x)}^{b(x)} \gamma(t, x, y) q(t, y) \, \mathrm{d}y$$

The spatial derivative of W might blow up, resulting in a blowup of the Lipschitz-property of the flux function $\lambda[W]$.

Theorem: Existence of the solution for larger time

Suppose that in addition one of the following items holds:

- $a(x) = a$ with $a \in \mathbb{R} \cup \{\pm\infty\}$ and $b(x) = b$ with $b \in \mathbb{R} \cup \{\pm\infty\}$
- $\operatorname{supp}(\gamma(t, x, \cdot)) \subsetneq (a(x), b(x))$

Then, the solution exists on every finite time horizon.

Sketch of the proof

Compute the spatial derivative of W.

Remarks and Regularity of the Solution

Remark: Properties of the solution

For the solution q as constructed we have for $t \in [0, T]$

$$\|q(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|q_0\|_{L^1(\mathbb{R})} + \|h\|_{L^1((0,t)\times\mathbb{R})}$$

and for $q \geq 0$ and $h \geq 0$ by construction $q \geq 0$ and

$$\|q(t, \cdot)\|_{L^1(\mathbb{R})} = \|q_0\|_{L^1(\mathbb{R})} + \|h\|_{L^1((0,t)\times\mathbb{R})}.$$

Remark: Pure L^1, L^∞ solutions

For the assumptions in the previous Theorem, we only need $q_0 \in L^1(\mathbb{R})$ and the solution will satisfy the regularity $C([0, T]; L^1(\mathbb{R}))$.
On the other hand, if $\|a - b\|_{L^\infty(\mathbb{R})}$ is finite, we also obtain a solution for $q_0 \in L^\infty(\mathbb{R})$ and $h \in L^\infty((0, T); L^\infty(\mathbb{R}))$ only.

Remark: Regularity

Since there is no "loss of information", no emerging shocks, we can obtain more regular solutions in case initial datum, source term, velocity and boundary terms of the nonlocal term are sufficiently regular.

Example – space dependent nonlocal term

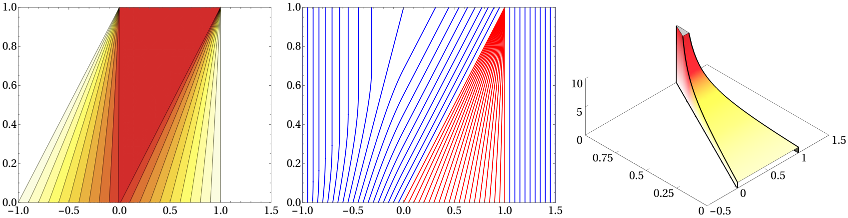

Figure 1: $W(t,x)$ (left), characteristics (middle) and solution (right)

$$q_t(t,x) + \partial_x \left(\int_x^{x+1} q(t,y)\mathrm{d}y \, q(t,x) \right) = 0$$
$$q_0(x) = \chi_{[0,1]}(x),$$

i.e. $\lambda[W](t,x) = W(t,x)$, $h \equiv 0$, $\gamma \equiv 1$, $a(x) = x$, $b(x) = x+1$. Solution can be computed explicitly and is illustrated in the above Figure 1. Solution blows up at $t = 1$.

Example – space independent nonlocal term

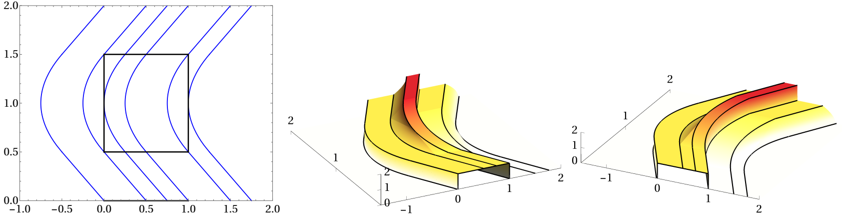

Figure 2: Characteristics (left) and solution (middle, right), the black box and the black line from $(0,0)$ to $(1,0)$ in the left picture represent the support of the source term and initial data.

$$
\begin{aligned}
q_t(t,x) + \Big((2w(t)-3)\ q(t,x)\Big)_x &= \chi_{[\frac{1}{2},\frac{3}{2}]\times[0,1]}(t,x)\\
q(0,x) &= \chi_{[0,1]}(x)\\
w(t) &= \int_{\mathbb{R}} q(t,s)\mathrm{d}s
\end{aligned}
$$

Solution can be computed explicitly and is illustrated in the above Figure 2.

Open Problems

Future work

- Extension to nonlocal balance laws with spatial variable $x \in \mathbb{R}^n$.
- Nonlocal balance laws on bounded domains, many applications in traffic flow on networks with strictly positive velocity.
- Convergence of nonlocal conservation laws to the classical "local" conservation laws $q_t + f(q)_x = 0$ by choosing as boundary in the nonlocal term something like $a(x) = x, b(x) = x + \epsilon$ and $\gamma = \frac{1}{\epsilon}$ for $\epsilon \to 0$.
- Optimal Control and adjoint PDE.
- Nonlocal balance laws with nonlocal source term.
- Numerical methods using the above developed formula of the solution.

Some Advertisement

A. Keimer and L. Pflug.
Existence, uniqueness and regularity results on nonlocal balance laws.
Journal of Differential Equations, 2017.

M. Gugat, A. Keimer, G. Leugering, and Z. Wang.
Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks.
NHM, 10(4):749–785, 2015.

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Regularity theory and adjoint-based optimality conditions for a nonlinear transport equation with nonlocal velocity.
SIAM Journal on Control and Optimization, 52(4):2141–2163, 2014.

A. Keimer, G. Leugering, and T. Sarkar.
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submitted, 2017.

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in preparation, 2017.

The End?

Thank you very much!