# Extension of wavelets to networks

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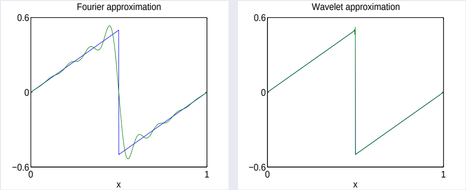
## Wavelets

A wavelet (due to Morlet and Grossmann in the early 1980s) is a mathematical function used to divide a given function or continuous-time signal into different scale components. They used the French word ondelette, meaning "small wave". Soon it was transferred to English by translating "onde" into "wave", giving "wavelet".

The study of wavelets has attained the present growth due to mathematical analysis of wavelets by Stromberg (Proceedings of Harmonic Analysis, Univ. of Chicago, pp. 475-494, 1981), Grossmann and Morlet (SIAM J. Math. Anal., pp. 723-736, vol. 15, 1984) and Meyer (Cambridge University Press, Cambridge, 1989).

## Example




Left: Initial signal Middle: Fourier approximation with only 17 terms Right: Wavelet approximation with only 17 terms

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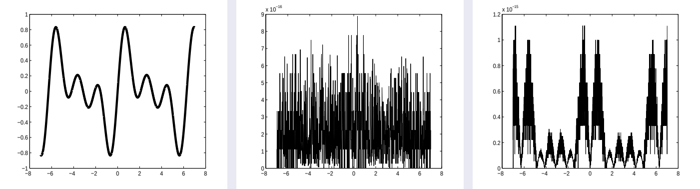


Left: Signal Middle: error in Fourier approximation Right: error in wavelet approximation

# Multiresolution Analysis of $L_2(\mathbb{R})$

## Definition

MRA is characterized by the following axioms

- $\{0\} \subset \cdots \subset \mathcal{V}^{-1} \subset \mathcal{V}^0 \subset \mathcal{V}^1 \cdots \subset L_2(\mathbb{R})$
- $\overline{\bigcup_{j\in\mathbb{Z}} \mathcal{V}^j} = L_2(\mathbb{R})$
- $\bigcap_{j\in\mathbb{Z}} \mathcal{V}^j = \{0\}$
- Invariance to dilations, *i.e* $f \in \mathcal{V}^j$ iff $f(2(.)) \in \mathcal{V}^{j+1}$
- Invariance to translations, *i.e* $\{\phi_k^0$ (scaling function) $= \phi(x-k) | k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{V}^0$

Now the sequence $\phi_k^j(x) = 2^{j/2}\phi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{V}^j$.

Since $\phi_0^0(x) = \phi(x) \in \mathcal{V}^0 \subset \mathcal{V}^1$, so

$$\phi(x) = \sum_{k=-\infty}^{\infty} h_k \phi_k^1(x).$$

This is called dialation equation and for Daubechies compactly supported scaling function only finitely many $h_k, k = 0, 1, \cdots D-1$ will be nonzero. Where $D$ is even positive integer called the wavelet genus and $h_0, h_1, \cdots, h_{D-1}$ are called low pass filter coefficients.

Define $\mathcal{W}^j = \{\psi_k^j \text{ (wavelet) } | k \in \mathbb{Z}\}$ to be the complement of $\mathcal{V}^j$ in $\mathcal{V}^{j+1}$, where $\mathcal{V}^{j+1} = \mathcal{V}^j + \mathcal{W}^j$.
Now the sequence $\psi(x) \in \mathcal{W}^0$ (which is called mother wavelet) such that $\psi_k^j(x) = 2^{j/2}\psi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{W}^j$. For, Daubechies compactly supported wavelet $\psi(x) \in \mathcal{W}^0 \subset \mathcal{V}^1$, therefore

$$\psi(x) = \sum_{k=0}^{D-1} g_k \phi_k^1(x).$$

This is called wavelet equation and $g_0, g_1, \cdots, g_{D-1}$ are called high pass filter coefficients connected by the relation $g_k = (-1)^k h_{D-1-k}, k = 0, 1, \cdots, D-1$.

## Why second generation wavelets?

- Second generation wavelet was developed by W. Swelden in 1996.
- Fast $O(\mathcal{N})$ transform.
- Dynamic grid adaption to the local irregularities of the solution.
  (This situation arises e.g. in the tracking of storms or fronts for the simulation of global atmospheric dynamics).

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## Why general manifolds?(e.g. Sphere)

- Application of adaptive wavelet collocation method (AWCM) to the problems of geodesy, climatology, meteorology (Representative examples include forecasting the moisture and cloud water fields in numerical weather prediction).
- Many PDEs arise from mean curvature flow, surface diffusion flow and Willmore flow on the sphere.

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## Why wavelets on manifolds? (e.g. spherical wavelets)

- Spherical triangular grids (quasi uniform triangulations) avoid the pole problem.
  Conventional grids–uniform longitude-latitude grid
- Problem- Singularity of coordinate system near the poles
- Solution-
  - Necessary to introduce auxiliary coordinate system.
  - Another solution is to avoid the introduction of the 'metric term' which are unbounded near the poles.
- To solve PDEs efficiently using adaptivity on general manifold by wavelet methods was an open problem till 2000.
- Past applications of wavelets to turbulence have been mainly restricted to flat geometries which severely limits for geophysical applications.

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# Wavelet multiresolution analysis of $L_2(S)$

**Definition**

MRA is characterized by the following axioms

- $V^j \subset \mathcal{V}^{j+1}$ (subspaces are nested).
- $\overline{\bigcup_{j=-\infty}^{j=\infty} \mathcal{V}^j} = L_2(S)$.
- Each $\mathcal{V}^j$ has a Riesz basis of scaling function $\{\phi_k^j | k \in \mathcal{K}^j\}$.

Define $\mathcal{W}^j = \{\psi_m^j(\textit{wavelets}) | m \in \mathcal{M}^j\}$ to be the complement of $\mathcal{V}^j$ in $\mathcal{V}^{j+1}$, where $\mathcal{V}^{j+1} = \mathcal{V}^j \oplus \mathcal{W}^j$.

$$\phi_k^j = \sum_{l \in \mathcal{K}^{j+1}} h_{k,l}^j \phi_l^{j+1} \text{ (dilation equation)}$$

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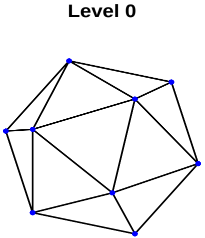
# Construction of spherical wavelets based on spherical triangular grids

The set of all vertices

$\mathcal{S}^j = \{p_k^j \in S : p_k^j = p_{2k}^{j+1} | k \in \mathcal{K}^j\}$ and $\mathcal{M}^j = \mathcal{K}^{j+1}/\mathcal{K}^j$.




Dyadic icosahedral triangulation of the sphere

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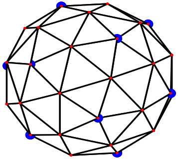
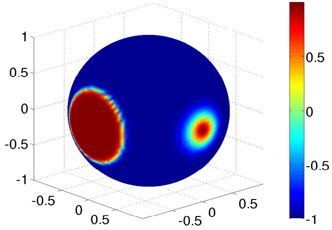
**Level 1**



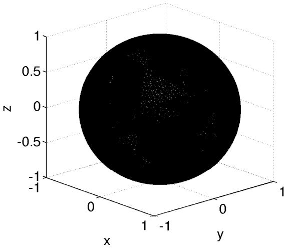
Dyadic icosahedral triangulation of the sphere

# Wavelet compression

$$u^J(p) = \sum_{k\in\mathcal{K}^0} c_k^{J_0}\phi_k^{J_0}(p) + \sum_{j=J_0}^{j=J-1}\sum_{m\in\mathcal{M}^j} d_m^j\psi_m^j(p)$$



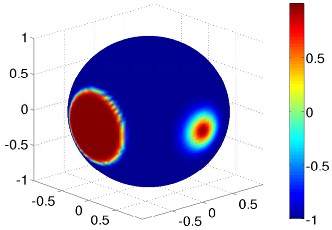

Test function




Wavelet locations $x_k^J$ without compression at $J=6$, $\#\mathcal{K}^6 = 40962$

# Wavelet compression

$$u^J_{\geq}(p) = \sum_{k\in\mathcal{K}^0} c_k^{J_0}\phi_k^{J_0}(p) + \sum_{j=J_0}^{j=J-1}\Big(\sum_{\substack{m\in\mathcal{M}^j\\ |d^j_m|\geq\epsilon}} d^j_m\psi^j_m(p) + \sum_{\substack{m\in\mathcal{M}^j\\ |d^j_m|\leq\epsilon}} d^j_m\psi^j_m(p)\Big)$$

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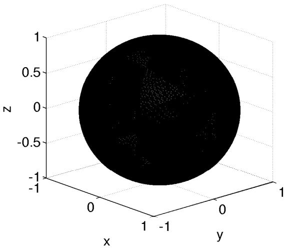


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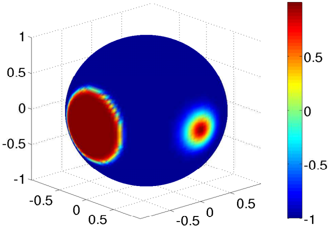


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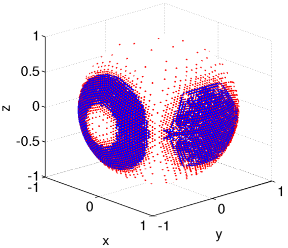


Wavelet locations $x_k^J$ st $J=6$, $\epsilon = 10^{-5}$, $N(\epsilon) = 8175$ and ratio $\frac{\#\mathcal{K}^6}{N(\epsilon)} \approx 5$

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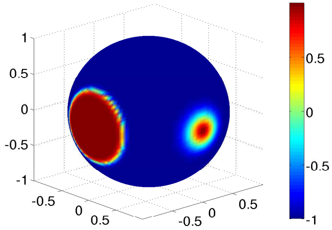


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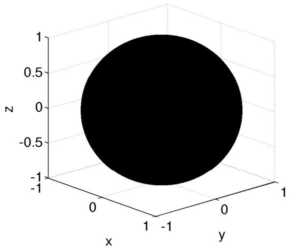
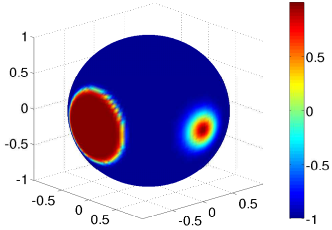



Wavelet locations $x_k^J$ without compression at $J = 7$, $\#\mathcal{K}^7 = 163842$

# Wavelet compression

$$u^J_{\geq}(p) = \sum_{k\in\mathcal{K}^0} c^{J_0}_k \phi^{J_0}_k(p) + \sum_{j=J_0}^{j=J-1} \sum_{\substack{m\in\mathcal{M}^j \\ |d^j_m|\geq\epsilon}} d^j_m \psi^j_m(p) + \text{Discarded term}$$

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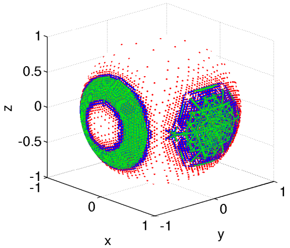


Wavelet locations $x^J_k$ at $J = 7$, $\epsilon = 10^{-5}$, $N(\epsilon) = 20353$ and ratio $\frac{\#\mathcal{K}^7}{N(\epsilon)} \approx 8$

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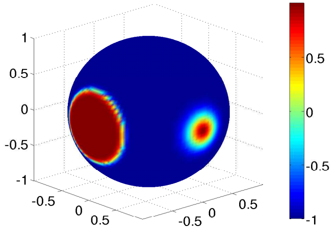


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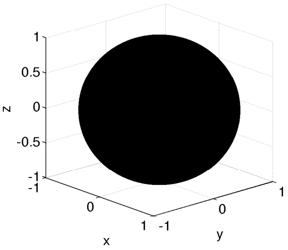


Wavelet locations $x_k^J$ without compression at $J = 8$, $\#\mathcal{K}^8 = 655362$

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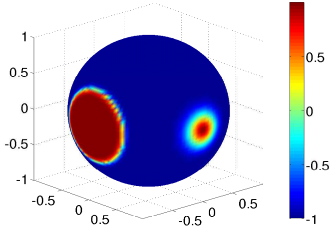


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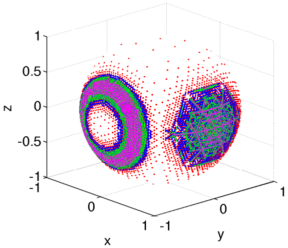


Wavelet locations $x^J_k$ at $J = 8$, $\epsilon = 10^{-5}$, $N(\epsilon) = 64231$ and ratio $\frac{\#\mathcal{K}^8}{N(\epsilon)} \approx 10$

- Diffusion wavelet was introduced by R. R. Coifman et al. in 2006 (Ref:, R. R. Coifman and M. Maggioni, Diffusion Wavelets, Appl. Comput. Harmon. Anal., Vol. 21, 2006).
- Given a manifold $X$ and a diffusion operator $T$ on $\mathcal{L}_2(X)$ such that high powers of $T$ have low numerical rank, an MRA can be constructed for $\mathcal{L}_2(X)$ which leads to the construction of diffusion wavelet.
- Classes of operators which can be used for the construction of diffusion wavelet include approximation of second order differential operators.
- A precision $\tau > 0$ is fixed and $X^J = \{x_1, x_2, \cdots, x_N\}$ is the discretization of $X$ using $N$ points.
- $\Phi^J = \{\delta_k\}_{k \in X^J}$, $\delta_k$ is an $N \times 1$ vector having 1 at $k^{th}$ place and 0 otherwise, then space $\mathcal{V}^J$ is defined as:

$$\mathcal{V}^J = \overline{\text{span}\{\Phi^J\}}.$$

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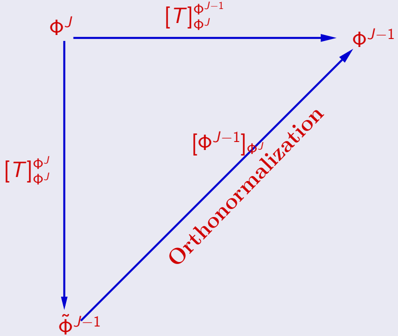
$[T]_{\Phi^J}^{\Phi^{J-1}}$ is used to indicate the matrix representing the linear operator $T$ with respect to basis $\Phi^J$ in the domain and $\Phi^{J-1}$ in the range.



- For the operator $T$ it is clear that

$$\cdots \subseteq \overline{\text{range}_\tau(T^{1+2+\cdots+2^j-1})} \subseteq \cdots \subseteq \overline{\text{range}_\tau(T)} \subseteq \overline{\text{span}\{\Phi^J\}} \subseteq \cdots \subseteq \mathcal{L}_2(X),$$

so that we have

$$\cdots \subseteq \mathcal{V}^{J-j} \subseteq \cdots \subseteq \mathcal{V}^{J-1} \subseteq \mathcal{V}^J \subseteq \cdots \subseteq \mathcal{L}_2(X),$$

which is analogous to the axiom (1) of MRA.

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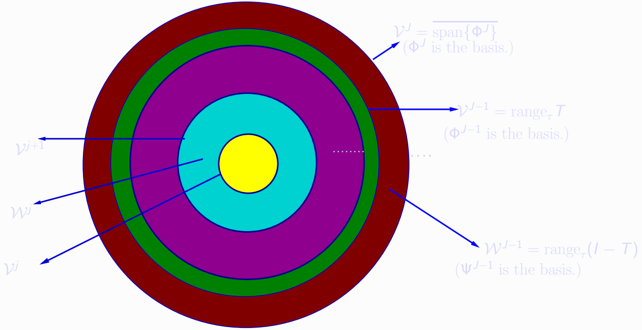
**Construction of the MRA can be visualized with following diagram**



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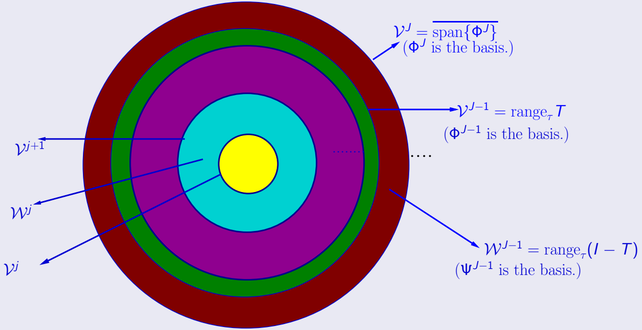

- Wavelet basis allowed to represent objects with singularities of complex structures with low number of degrees of freedom, a property that is particularly promising when thinking of an application to the numerical solutions of PDEs.
- Good approximation properties
- Efficient multiscale decompositions
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- Despite the vast literature available, the wavelet theory for numerical solution of PDEs on general manifold is still in nascent stage.
- We have developed adaptive meshfree diffusion wavelet method for solving PDEs on the sphere and this method can be easily generalised onto general manifolds.

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- The wavelet is constructed on an arbitrary finite weighted graph $G = \{V, E, \omega\}$.
- The adjacency matrix $A = \{a_{m,n}\}$ for the weighted graph $G$ has the entries

$$a_{m,n} = \begin{cases} \omega(e) & \text{if } e \in E \text{ connects vertices } m \text{ and } n \\ 0 & \text{otherwise.} \end{cases}$$

- For a weighted graph, the degree of each vertex $m$, written as $d(m)$, is defined as the sum of weights of all the edges incident to it, i.e., $d(m) = \sum_n a_{m,n}$. A matrix $D$ is defined as a diagonal matrix with $d(m)$ as the diagonal entries.
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- For any $f \in \mathbb{R}^N$ defined on the vertices of the graph $G$, its graph Fourier transform $\hat{f}$ is defined by

$$\hat{f}(l) = \langle \chi_l, f \rangle = \sum_{n=1}^{N} \chi_l^*(n) f(n),$$

where $\{\chi_l, \quad l = 0, 1, 2, \cdots, N-1\}$ are the eigenvectors corresponding to the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_{N-1}$ of the matrix $\mathcal{L}$ (like $e^{i\omega x}$ used in defining the Fourier transform of the function defined on $\mathbb{R}$ are eigenfunctions of the one–dimensional Laplacian operator $\frac{d^2}{dx^2}$).

- The inverse graph Fourier transform is

$$f(n) = \sum_{l=0}^{N-1} \hat{f}(l) \chi_l(n).$$

- To define spectral graph wavelet transform, initially a kernel function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is chosen satisfying $g(0) = 0$ and $\lim_{x\to\infty} g(x) = 0$ (we will refer $g$ as wavelet kernel).
- Then, for the given wavelet kernel $g$, the wavelet operator $T_g = g(\mathcal{L})$ acts on a given function $f$ by modulating each Fourier mode as

$$\widehat{T_g f}(l) = g(\lambda_l)\hat{f}(l),$$

which implies

$$(T_g f)(m) = \sum_{l=0}^{N-1} g(\lambda_l)\hat{f}(l)\chi_l(m).$$

The wavelet operator at scale $t$ is then defined by $T_g^t = g(t\mathcal{L})$.
- The spectral graph wavelets are defined as

$$\psi_n^t = T_g^t \delta_n.$$

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- The wavelet coefficients of a function $f$ are obtained by taking the inner product of that function with these wavelets, as

  $W_f(t,n) = \langle \psi_n^t, f \rangle$ **(Spectral graph wavelet transform (SGWT))**.

- Spectral graph scaling functions are determined by a single real valued function $h : \mathbb{R}^+ \to \mathbb{R}$ which satisfies $h(0) > 0$ and $\lim_{x \to \infty} h(x) = 0$ (we will refer $h$ as scaling function kernel).

$$\phi_n = h(\mathcal{L})\delta_n.$$

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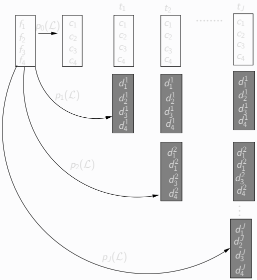
**Fast SGST and SGWT.**



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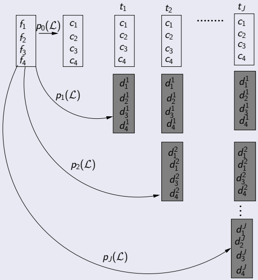
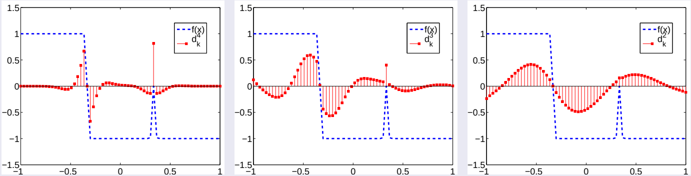
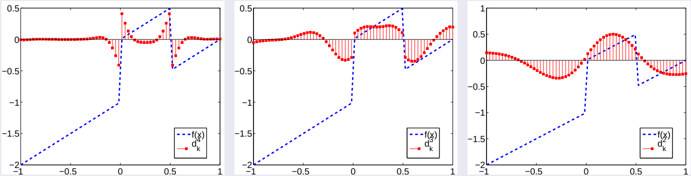


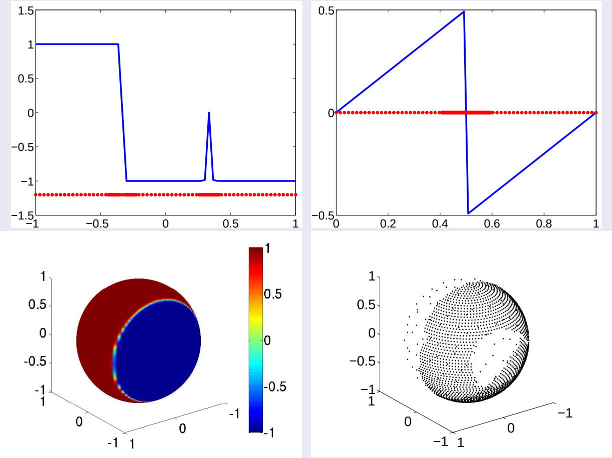
## $d_k^j$ for different value of $j$ for $f(x) = -\tanh\left(\frac{x+x_0}{2\nu}\right) + e^{-64^2(x-x_0)^2}, x_0 = \frac{1}{3}, \nu = 10^{-3}, (J = 4)$.




## $f(x) = x$, if $0 < x < 0.5$ and $x - 1$ otherwise.



## Functions and the corresponding adaptive node arrangements using SGW with $R = 0.1$ and $M = 4$.



## Turing patterns on the sphere

- In 1952, A. M. Turing settled the basis for explaining biological patterns using two interacting chemicals, which under certain conditions, can generate stable patterns from an initial near-homogeneity. This phenomenon has now been shown to occur in chemistry and biology.
- The Turing patterns are governed by a system of nonlinear reaction-diffusion equations. We solve the following system

$$\frac{\partial u}{\partial t} = D\delta\nabla^2 u + \alpha u(1 - r_1 v^2) + v(1 - r_2 u),$$
$$\frac{\partial u}{\partial t} = \delta\nabla^2 v + \beta v\left(1 + \frac{\alpha r_1}{\beta} uv\right) + u(\gamma + r_2 v).$$

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- At initial state, i.e., at $t = 0$ we consider $u = v = 0$, except on a narrow band.

- The stable patterns can be either stripes or spots, depending on the parameters $r_1$ and $r_2$. The parameter $r_1$ favours stripes while $r_2$ favours spots.
- We fix the parameters $D = 0.516, \delta = 0.0045, \alpha = 0.899, \beta = -0.91$ and $\gamma = -\alpha$.
- As **case** 1, we take $r_1 = 3.5, r_2 = 0$.
- As **case** 2, we take $r_1 = .002, r_2 = 0.2$.

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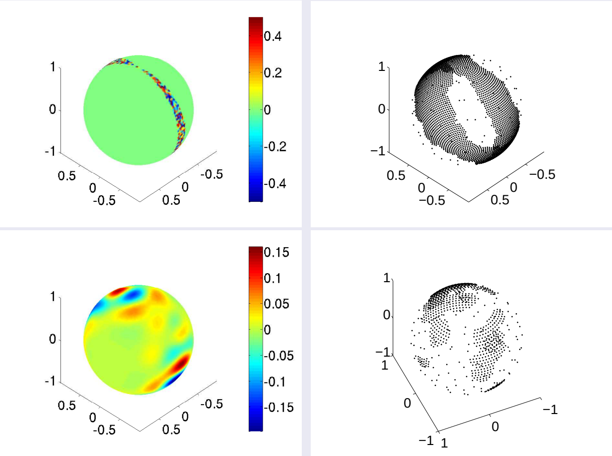
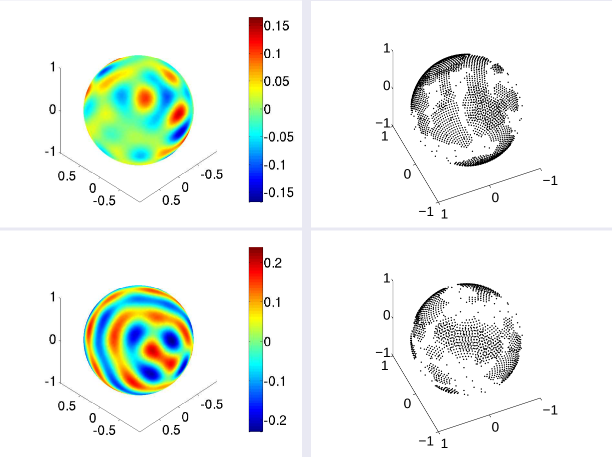
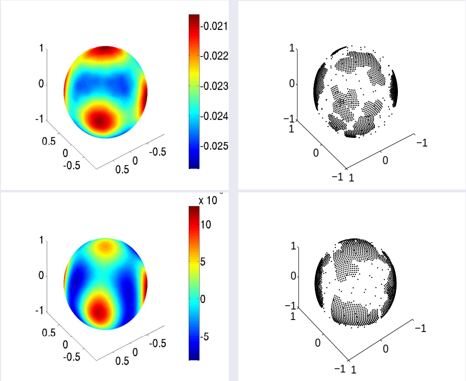
## Solution ($u$ component) and dynamically adapted node arrangement for case 1 at $t = 0$ and $t = 18$.



## Solution ($u$ component) and dynamically adapted node arrangement for case 1 at $t = 50$ and $t = 1000$.



## Solution ($u$ component) and dynamically adapted node arrangement for case 2 at $t = 250$ and $t = 1000$.




Ref: An adaptive meshfree spectral graph wavelet method for PDEs (with Kavita Goyal), **Applied Numerical Mathematics**, Vol. 113 (2017) pp. 168–185.

## Collaborator

- Günter Leugering (FAU Erlangen)
- Ankita Shukla (IIT Delhi)

- Mathematical solutions of PDEs on network–like structure modelling many real life phenomenon (i.e water wave propogation in open channels) exhibit singularities and these singularities are of physical relevance.
- To discover all the features of the solution we need a large set of node points but this will increase the computational as well as storage cost.
- In some cases the set required to capture all the features of the solution may exceed the practical limitations.
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- Adaptive mesh refinement (AMR) is the most typical technique used for adaptivity. In AMR the entire computational domain is covered with a coarse Cartesian grid. Individual grid cells are selected for refinement in moving from one step of the numerical algorithm to the next step based on a posteriori criterion.
- These methods are, no doubt, computationally efficient but the theory proving their advantages over their corresponding non adaptive counterparts is not well developed. In particular, the rate of convergence of the adaptive algorithm, which describes the trade off between the accuracy and complexity of the approximation is not clearly understood.
- One of the important property of wavelet is that the wavelet coefficients $d_k^j$ decrease rapidly for smooth functions. Moreover, if a function has a discontinuity in one of its derivatives then the wavelet coefficients will decrease slowly only near the point of discontinuity and maintain fast decay where the function is smooth.

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- This property of wavelet makes it suitable to detect where in the numerical solution of a PDEs on large networks the singularities are located.
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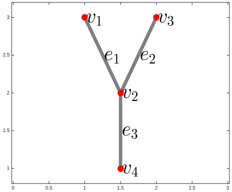
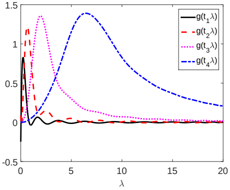



Topology of network

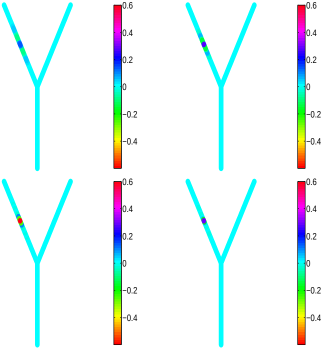
For our numerical computation, we have taken the following wavelet kernel

$$g(\lambda) = \begin{cases} \lambda^2 & \text{for } \lambda < 1 \\ -5 + 11\lambda - 6\lambda^2 + \lambda^3 & 1 \leq \lambda \leq 2 \\ 4\lambda^{-2} & \text{for } \lambda > 2. \end{cases}$$

The wavelet kernels are plotted in figure against different $\lambda$ using $t_1 = 8.80$, $t_2 = 2.57$, $t_3 = .75$ and $t_4 = .22$ for the star shaped network.

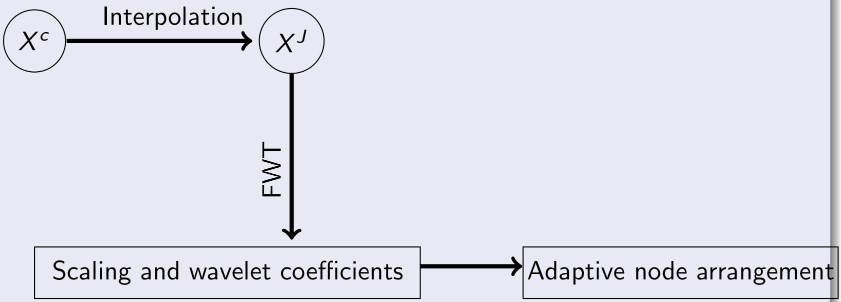



The wavelet scales $t_j$ are selected to be logarithmically equispaced between the minimum $(t_J)$ and maximum $(t_1)$ scales. All the properties (1, 2 and 3) of $g$ are satisfied as it could be observed from above figure.

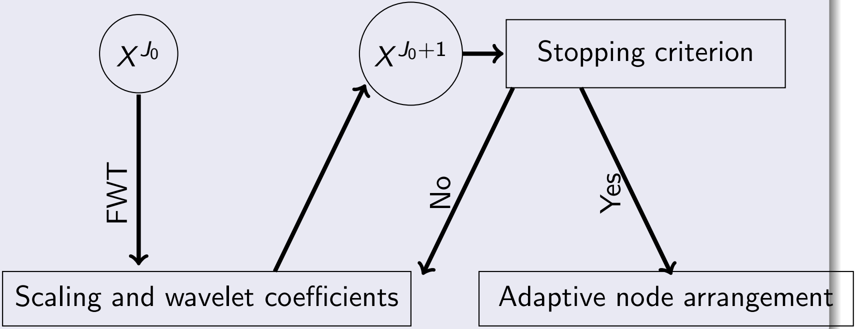



The wavelet functions for $J = 4$ at $t_1 = 8.80$, $t_2 = 2.57$, $t_3 = .75$ and $t_4 = .22$. The space localization is apparent from the figure as $t_j \to 0$.

## Standard adaptation technique



## Modified adaptation technique



- There is some sort of similarity in diffusion and spectral graph wavelet, for example, both require a diffusion operator for their construction.
- The largest difference between the two is that the diffusion wavelet is designed to be orthonormal whereas the spectral graph wavelet is not. The orthogonalising technique in the construction of diffusion wavelet complicates the construction procedure. On the other hand the approach used for the spectral graph wavelet is much simpler.
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**Thank you very much for attention!**