MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Feedback Control, Moving Interfaces, and Non-Autonomous Riccati Equations

VII Partial differential equations, optimal design and numerics

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Given a coupled nonlinear system:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathcal{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \mathbf{u}),\\ \dot{\mathbf{w}} &= \mathcal{F}_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \mathbf{u}),\end{aligned}$$

together with a reference solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}})$ obtained with an open loop control.

Goal: Stabilization of $(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}})$ by Riccati feedback.

Motivation: The open loop control $\tilde{\mathbf{u}}$ is not robust against perturbations and uncertainties.

Strategy: Linearization around $(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}})$ leads to linear system $(\mathcal{M}, \mathcal{A}, \mathcal{B}, \mathcal{C})$.

Linear Quadratic Regulator Approach

Minimize
$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty ||\mathbf{y}-\mathbf{y_d}||^2 + \lambda||\mathbf{u}||^2 \,\mathrm{dt}$$

subject to
$$\begin{aligned}\mathcal{M}\frac{d}{dt}\mathbf{x}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathcal{C}\mathbf{x}(t).\end{aligned}$$

Linear Quadratic Regulator Approach

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$$\mathbf{y}(t) = \mathcal{C}\mathbf{x}(t).$$

Riccati Based Feedback Approach

e.g.,[Locatelli '01]

- Feedback: $\mathcal{K} = \mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}$,

  where $\mathbf{X}$ is the solution of the generalized algebraic Riccati equation

  $$\mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{A} - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M} = 0.$$

- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\mathbf{x}(t)$.

Linear Quadratic Regulator Approach

Minimize $$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty ||\mathbf{y}-\mathbf{y_d}||^2 + \lambda||\mathbf{u}||^2 \,\mathrm{dt}$$

subject to $$\boxed{\begin{aligned} \mathcal{M}\frac{d}{dt}\mathbf{x}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathcal{C}\mathbf{x}(t). \end{aligned}}$$

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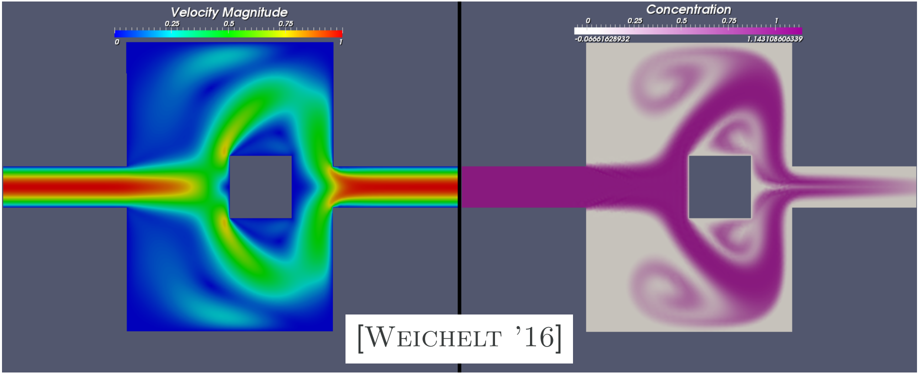
- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\mathbf{x}(t)$.

$$\boxed{\begin{aligned}\mathcal{M}\frac{d}{dt}\mathbf{x}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathcal{C}\mathbf{x}(t).\end{aligned}}$$

Convection-Diffusion Models: Concentration / Heat Equation

$$\begin{aligned}\partial_t\vartheta + v\cdot\nabla\vartheta - \alpha\Delta\vartheta &= 0,\\ \partial_t v + (v\cdot\nabla)v - \eta\Delta v + \nabla p &= 0,\\ \nabla\cdot v &= 0.\end{aligned}$$



Convection-Diffusion Models: Concentration / Heat Equation

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$$\partial_t v + (v \cdot \nabla) v - \eta \Delta v + \nabla p = 0,$$
$$\nabla \cdot v = 0.$$

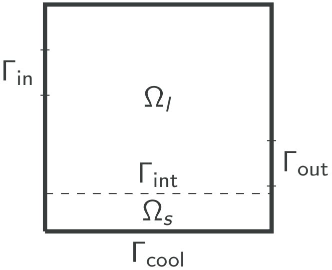
Stefan Problem

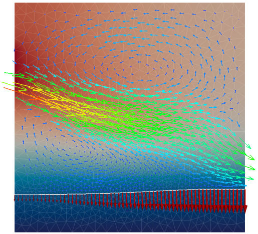
$$\boxed{\begin{aligned} \mathcal{M}\frac{d}{dt}\mathbf{x}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathcal{C}\mathbf{x}(t). \end{aligned}}$$

Phase Change Model: Stefan Problem

$$\begin{aligned} \partial_t T + v \cdot \nabla T - \alpha \Delta T &= 0, && \text{on } \Omega_s \cup \Omega_l, \\ [k_s(\nabla T)_s - k_l(\nabla T)_l] &= L \cdot V_{\text{int}}, && \text{on } \Gamma_{\text{int}}, \\ \partial_t v + (v \cdot \nabla) v - \eta \Delta v + \nabla p &= 0, && \text{on } \Omega_l, \\ \nabla \cdot v &= 0, && \text{on } \Omega_l \\ p \cdot \boldsymbol{n} - \eta \partial_{\boldsymbol{n}} v &= \mathbf{u} \cdot \boldsymbol{n}, && \text{on } \Gamma_{\text{in}}. \end{aligned}$$

Stefan Problem





Phase Change Model: Stefan Problem

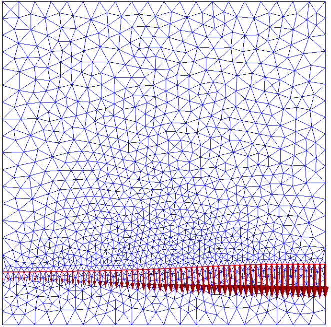
$$
\begin{aligned}
\partial_t T + v \cdot \nabla T - \alpha \Delta T &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
[k_s(\nabla T)_s - k_l(\nabla T)_l] &= L \cdot V_{\text{int}}, && \text{on } \Gamma_{\text{int}},\\
\partial_t v + (v \cdot \nabla) v - \eta \Delta v + \nabla p &= 0, && \text{on } \Omega_l,\\
\nabla \cdot v &= 0, && \text{on } \Omega_l\\
p \cdot \boldsymbol{n} - \eta \partial_{\boldsymbol{n}} v &= \mathbf{u} \cdot \boldsymbol{n}, && \text{on } \Gamma_{\text{in}}.
\end{aligned}
$$

Stefan Problem

Difficulties

- discontinuity of the temperature gradient along the interface
  ↳ resolve interface with mesh edges, mesh movement
- linearization of the system
  ↳ use reference trajectory

Stefan Problem



Mesh Movement via Harmonic Extension

$$
\begin{aligned}
\Delta V_{\text{mesh}} &= 0, && \text{on } \Omega_s \cup \Omega_l, \\
V_{\text{mesh}} - V_{\text{int}} \cdot \boldsymbol{n}_{\text{int}} &= 0, && \text{on } \Gamma_{\text{int}}.
\end{aligned}
$$

Stefan Problem

Stefan Problem with Mesh Movement

$$
\begin{aligned}
\partial_t T + (v - V_{\text{mesh}}) \cdot \nabla T - \alpha \Delta T &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
[k_s (\nabla T)_s - k_l (\nabla T)_l] &= L \cdot V_{\text{int}}, && \text{on } \Gamma_{\text{int}},\\
\Delta V_{\text{mesh}} &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
V_{\text{mesh}} - V_{\text{int}} \cdot \boldsymbol{n}_{\text{int}} &= 0, && \text{on } \Gamma_{\text{int}},\\
\partial_t v + ((v - V_{\text{mesh}}) \cdot \nabla) v - \eta \Delta v + \nabla p &= 0, && \text{on } \Omega_l,\\
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p \cdot \boldsymbol{n} - \eta \partial_{\boldsymbol{n}} v &= \mathbf{u} \cdot \boldsymbol{n}, && \text{on } \Gamma_{\text{in}}.
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\partial_t T + (v - V_{\text{mesh}}) \cdot \nabla T - \alpha \Delta T &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
[k_s(\nabla T)_s - k_l(\nabla T)_l] &= L \cdot \boxed{V_{\text{int}}}, && \text{on } \Gamma_{\text{int}},\\
\Delta V_{\text{mesh}} &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
V_{\text{mesh}} - \boxed{V_{\text{int}}} \cdot \boldsymbol{n}_{\text{int}} &= 0, && \text{on } \Gamma_{\text{int}},\\
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\Delta V_{\text{mesh}} &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
V_{\text{mesh}} - \left(\frac{1}{L}[k_s(\nabla T)_s - k_l(\nabla T)_l]\right) \cdot \boldsymbol{n}_{\text{int}} &= 0, && \text{on } \Gamma_{\text{int}},\\
\partial_t v + ((v - V_{\text{mesh}}) \cdot \nabla)v - \eta \Delta v + \nabla p &= 0, && \text{on } \Omega_l,\\
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Difficulties

- discontinuity of the temperature gradient along the interface
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- linearization of the system
  ↳ use reference trajectory

+

Stefan Problem

For linearization use known reference trajectories: $\tilde{T}$, $\tilde{V}_{\text{mesh}}$, $\tilde{v}$

Linearized Stefan Problem

$$
\begin{aligned}
\partial_t T + \boxed{(v - V_{\text{mesh}}) \cdot \nabla T} - \alpha \Delta T &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
\Delta V_{\text{mesh}} &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
V_{\text{mesh}} - \left(\frac{1}{L}[k_s(\nabla T)_s - k_l(\nabla T)_l]\right) \cdot \boldsymbol{n}_{\text{int}} &= 0, && \text{on } \Gamma_{\text{int}},\\
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\partial_t T + (v - V_{\text{mesh}}) \cdot \nabla \tilde{T} - \alpha \Delta T &= 0, && \text{on } \Omega_s \cup \Omega_l,\\
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\end{aligned}
$$

The Riccati Equations

Autonomous Case

$$\mathcal{M}\frac{d}{dt}\mathbf{x}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathcal{C}\mathbf{x}(t).$$

Algebraic Riccati equation:

$$0 = \mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{A} - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}.$$

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Non-autonomous Case

$$\mathcal{M}(t)\frac{d}{dt}\mathbf{x}(t) = \mathcal{A}(t)\mathbf{x}(t) + \mathcal{B}(t)\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathcal{C}(t)\mathbf{x}(t).$$

autonomous Differential Riccati equation (DRE):

$$-\mathcal{M}^{\mathsf{T}}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^{\mathsf{T}}\mathcal{C} + \mathcal{A}^{\mathsf{T}}\mathbf{X}\mathcal{M} + \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{A} - \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{B}\mathcal{B}^{\mathsf{T}}\mathbf{X}\mathcal{M}.$$

The Riccati Equations

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$$\mathcal{M}\frac{d}{dt}\mathbf{x}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t),$$
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Algebraic Riccati equation:

$$0 = \mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{A} - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}.$$

Non-autonomous Case

$$\mathcal{M}(t)\frac{d}{dt}\mathbf{x}(t) = \mathcal{A}(t)\mathbf{x}(t) + \mathcal{B}(t)\mathbf{u}(t),$$
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Non-autonomous Differential Riccati equation (DRE):

$$-\mathcal{M}^\mathsf{T}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^\mathsf{T}\mathcal{C} + (\dot{\mathcal{M}} + \mathcal{A})^\mathsf{T}\mathbf{X}\mathcal{M} + \mathcal{M}^\mathsf{T}\mathbf{X}(\dot{\mathcal{M}} + \mathcal{A}) - \mathcal{M}^\mathsf{T}\mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}\mathcal{M}.$$

The Riccati Equations

An autonomous generalized DRE

$$-\mathcal{M}^{\mathsf{T}}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^{\mathsf{T}}\mathcal{C} + \mathcal{A}^{\mathsf{T}}\mathbf{X}\mathcal{M} + \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{A} - \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{B}\mathcal{B}^{\mathsf{T}}\mathbf{X}\mathcal{M}$$

can be solved with, e.g.,

- BDF and Rosenbrock methods, [Mena, 2007], [Lang et al., 2015]
- splitting methods, [Stillfjord, 2015]
- peer methods. [Lang, 2017]

The Riccati Equations

An autonomous generalized DRE

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The Riccati Equations

An autonomous generalized DRE

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can be solved with, e.g.,

- BDF and Rosenbrock methods, [Mena, 2007], [Lang et al., 2015]
- splitting methods, [Stillfjord, 2015]
- peer methods. [Lang, 2017]

For non-autonomous generalized DREs

$$-\mathcal{M}^{\mathsf{T}}\dot{\mathbf{X}}\mathcal{M} = \mathcal{C}^{\mathsf{T}}\mathcal{C} + (\dot{\mathcal{M}} + \mathcal{A})^{\mathsf{T}}\mathbf{X}\mathcal{M} + \mathcal{M}^{\mathsf{T}}\mathbf{X}(\dot{\mathcal{M}} + \mathcal{A}) - \mathcal{M}^{\mathsf{T}}\mathbf{X}\mathcal{B}\mathcal{B}^{\mathsf{T}}\mathbf{X}\mathcal{M},$$

the methods above lead to large requirements of memory and computational time.

Time Invariant Subspace

Autonomous DRE: $$-\dot{\mathbf{X}} = \mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\mathbf{X} + \mathbf{X}\mathcal{A} - \mathbf{X}\mathcal{B}\mathcal{B}^\mathsf{T}\mathbf{X}.$$

Theorem [Anderson, Moore, Linear Optimal Control '71]

Let $(\mathcal{A}, \mathcal{B})$ be stabilizable, $(\mathcal{C}, \mathcal{A})$ be observable, and $\mathbf{X}(0) > 0$.
$\tilde{\mathbf{X}} > 0$ is the solution of

$$\mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathcal{A} - \tilde{\mathbf{X}}\mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}} = 0.$$

For $\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}}$, $\mathbf{P} > 0$ is the solution of

$$-\mathcal{B}\mathcal{B}^\mathsf{T} + \tilde{\mathcal{A}}^\mathsf{T}\mathbf{P} + \mathbf{P}\tilde{\mathcal{A}} = 0.$$

The DRE has the unique solution

$$\mathbf{X}(t) = \tilde{\mathbf{X}} + e^{t\tilde{\mathcal{A}}^\mathsf{T}} \left(e^{t\tilde{\mathcal{A}}}\mathbf{P}e^{t\tilde{\mathcal{A}}^\mathsf{T}} + (\mathbf{X}(0) - \tilde{\mathbf{X}})^{-1} - \mathbf{P} \right)^{-1} e^{t\tilde{\mathcal{A}}}$$

Time Invariant Subspace

$$\mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathcal{A} - \tilde{\mathbf{X}}\mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}} = 0,$$
$$\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}}, \qquad -\mathcal{B}\mathcal{B}^\mathsf{T} + \tilde{\mathcal{A}}^\mathsf{T}\mathbf{P} + \mathbf{P}\tilde{\mathcal{A}} = 0.$$

$$\mathbf{X}(t) = \tilde{\mathbf{X}} + e^{t\tilde{\mathcal{A}}^\mathsf{T}} \left(e^{t\tilde{\mathcal{A}}}\mathbf{P}e^{t\tilde{\mathcal{A}}^\mathsf{T}} + (\mathbf{X}(0) - \tilde{\mathbf{X}})^{-1} - \mathbf{P} \right)^{-1} e^{t\tilde{\mathcal{A}}}$$

Time Invariant Subspace

$$\mathcal{C}^{\mathsf{T}}\mathcal{C} + \mathcal{A}^{\mathsf{T}}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathcal{A} - \tilde{\mathbf{X}}\mathcal{B}\mathcal{B}^{\mathsf{T}}\tilde{\mathbf{X}} = 0,$$
$$\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{B}^{\mathsf{T}}\tilde{\mathbf{X}}, \qquad -\mathcal{B}\mathcal{B}^{\mathsf{T}} + \tilde{\mathcal{A}}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\tilde{\mathcal{A}} = 0.$$

$$\mathbf{X}(t) = \underset{\substack{\downarrow \\ \text{not time dependent}}}{\tilde{\mathbf{X}}} + e^{t\tilde{\mathcal{A}}^{\mathsf{T}}}\left(e^{t\tilde{\mathcal{A}}}\mathbf{P}e^{t\tilde{\mathcal{A}}^{\mathsf{T}}} + (\mathbf{X}(0) - \tilde{\mathbf{X}})^{-1} - \mathbf{P}\right)^{-1} e^{t\tilde{\mathcal{A}}}$$

$$
\begin{gathered}
\mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathcal{A} - \tilde{\mathbf{X}}\mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}} = 0,\\
\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}}, \qquad -\mathcal{B}\mathcal{B}^\mathsf{T} + \tilde{\mathcal{A}}^\mathsf{T}\mathbf{P} + \mathbf{P}\tilde{\mathcal{A}} = 0.
\end{gathered}
$$

$\rightarrow 0$, for $t \rightarrow \infty$

$$
\mathbf{X}(t) = \tilde{\mathbf{X}} + e^{t\tilde{\mathcal{A}}^\mathsf{T}} \left(e^{t\tilde{\mathcal{A}}}\mathbf{P}e^{t\tilde{\mathcal{A}}^\mathsf{T}} + (\mathbf{X}(0) - \tilde{\mathbf{X}})^{-1} - \mathbf{P} \right)^{-1} e^{t\tilde{\mathcal{A}}}
$$

not time dependent

Time Invariant Subspace

$$\mathcal{C}^\mathsf{T}\mathcal{C} + \mathcal{A}^\mathsf{T}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathcal{A} - \tilde{\mathbf{X}}\mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}} = 0,$$
$$\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{B}^\mathsf{T}\tilde{\mathbf{X}}, \qquad -\mathcal{B}\mathcal{B}^\mathsf{T} + \tilde{\mathcal{A}}^\mathsf{T}\mathbf{P} + \mathbf{P}\tilde{\mathcal{A}} = 0.$$

$\rightarrow 0$, for $t \rightarrow \infty$

$$\mathbf{X}(t) = \tilde{\mathbf{X}} + e^{t\tilde{\mathcal{A}}^\mathsf{T}} \left(e^{t\tilde{\mathcal{A}}}\mathbf{P}e^{t\tilde{\mathcal{A}}^\mathsf{T}} + (\mathbf{X}(0) - \tilde{\mathbf{X}})^{-1} - \mathbf{P} \right)^{-1} e^{t\tilde{\mathcal{A}}}$$

not time dependent

approximate with, e.g., extended Krylov subspace

$$\mathcal{K}_{2k-1}(\mathcal{A}^\mathsf{T}, (\mathcal{A}^\mathsf{T})^{-k+1}\mathcal{C}^\mathsf{T})$$
$$= \mathrm{range}([(\mathcal{A}^\mathsf{T})^{-k+1}\mathcal{C}^\mathsf{T}, \ldots, \mathcal{C}^\mathsf{T}, \mathcal{A}^\mathsf{T}\mathcal{C}^\mathsf{T}, \ldots, (\mathcal{A}^\mathsf{T})^{k-1}\mathcal{C}^\mathsf{T}])$$

Time Invariant Subspace

Let V_r be an orthonormal basis of $\mathcal{K}_{2k-1}(\mathcal{A}^\mathsf{T}, (\mathcal{A}^\mathsf{T})^{-k+1}\mathcal{C}^\mathsf{T})$.

$$\mathcal{A}_r := V_r^\mathsf{T} \mathcal{A} V_r, \quad \mathcal{B}_r := V_r^\mathsf{T} \mathcal{B}, \quad \mathcal{C}_r := \mathcal{C} V_r.$$

Projected DRE:

$$\begin{aligned}
-\dot{\mathbf{X}}_r &= \mathcal{C}_r^\mathsf{T}\mathcal{C}_r + \mathcal{A}_r^\mathsf{T}\mathbf{X}_r + \mathbf{X}_r\mathcal{A}_r - \mathbf{X}_r\mathcal{B}_r\mathcal{B}_r^\mathsf{T}\mathbf{X}_r,\\
\mathbf{X} &\approx V_r \mathbf{X}_r V_r^\mathsf{T}.
\end{aligned}$$

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Projected DRE:

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Can this approach be extended to non-autonomous DREs?

Conclusions

Presented

- Numerical solution of the Stefan Problem with mesh movement and finite elements.
- Steering of the interface position with open loop control and computation of reference trajectories.
- Linearization of the Stefan Problem around a given working trajectory.
- Extension of the linear-quadratic regulator approach for convection-diffusion(-reaction) and Navier-Stokes models to Stefan problems.

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Observations

- The Stefan problem results in a complicated Riccati equation.
- It has a differential-algebraic structure and is non-autonomous.

Outlook

- The goal is to apply the linear-quadratic regulator approach for the Stefan problem.
- The first step is a simplified model without Navier–Stokes equations.
- The existing solvers for solving the Riccati equation (like BDF, Rosenbrock method, and Newton-ADI) can be adjusted to the problem.
- Further investigation of projection-based approach.

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Thank you!

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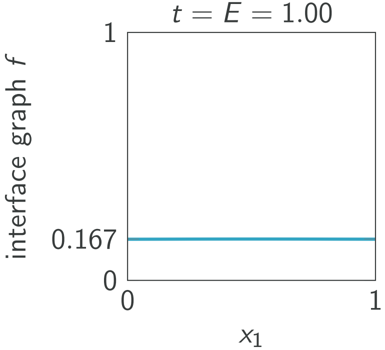
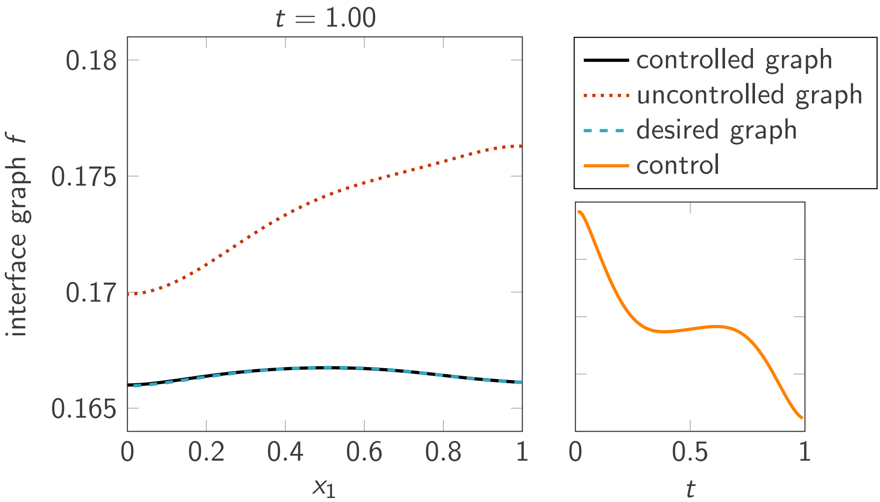
Open Loop Control

Cost Functional and Desired Interface Position for Open Loop Control

$$\mathcal{J}(\mathbf{x},\mathbf{u}) := ||f(E) - f_d(E)||^2 + \frac{\lambda}{2} \int_0^E ||\mathbf{u}(t)||^2 \, dt.$$



Open Loop Control



Riccati Equation

$$\begin{bmatrix} \\ \\ \mathcal{M}(t) \\ \\ \\ \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \\ \\ \mathbf{x}(t) \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \mathcal{A}(t) \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \mathbf{x}(t) \\ \\ \\ \end{bmatrix} + \mathcal{B}(t)\mathbf{u}(t),$$

$$\mathbf{y} = \begin{bmatrix} & \mathcal{C}(t) & \end{bmatrix} \begin{bmatrix} \\ \\ \mathbf{x}(t) \\ \\ \\ \end{bmatrix}.$$

Riccati Equation

$$\begin{bmatrix} \mathcal{M}_v & 0 & 0 & 0 \\ 0 & \mathcal{M}_T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ T \\ V_{\text{mesh}} \\ p \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 & J^T \\ A_2 & A_3 & A_4 & 0 \\ 0 & A_5 & A_6 & 0 \\ J & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ T \\ V_{\text{mesh}} \\ p \end{bmatrix} + \mathcal{B}(t)\mathbf{u}(t),$$

$$\mathbf{y} = \begin{bmatrix} 0 & \mathcal{C}_f & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ T \\ V_{\text{mesh}} \\ p \end{bmatrix}.$$

Riccati Equation

$$\begin{bmatrix} \mathcal{M}_v & 0 & 0 & 0 \\ 0 & \mathcal{M}_T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ T \\ V_{mesh} \\ p \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 & J^T \\ A_2 & A_3 & A_4 & 0 \\ 0 & A_5 & A_6 & 0 \\ J & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ T \\ V_{mesh} \\ p \end{bmatrix} + \mathcal{B}(t)\mathbf{u}(t),$$

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$$-\mathcal{M}^T \dot{\mathbf{X}} \mathcal{M} = \mathcal{C}^T \mathcal{C} + (\dot{\mathcal{M}} + \mathcal{A})^T \mathbf{X} \mathcal{M} + \mathcal{M}^T \mathbf{X} (\dot{\mathcal{M}} + \mathcal{A}) - \mathcal{M}^T \mathbf{X} \mathcal{B} \mathcal{B}^T \mathbf{X} \mathcal{M}.$$