# Feedback Stabilization of a Fluid Structure Model.

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Benasque 2017

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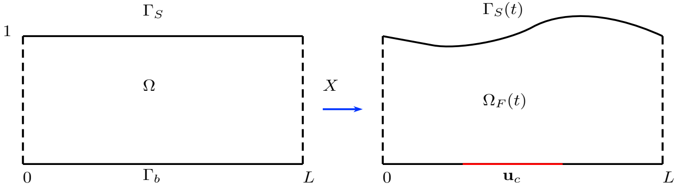
# Setting up the problem

The fluid-structure interaction problems appear naturally in aerodynamics, aeroacoustics and biology. They are of two types of fluid-structure interaction:

- A solid is immersed in a fluid : movement of fish or a submarine in a river or ocean, flow around an aircraft or formula 1 car.
- A fluid is contained in a domain and all or part of the boundary is deformable (blood flow in an artery or the respiratory movement mechanism)

# Fluid Structure Interaction




- $\Omega = (0, L) \times (0, 1) \rightsquigarrow$ The domain in the reference configuration.
- $\Gamma_s = (0, L) \times \{1\} \rightsquigarrow$ The elastic part of the fluid boundary in the reference configuration.
- For $t \geqslant 0$ and $x \in (0, L)$, $\eta(t, x)$ denotes the vertical displacement of the elastic structure.
- The domain occupied by the fluid at time $t > 0$ is

$$\Omega_F(t) = \Big\{(x, y) \mid x \in (0, L),\ 0 < y < 1 + \eta(t, x)\Big\}.$$

- $\Gamma_S(t) = \Big\{(x, y) \mid x \in (0, L),\ y = 1 + \eta(t, x)\Big\}.$

# Model Problem:

- Fluid equation : written in unknown moving domain $\Omega_F(t)$.
  $\Longrightarrow$ Navier-Stokes equations

$$\rho_f(\partial_t \mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u}) - \mathrm{div}\ \sigma(\mathbf{u}, p) = 0, \quad \mathrm{div}\,\mathbf{u} = 0$$
$$\sigma(\mathbf{u}, p) = \nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - pI$$

- Structure equation : written in reference configuration

$$\rho_s \partial_{tt}\eta + \alpha \partial_x^4 \eta - \beta \partial_x^2 \eta - \gamma \partial_x^2 \partial_t \eta = -\sqrt{1 + (\partial_x \eta)^2}\sigma(\mathbf{u}, p)\widetilde{\mathbf{n}} \cdot \mathbf{n}$$

- Coupling condition : The fluid sticks to the boundary of the structure and consequently the fluid velocity and the structure velocity are equal at the interface.

$$\mathbf{u}(t, x, 1 + \eta(t, x)) = \partial_t \eta(t, x)\mathbf{e_2} \quad \text{for } t \geqslant 0, x \in (0, L).$$

# Boundary and Initial conditions

Fluid Boundary Conditions :

- Enclosed Cavity : $\mathbf{u} = 0$ on $\partial\Omega \setminus \Gamma_S(t)$.
- Inflow and Outflow Boundary conditions :

$$\sigma(\mathbf{u}, p)\mathbf{n} = -p_{in\setminus out}\mathbf{n} \text{ on } \Gamma_{in\setminus out}$$

- Periodic boundary conditions.

Structure Boundary Conditions :

- Clamped \Periodic

Initial Conditions :

$$(\eta(0), \partial_t \eta(0)) = (\eta_1^0, \eta_2^0) \text{ in } \Gamma_s,$$
$$\mathbf{u}(0) = \mathbf{u}^0 \text{ in } \Omega_F(0).$$

# State of the Art

$\rho_s \partial_{tt}\eta + \alpha\partial_x^4\eta - \beta\partial_x^2\eta - \gamma\partial_x^2\partial_t\eta = \cdots$

- Existence of at least one weak solution
  - 3D/2D coupling with damped plate ($\alpha, \gamma > 0$) - Chambolle, Desjardins, Esteban, Grandmont, 05
  - 3D/2D coupling with plate ($\alpha > 0, \gamma = 0$) - Grandmont, 09 . Also true for 2D/1D coupling if $\alpha = \gamma = 0$ and $\beta > 0$.
  - 2D/1D coupling with $\alpha > 0$ and $\gamma \geqslant 0$ - Muha, Canić, 13
- Existence of unique strong solution
  - 2D/1D coupling with $\alpha \geqslant 0, \gamma > 0$ - local in time existence for small data - Beirao Da Veiga, 04
  - 2D/1D or 3D/2D coupling $\alpha \geqslant 0, \gamma > 0$ - local in time existence for any initial data - Julien Lequeurre, 11 and 13
  - 2D/1D coupling with $\alpha > 0, \gamma > 0$ - global in time strong solution - Grandmond and Hillairet, 16

# State of the Art

- Local Stabilization - Damped Plate equation
  - Control acts everywhere on the structure equation - J.-P. Raymond, 10
  - Boundary Control - Ndiaye, Matignon and Raymond - 14
  - Boundary control for weak solutions - Badra and Takahashi - 17

# Controlled System

The controlled system that we consider is

$$
\begin{aligned}
&\rho_f(\mathbf{u}_t + (\mathbf{u}.\nabla)\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{ in } (0,\infty)\times\Omega_F(t),\\
&\mathbf{u}(t,x,1+\eta(t,x)) = \partial_t\eta(t,x)\mathbf{e_2} \quad \text{for } t \geqslant 0, x \in (0,L)\\
&\mathbf{u} = L\mathbf{u}_c \quad \text{on } \Sigma^b_\infty, \quad \mathbf{u}(0) = \mathbf{u}^0 \text{ in } \Omega_{\eta(0)},\\
&\rho_s\partial_{tt}\eta - \Delta_s\eta + (-\Delta_s)^{\frac{1}{2}}\partial_t\eta = \mathcal{H}(\mathbf{u},p,\eta) \qquad (1.1)\\
&(\eta(0),\eta_t(0)) = (\eta_1^0,\eta_2^0) \text{ in } \Gamma_s,\\
&\mathbf{u}(\cdot,t),\ p(\cdot,t),\ \text{and } \eta(\cdot,t) \quad \text{are } L\text{-periodic with respect to } x,
\end{aligned}
$$

where

$$
\mathcal{H}(\mathbf{u},p,\eta) = p|_{\Gamma_S(t)} - \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)|_{\Gamma_S(t)}(-\eta_x\mathbf{e_1} + \mathbf{e_2})\cdot\mathbf{e_2}.
$$

The operator $L$ localizes the action of the control $\mathbf{u}_c$ in a relatively compact subset of $\Gamma_b$ and such that

$$
\int_{\Gamma_b} L\mathbf{u}_c\cdot\mathbf{n} = 0.
$$

# Goal

- We choose a control finite dimension of the form

$$\mathbf{u}_c(t,x) = \sum_{i=1}^{N_c} g_i(t)\mathbf{w}_i(x),$$

  where $(\mathbf{w}_i(x))_{1\leqslant i\leqslant N_c}$ is chosen suitably and and the control variable is $\mathbf{g} = (g_1, g_2, \cdots g_{N_c})$.

- To determine a control $\mathbf{g}$ in feedback form, able to stabilize, with any exponential decay rate $-\omega < 0$, the system (1.1) in some appropriate space locally around $(\mathbf{u}, p, \eta) = (\mathbf{0}, 0, 0)$.

# Some Remarks

- The incompressibility condition together with boundary conditions imply:

$$\int_0^L \partial_t \eta = 0.$$

- For simplicity we assume

$$\eta(t,\cdot) \in L^2_{\#,0}(\Gamma_S) = \left\{ f \in L^2_{\#}(\Gamma_S) \mid \int_0^L f = 0 \right\}.$$

- Consequently, for any regular solution $(\mathbf{u}, p, \eta)$, we have

$$\int_{\Gamma_s} \mathcal{H}(\mathbf{u}, p, \eta) = 0.$$

- We introduce the orthogonal projection $M_s \in \mathcal{L}(L^2_{\#}(\Gamma_s), L^2_{\#,0}(\Gamma_s))$, and rewrite the structure equation as

$$\rho_s \partial_{tt}\eta - \Delta_s \eta + (-\Delta_s)^{\frac{1}{2}} \partial_t \eta = M_s(\mathcal{H}(\mathbf{u}, p, \eta)).$$

# Method

- Rewrite the system in the reference configuration.
-

$$
\begin{aligned}
X(t,\cdot) : \Omega &\longmapsto \Omega_F(t) \\
(x,z) &\longmapsto (x,y) = (x,(1+\eta(t,x))z)\,.
\end{aligned}
$$

- Linearize the fluid structure interaction system.
- Find a feedback control stabilizes the linearized system.
- Stabilization of nonlinear system in reference configuration.
- Come back to the original configuration.

# System in the reference configuration:

We set

$$\widehat{\mathbf{u}}(t,x,z)=\mathbf{u}(t,X(t,x,z)),\quad \widehat{p}(t,x,z)=p(t,X(t,x,y)).$$

$$\begin{aligned}
&\rho_f(\partial_t\widehat{\mathbf{u}}+(\widehat{\mathbf{u}}.\nabla)\widehat{\mathbf{u}})-\nu\Delta\widehat{\mathbf{u}}+\nabla\widehat{p}=\widehat{F}(\widehat{\mathbf{u}},\widehat{p},\eta),\ \text{in } (0,\infty)\times\Omega_F(0)\\
&\mathrm{div}\ \widehat{\mathbf{u}}=\widehat{G}(\widehat{\mathbf{u}},\eta)\quad \text{in}\quad (0,\infty)\times\Omega_F(0),\\
&\widehat{\mathbf{u}}=\partial_t\eta(t,x)\mathbf{e}_2\quad \text{on } (0,\infty)\times\Gamma_S,\\
&\widehat{\mathbf{u}}=\sum_{i=1}^{N_c}g_i(t)L\mathbf{w}_i(x)\quad \text{on } (0,\infty)\times\Gamma_b, \qquad (1.2)\\
&\rho_S\eta_{tt}-\beta\Delta_s\eta+(-\Delta_s)^{\frac{1}{2}}\eta_t=M_s(\widehat{p}-2\nu\hat{u}_{2,z}+\widehat{H}(\widehat{\mathbf{u}},\eta))\quad \text{on } (0,\infty)\times\Gamma_s,\\
&+\ \text{initial conditions}
\end{aligned}$$

# Nonlinear Terms

$$\hat{F}(\hat{\mathbf{u}},\hat{p},\eta) = -\eta\hat{\mathbf{u}}_t + \left(z\eta_t + \nu z\left(\frac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right)\hat{\mathbf{u}}_z$$
$$+\nu\left(-2z\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \left(\frac{z^2\eta_x^2-\eta}{1+\eta}\right)\hat{\mathbf{u}}_{zz}\right)$$
$$+z(\eta_x\hat{p}_z - \eta\hat{p}_x)\mathbf{e}_1 - (1+\eta)\hat{u}_1\hat{\mathbf{u}}_x + (z\eta_x\hat{u}_1 - \hat{u}_2)\hat{\mathbf{u}}_z,$$

$$\hat{G}(\hat{\mathbf{u}},\eta) = -\eta\hat{u}_{1,x} + z\eta_x\hat{u}_{1,z} = \mathrm{div}\;\hat{\boldsymbol{\xi}} \;\text{ with }\; \hat{\boldsymbol{\xi}} = -\eta\hat{u}_1\mathbf{e}_1 + z\eta_x\hat{u}_1\mathbf{e}_2,$$

and

$$\hat{H}(\hat{\mathbf{u}},\eta) = \nu\left(\frac{\eta_x}{1+\eta}\hat{u}_{1,z} + \eta_x\hat{u}_{2,x} - \frac{\eta_x^2-2\eta}{1+\eta}\hat{u}_{2,z}\right).$$

# Linearized Model

Set $\eta_1 = \eta$ and $\eta_2 = \partial_t \eta$. The system linearized around $(\mathbf{0}, 0, 0, 0)$, is

$$
\begin{aligned}
&\rho_f \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = 0, \quad \mathrm{div}\ \mathbf{v} = 0 \quad \text{ in } (0,\infty) \times \Omega_F(0),\\
&\mathbf{v} = \eta_2 \mathbf{e}_2 \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = \mathbf{u}_c \text{ on } \Sigma^b_\infty,\\
&\mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 \text{ on } \Sigma^s_\infty,\\
&\rho_s \eta_{2,t} - \beta \Delta_s \eta_1 + (-\Delta_s)^{\frac{1}{2}} \eta_2 = \gamma_s (p - 2\nu v_{2,z}) \text{ on } \Sigma^s_\infty,\\
&\eta_1(0) = \eta_1^0, \quad \eta_2(0) = \eta_2^0 \text{ in } \Gamma_s.
\end{aligned}
\tag{1.3}
$$

- $\gamma_s p = M_s p|_{\Gamma_s}$.
- $v_1 = 0$ on $\Gamma_s$ and $\mathrm{div}\ \mathbf{v} = 0$ implies $v_{2,z} = 0$ on $\Gamma_S$.

# Damped Wave Equation

We consider

$$\eta_{1,t} = \eta_2 \text{ in } (0,\infty)\times(0,L),$$
$$\rho_s \eta_{2,t} - \beta\Delta_s \eta_1 + (-\Delta_s)^{\frac{1}{2}}\eta_2 = h \text{ in } (0,\infty)\times(0,L),$$

- $A_S$ generates an analytic semigroup on $H^1_\# \times L^2_\#$ with $D(A_S) = H^2_\# \times H^1_\#$.
- For $h \in L^2(L^2_\#)$ and regular initial conditions we have $\eta_2 \in L^2(H^1_\#)$

$$\rho_f \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = 0, \quad \mathrm{div}\, \mathbf{v} = 0 \quad \text{in } (0,\infty)\times\Omega_F(0),$$
$$\mathbf{v} = \eta_2 \mathbf{e}_2 \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = \mathbf{u}_c \text{ on } \Sigma^b_\infty,$$

- Consequently $\mathbf{v} \in L^2(H^{3/2}_\#)$ and $p \in L^2(H^{1/2}_\#)$.
- Qn : How to define $p|_{\Gamma_s}$?
- In case of damped plate equation $\eta_2 \in L^2(H^2_\#)$ and $p \in L^2(H^1_\#)$

# Stabilizability of Abstract Linear control System

We consider the following system

$$\frac{d}{dt}z(t) = Az(t) + Bu(t), \quad z(0) = z_0.$$

- $A : \mathcal{D}(A) \subset H \mapsto H$ generates a $C^0$ semigroup and $B \in \mathcal{L}(U, H)$.
- $A$ generates an analytic semigroup and has compact resolvent.
- $(A, B)$ is stabilazable if and only if

$$\ker(\lambda I - A^*) \cap \ker B^* = \{0\} \text{ for all } \mathrm{Re}\lambda \geqslant 0.$$

- There exists $K \in \mathcal{L}(H, U)$ such that $(A + BK)$ is stable.

# Rewriting as an evolution equation

$$\mathbf{L}^2_{\#}(\Omega) = \mathbf{V}^0_{\#,n}(\Omega) \oplus \nabla H^1_{\#}(\Omega),$$

where

$$\mathbf{V}^0_{\#,n}(\Omega) = \left\{ \mathbf{y} \in \mathbf{L}^2_{\#}(\Omega) \mid \operatorname{div} \mathbf{y} = 0, \ \mathbf{y}.\mathbf{n} = 0 \text{ on } \Gamma_b \cup \Gamma_s \right\}$$

and

$$\nabla H^1_{\#}(\Omega) = \left\{ \nabla f \mid f \in H^1_{\#}(\Omega) \right\}.$$

The orthogonal projection in $\mathbf{L}^2_{\#}(\Omega)$ onto $\mathbf{V}^0_{\#,n}$ is denoted by $P$. The fluid equation can be written as

$$\begin{aligned} &P\mathbf{v}' = A_0 P\mathbf{v} + (-A_0)PD_s\eta_2 + (-A_0)PD_b\mathbf{u}_c, \quad , \quad \mathbf{v}(0) = \mathbf{v}^0 \\ &(I-P)\mathbf{v}(t) = (I-P)D_s\eta_2(t) + (I-P)D_b\mathbf{u}_c. \end{aligned}$$

- $A_0 = P\Delta$, $D(A_0) = \{H^2 \cap V^0_n \mid \mathbf{v} = 0 \text{ on } \Gamma_b \cup \Gamma_S\}$.
- The above system is well posed in $D(A_0^*)'$.

# The pressure term

Taking divergence and normal trace of the fluid equation we obtain

$$\Delta p = 0 \text{ in } \Omega,$$
$$\frac{\partial p}{\partial \mathbf{n}} = \nu \Delta \mathbf{v} \cdot n - \rho_f (\mathbb{1}_{\Gamma_s} \partial_t \eta_2) - \mathbb{1}_{\Gamma_b} \partial_t \mathbf{u}_c \cdot n_b$$

Thus $p = N(\nu \Delta P \mathbf{v} \cdot \mathbf{n}) - \rho_f N_s(\partial_t \eta_2) - N_b(\partial_t \mathbf{u}_c \cdot \mathbf{n}_b)$.

The structure equation becomes

$$\eta_{1,t} = \eta_2$$
$$(\rho_s + \rho_f \gamma_s N_s)\eta_{2,t} - \beta \Delta_s \eta_1 + (-\Delta_s)^{\frac{1}{2}} \eta_2 = \gamma_s N(\nu \Delta P \mathbf{v} \cdot \mathbf{n}) - \gamma_s N_b(\partial_t \mathbf{u}_c \cdot \mathbf{n}_b)$$

- The "added mass" operator $K_s = (\rho_s + \rho_f \gamma_s N_s)$ is an automorphism on $L^2_{\#,0}$.

# Evolution equation

$$\frac{d}{dt}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_{FS}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}_1\mathbf{g} + \mathcal{B}_2\mathbf{g}_t,$$

$$(I-P)\mathbf{v} = (I-P)D_s\eta_2 + \sum_{i=1}^{N_c} g_i(I-P)D_b\mathbf{w}_i,$$

$$\mathcal{A}_{FS} = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ K_s^{-1}\gamma_s N(\nu\Delta(\cdot)\cdot\mathbf{n}) & K_s^{-1}\beta\Delta_s & -K_s^{-1}(-\Delta_s)^{\frac{1}{2}} \end{pmatrix}.$$

$$\mathcal{B}_1\mathbf{g} = \begin{pmatrix} \sum_{i=1}^{N_c} g_i(-A_0)PD_bL\mathbf{w}_i \\ 0 \\ 0 \end{pmatrix} \quad \mathcal{B}_2\mathbf{g}_t = \begin{pmatrix} 0 \\ 0 \\ -\sum_{i=1}^{N_c} g_{i,t}K_s^{-1}\gamma_s N_b(L\mathbf{w}_i\cdot\mathbf{n}) \end{pmatrix}.$$

We equip the space

$$\mathbf{Z} = \mathbf{V}^0_{\#,n}(\Omega) \times \mathcal{H}^1_{\#}(\Gamma_s) \times L^2_{\#,0}(\Gamma_s), \tag{1.4}$$

We now consider the unbounded operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}; \mathbf{Z}))$ in $\mathbf{Z}$ with

$$\mathcal{D}(\mathcal{A}; \mathbf{Z}) = \Big\{ (P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{V}^0_{\#,n} \times \mathcal{H}^2_{\#}(\Gamma_s) \times \mathcal{H}^1_{\#}(\Gamma_s) \\ \mid A_0(P\mathbf{v} - PD_s\eta_2) \in \mathbf{V}^0_{\#,n}(\Omega) \Big\}.$$

- Qn: How to make sense of the term $\gamma_s N(\nu \Delta(P\mathbf{v}) \cdot \mathbf{n})$ ? $P\mathbf{v} \notin H^2$.
- Perturbation argument and transposition method.

Rewrite $\mathcal{A}_{FS}$ in the form $\mathcal{A}_{FS} = \mathcal{A}_1 + \widetilde{B}$, with

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & K_s^{-1}\beta\Delta_s & -K_s^{-1}(-\Delta_s)^{\frac{1}{2}} \end{pmatrix}$$

and

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K_s^{-1}\gamma_s N(\Delta(\cdot)\cdot\mathbf{n}) & 0 & 0 \end{pmatrix}$$

- The operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A};\mathbf{Z}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$, and the resolvent of $\mathcal{A}_1$ is compact in $\mathbf{Z}$. We show
$$\|\lambda(\lambda I - \mathcal{A}_1)^{-1}\|_{\mathcal{L}(\mathbf{Z})} \leqslant C.$$
- To show $\gamma_s N(\nu\Delta(P\mathbf{v})\cdot\mathbf{n}) \in L^2_{\#,0}(\Gamma_S)$ for all $(P\mathbf{v},\eta_1,\eta_2) \in \mathcal{D}(\mathcal{A}_1)$.
- For all $\epsilon > 0$, there exists $C_\epsilon > 0$, such that
$$\|\gamma_s N(\Delta\mathbf{v}\cdot\mathbf{n})\|_{L^2_{\#,0}(\Gamma_s)} \leq \epsilon\|\mathcal{A}_1(P\mathbf{v},\eta_1,\eta_2)\|_{\mathbf{Z}} + C_\epsilon\|P\mathbf{v},\eta_1,\eta_2)\|_{\mathbf{Z}}$$

Rewrite $\mathcal{A}_{FS}$ in the form $\mathcal{A}_{FS} = \mathcal{A}_1 + \widetilde{B}$, with

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & K_s^{-1}\beta\Delta_s & -K_s^{-1}(-\Delta_s)^{\frac{1}{2}} \end{pmatrix}$$

and

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K_s^{-1}\gamma_s N(\Delta(\cdot)\cdot\mathbf{n}) & 0 & 0 \end{pmatrix}$$

- The operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A};\mathbf{Z}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$, and the resolvent of $\mathcal{A}_1$ is compact in $\mathbf{Z}$. We show
$$\|\lambda(\lambda I - \mathcal{A}_1)^{-1}\|_{\mathcal{L}(\mathbf{Z})} \leqslant C.$$
- To show $\gamma_s N(\nu\Delta(P\mathbf{v})\cdot\mathbf{n}) \in L^2_{\#,0}(\Gamma_S)$ for all $(P\mathbf{v}, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}_1)$.
- For all $\epsilon > 0$, there exists $C_\epsilon > 0$, such that

$$\|\gamma_s N(\Delta\mathbf{v}\cdot\mathbf{n})\|_{L^2_{\#,0}(\Gamma_s)} \leq \epsilon\|\mathcal{A}_1(P\mathbf{v}, \eta_1, \eta_2)\|_{\mathbf{Z}} + C_\epsilon\|P\mathbf{v}, \eta_1, \eta_2)\|_{\mathbf{Z}}$$

$(P\mathbf{v}, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}_1)$, iff $(\mathbf{v}, \eta_1, \eta_2)$

$$\begin{aligned}
&\lambda \mathbf{v} - \mathrm{div}\sigma(\mathbf{v}, q) = f, \quad \mathrm{div}\ \mathbf{v} = 0 \text{ in } \Omega, \\
&\mathbf{v} = \eta_2 \mathbf{e}_2 \text{ on } \Gamma_S, \quad \mathbf{v} = 0 \text{ on } \Gamma_b \\
&\lambda \eta_1 - \eta_2 = g \text{ in } \Gamma_s \\
&\lambda \eta_2 - \Delta_s \eta_1 + (-\Delta_s)^{\frac{1}{2}} \eta_2 = h \text{ in } \Gamma_s,
\end{aligned}$$

for some $\lambda > 0$ and $f \in V^0_{\#,n}(\Omega), g \in H^1(\Gamma_S)$ and $h \in L^2(\Gamma_S)$.

- $q|_\Gamma = (\sigma(\mathbf{v}, q)\mathbf{n} \cdot \mathbf{n})|_\Gamma \in L^2_{\#,0}(\Gamma_S)$.
- $\Delta \mathbf{v} \cdot \mathbf{n} = \frac{\partial q}{\partial \mathbf{n}}$ where

$$-\Delta q = 0 \text{ in } \Omega, \quad q|_\Gamma \in L^2(\Gamma).$$

- $\dfrac{\partial q}{\partial \mathbf{n}} \in H^{-1}(\Gamma)$, by transposition method.
- Hence, $\gamma_s N(\Delta \mathbf{v} \cdot \mathbf{n}) \in L^2_{\#,0}(\Gamma_s)$.

**Theorem**

*The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}; \mathbf{Z}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}$, and the resolvent of $\mathcal{A}$ is compact.*

## Extended System

- Time derivative of the control variable $\mathbf{g}$ appears. We want to obtain an evolution equation without the time derivative of the control variable.
- We choose $\mathbf{g}$ as a new state variable and by introducing $\mathbf{f} = \mathbf{g}_t - \Lambda \mathbf{g}$ as a new control variable, where $\Lambda$ is a diagonal matrix that we choose later on.
- The extended system:

$$\frac{d}{dt}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \mathcal{A}_e \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} + \mathcal{B}_e \mathbf{f}, \tag{1.5}$$

$(\mathcal{A}_e, \mathcal{D}(\mathcal{A}_e; \mathbf{Z}_e))$ is the unbounded operator on $\mathbf{Z}_e = \mathbf{Z} \times \mathbb{R}^{N_c}$ defined by

$$\mathcal{A}_e = \begin{pmatrix} \mathcal{A}_{FS} & \mathcal{B}_1 + \mathcal{B}_2 \Lambda \\ 0 & \Lambda \end{pmatrix},$$

The operator $\mathcal{B}_e \in \mathcal{L}(\mathbb{R}^{N_c}, \mathbf{Z}_e)$ is defined by

$$\mathcal{B}_e \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ -\sum_{i=1}^{N_c} f_i K_s^{-1} \gamma_s N_b(\mathbf{w}_i \cdot \mathbf{n}) \\ \mathbf{f} \end{pmatrix}.$$

- The operator $(\mathcal{A}_e, \mathcal{D}(\mathcal{A}_e; \mathbf{Z}_e))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{Z}_e$, and its resolvent is compact.
- The spectrum of $\mathcal{A}$, is a discrete spectrum, the eignevalues are isolated and of finite multiplicity.
- For simplicity let us assume that there is only one unstable eigenvalue which is real, say $\lambda$.

## Choice of $\Lambda$ and $(\mathbf{w})_i$

Let us assume $(\mathbf{v}, \eta_1, \eta_2, \mathbf{g}) \in \mathrm{Ker}(\lambda I - \mathcal{A}_e^*) \cap \mathrm{Ker}\mathcal{B}_e^*$.

$$(\lambda I - \mathcal{A}_{FS})(\mathbf{v}, \eta_1, \eta_2)^T = 0,$$

$$(\lambda I - \Lambda)\mathbf{g} = -\left(\int_{\Gamma_b} \sigma(\mathbf{v}, p) n \cdot L\mathbf{w}_i\right)_{1 \leqslant i \leqslant N_c}$$

$$\left(g_i - \int_{\Gamma_b} N_s(\eta_2)\mathbf{n} \cdot L\mathbf{w}_i\right) = 0.$$

- $\lambda \notin \sigma(\mathcal{A}) \implies (\mathbf{v}, p, \eta_1, \eta_2, \mathbf{g}) = 0.$
- $\sigma(\mathcal{A}) \cap \sigma(\Lambda) = \{0\}$. $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2)$.
- $\int_{\Gamma_b} [(\lambda - \alpha_1)(N_s \eta_2) n + \sigma(\mathbf{v}, p)\mathbf{n}] \cdot L\mathbf{w}_1 = 0$
- Unique continuation results (Fabre-Lebeau, Trigianni)

# Closed Loop Non homogeneous System

$$
\begin{aligned}
&\mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = F, \quad \mathrm{div}\ \mathbf{v} = G = \mathrm{div}\ \boldsymbol{\xi} \quad \text{ in } Q_\infty,\\
&\mathbf{v} = \eta_2\mathbf{e}_2 \text{ on } \Sigma_\infty^s, \quad \mathbf{v} = \sum_{i=1}^{N_c} g_i(t)\mathbf{w}_i(x) \text{ on } \Sigma_\infty^b,\\
&\mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 \text{ on } \Sigma_\infty^s,\\
&\eta_{2,t} - \beta\Delta_s\eta_1 + (-\Delta_s)^{\frac{1}{2}}\eta_2 = \gamma_s(p - 2\nu v_{2,z}) + H \text{ on } \Sigma_\infty^s,\\
&\eta_1(0) = \eta_1^0, \quad \eta_2(0) = \eta_2^0 \text{ in } \Gamma_s,\\
&\mathbf{g}_t - \Lambda\mathbf{g} = \mathcal{K}(\mathbf{v}(\cdot,t), \eta_1(\cdot,t), \eta_2(\cdot,t), \mathbf{g})^T, \quad \mathbf{g}(0) = 0.
\end{aligned}
$$

We define

$$
Y_{div} = \Big\{\boldsymbol{\xi} \in \mathbf{L}^2_\#(Q_\infty) | \mathrm{div}\ \boldsymbol{\xi} \in L^2(0,\infty; H^1_\#(\Omega)), \boldsymbol{\xi}_t \in \mathbf{L}^2_\#(Q_\infty),
\\ \langle \boldsymbol{\xi}(\cdot,t)\cdot\mathbf{n}, 1\rangle_{H_\#^{-1/2}(\Gamma), H_\#^{-1/2}(\Gamma)} = 0\Big\}.
$$

## Theorem

*Let* $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}^1_\#(\Omega) \cap \mathbf{V}^0_\#(\Omega) \times \mathcal{H}^2_\#(\Gamma_s) \times \mathcal{H}^1_\#(\Gamma_s)$ + *Compatibility condition. Let* $e^{\omega t} F \in L^2(0,\infty; \mathbf{L}^2_\#(\Omega))$, $e^{\omega t}\boldsymbol{\xi} \in Y_{div}$, *and* $e^{\omega t} H \in L^2(0,\infty; \mathcal{H}^{1/2}_\#(\Gamma_s))$. *Then above system has a unique solution*

$$e^{\omega t}\mathbf{v} \in L^2(0,\infty; H^2) \cap H^1(0,\infty; L^2)$$

$$e^{\omega t} p \in L^2(0,\infty; \mathcal{H}^1_\#(\Omega))$$

$$e^{\omega t}\eta_1 \in L^2(0,\infty; \mathcal{H}^{5/2}_\#(\Gamma_s)) \cap H^2(0,\infty; \mathcal{H}^{1/2}_\#(\Gamma_s)),$$

$$e^{\omega t}\eta_2 \in L^2(0,\infty; \mathcal{H}^{3/2}_\#(\Gamma_s)) \cap H^1(0,\infty; \mathcal{H}^{1/2}_\#(\Gamma_s)),$$

$$e^{\omega t}\mathbf{g} \in H^1(0,\infty; \mathbb{R}^{N_c}).$$

# Nonlinear closed loop system in the deformed configuration

$$
\begin{aligned}
&\rho_f \partial_t \mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} - \mathrm{div}\sigma(\mathbf{u}, p) = 0, \mathrm{div}\ \mathbf{u} = 0 \quad \text{in } (0,\infty) \times \Omega_F(t)\\
&\mathbf{u}(x, 1+\eta(x,t), t) = \eta_t(x,t)\mathbf{e}_2 \text{ for } (x,t) \in (0,L) \times (0,\infty)\\
&\mathbf{u} = \sum_{i=1}^{N_c} g_i(t)\mathbf{w}_i(x) \text{ on } \Sigma_\infty^b, \quad \mathbf{u}(0) = \mathbf{u}^0 \text{ in } \Omega_{\eta(0)},\\
&\rho_s \partial_{tt}\eta - \beta\Delta_s\eta + (-\Delta_s)^{\frac{1}{2}}\partial_t\eta = p + H(\mathbf{u}, \eta) \text{ on } \Sigma_\infty^s,\\
&(\eta(0), \eta_t(0)) = (\eta_1^0, \eta_2^0) \text{ in } \Gamma_s,\\
&\mathbf{g}_t - \Lambda\mathbf{g} + \omega\mathbf{g} = \mathcal{K}(\mathbf{u}\circ X^{-1}, \eta_1, \eta_2, \mathbf{g})^T, \quad \mathbf{g}(0) = 0.
\end{aligned}
$$

## Stabilization result:

### Theorem

*For all* $\omega > 0$, *there exists* $0 < \mu_0 < 1$ *a for all* $\mu \in (0, \mu_0)$ *and all initial data* $(\mathbf{u}_0, \eta_1^0, \eta_2^0) \in \mathbf{H}^1_\#(\Omega_F(0)) \times \mathcal{H}^2_\#(\Gamma_s) \times \mathcal{H}^1_\#(\Gamma_s)$, *satisfying*

$$\operatorname{div} \mathbf{u}^0 = 0 \quad \text{in } \Omega_F(0), \quad \mathbf{u}^0(x, 1 + \eta_1^0(x)) = \eta_2^0(x)\mathbf{e}_2 \quad \text{for } x \in (0, L)$$
$$1 + \eta_0(x) > 0 \text{and} \quad \mathbf{u}^0 = 0 \quad \text{on } \Gamma_b.$$

*and*

$$\|\mathbf{u}^0\|_{\mathbf{H}^1_\#(\Omega_F(0))} + \|\eta_1^0\|_{\mathcal{H}^2_\#(\Gamma_s)} + \|\eta_2^0\|_{\mathcal{H}^1_\#(\Gamma_s)} \leqslant \mu,$$

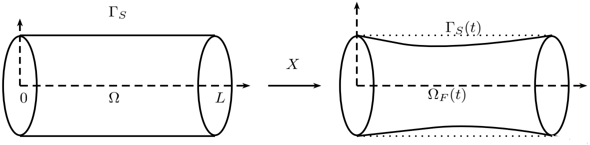
*there exists a control* $\mathbf{g} = (g_1, g_2, \cdots, g_{N_c}) \in H^1_0(0, \infty; \mathbb{R}^{N_c})$, *such that the solution to* (1.1) *satisfies*

$$\|e^{\omega t}\mathbf{u}(t, \cdot) \circ X^{-1}\|_{\mathbf{H}^1_\#(\Omega)} + \|e^{\omega t}\eta(t, \cdot)\|_{\mathcal{H}^2_\#(\Gamma_s)} + \|e^{\omega t}\partial_t\eta(t, \cdot)\|_{\mathcal{H}^1_\#(\Gamma_s)} \lesssim \mu.$$

*Moreover,* $1 + \eta > 0$ *for all* $t \in [0, \infty), x \in (0, L)$.

# Future Direction of Work

- 3D/3D coupling




- Koiter shell equation on $\Gamma_S$.
- Existence of weak solutions - Lengeler et.al, Buka? et. al, Muha-Canić, .....
- (with A. Roy and J.-P Raymond in preparation) : Local in time strong solution with inflow/outflow boundary conditions.
- Stabilization results....
- Existence of strong solution without damping.

**Thank you.**