New applications of Russell type controls for the perturbation and the approximation of infinite dimensional vibrating systems

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Russell’s contribution to Infinite Dimensional Systems Theory

- Definition and duality of various observability and controllability concepts
- First controllability results for heat and wave equations
- Hautus test in infinite dimensions
- From the wave to the heat equations
- Weak observability implies polynomial stabilizability
- Backwards and forwards stabilizability implies controllability (Russell’s principle)
- ….

Russell's principle: backwards and forwards exponential stabilizability implies exact controllability

Let X, U and Y be Hilbert spaces, A a semigroup generator, $B \in \mathcal{L}(U, X)$ a control operator and $C \in \mathcal{L}(X, Y)$ an observation operator.

Theorem. (Russell, 1973) Assume that there exist $K_f,\ K_b \in \mathcal{L}(Y, U)$ such that $A + BK_f$ and $-A + BK_b$ generate exponentially stable semigroups. Then the pair (A, B) is exactly controllable in some time $\tau > 0$.

Proof. Let $\tau > 0$ be such that $\|\mathrm{e}^{\tau(-A+BK_b)}\,\mathrm{e}^{\tau(A+BK_f)}\| < 1$. Let $z_0 \in X$ and let $z(t) = w_f(t) - w_b(t)$, where

$$\dot{w}_f(t) = (A + BK_f)w_f(t), \qquad w_f(0) = \left(I - \mathrm{e}^{\tau(-A+BK_b)}\,\mathrm{e}^{\tau(A+BK_f)}\right)^{-1} z_0,$$
$$\dot{w}_b(t) = (A - BK_b)w(t), \qquad w_b(\tau) = w_f(\tau).$$

Setting $u(t) = BK_f w(t) - BK_b w_b(t)$ we have

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0, \qquad z(\tau) = 0.$$

Some remarks

- This principle has been originally seen as providing a qualitative property (controllability) from another qualitative property (stabilizability). It also holds for unbounded control operators.

- The converse principle “controllability implies stabilizability also holds for bounded control operators. It does not hold, in general, for unbounded ones.

- It looks unobvious how to use Russell type controls computationally. Indeed, they require, in principle, the knowledge of τ, K_f, K_b.

- In this presentation we use a different perspective
  - We assume that (A,B) is exactly controllable in time τ .
  - With some extra structure, we obtain “explicit” Russell controls in time τ .
  - We prove that these controls have remarkable properties.

Outline

- A version of Russell's principle for vibrating systems
  - From exact controllability to uniform stabilizability
  - A regularity result

- From distributed control to boundary control :
  - Neumann boundary control
  - Dirichlet boundary control
  - a general singular perturbation approach

- Approximation by finite dimensional systems

A version of Russell’s principle for vibrating systems

Standing assumptions

Let $\mathcal{H}$ be a Hilbert space and let $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ be a strictly positive operator. For $\alpha \geqslant 0$, $\mathcal{H}_\alpha$ is $\mathcal{D}(A_0^\alpha)$ with the graph norm. $\mathcal{H}_{-\alpha} = (\mathcal{H}_\alpha)'$.

$\mathcal{Y}$ is another Hilbert space and $C \in \mathcal{L}(\mathcal{H}_{\frac{1}{2}}, \mathcal{Y})$ is an observation operator.

Assumptions:

(H1) There exists $\gamma > 0$ such that $\|sC(s^2 I + A_0)^{-1} C^*\|_{\mathcal{L}(\mathcal{U})} \leqslant d_\gamma$ if $\operatorname{Re} s = \gamma$.

(H1') The system (A_0, C^*, C) is well-posed.

(H2) There exists $\tau > 0$, $K_\tau > 0$ s. t. $K_\tau^2 \int_0^\tau \|C\dot{p}(t)\|^2 \, \mathrm{d}t \geqslant \|f\|_{\frac{1}{2}}^2 + \|g\|^2$, holds for every $f \in \mathcal{H}_1$, $g \in \mathcal{H}_{\frac{1}{2}}$ and p satisfying

$$\ddot{p}(t) + A_0 p(t) = 0 \qquad , p(0) = f, \qquad \dot{p}(0) = g.$$

From exact controllability to uniform stabilizability

Theorem 1. (Ammari and M.T. 2002) Assume that A_0 and C satisfy the assumption *(H1)* and *(H2)* above. Then there exist $m_{\tau,\gamma} \in (0,1)$, depending only on γ, d_γ, τ and K_τ such that the estimate

$$\|\dot{w}(\tau)\|^2 + \|w(\tau)\|^2_{\frac{1}{2}} \leqslant m_{\tau,\gamma} \left(\|\dot{w}(0)\|^2 + \|w(0)\|^2_{\frac{1}{2}} \right),$$

holds for every solution $w \in C([0,\infty);\mathcal{H}_{\frac{1}{2}}) \cap C^1([0,\infty);\mathcal{H})$ of

$$\ddot{w}(t) + A_0 w(t) + C^* \frac{\mathrm{d}}{\mathrm{d}t} C w(t) = 0.$$

Remark. The converse of the above result is true (with the same τ) by Russell's principle.

Construction of Explicit Russell Controls

Assume that A_0, C satisfy the assumption *(H1)* and *(H2)* and set $B = C^*$.
Solve:

$$\ddot{w}(t) + A_0 w(t) + B\frac{\mathrm{d}}{\mathrm{d}t}\left(B^* w(t)\right) = 0, \qquad w(0) = \psi_0, \quad \dot{w}(0) = \psi_1,$$

$$\ddot{w}^b(t) + A_0 w^b(t) - B\frac{\mathrm{d}}{\mathrm{d}t}\left(B^* w^b(t)\right) = 0, \qquad w^b(\tau) = w(\tau), \;\; \dot{w}^b(\tau) = \dot{w}(\tau).$$

$$L_\tau \begin{bmatrix}\psi_0\\ \psi_1\end{bmatrix} = \begin{bmatrix} w^b(0)\\ \dot{w}^b(0)\end{bmatrix} \Longrightarrow^{Th.1} \left\|(I - L_\tau)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{1}{1 - m_{\tau,\gamma}}.$$

Setting $\begin{bmatrix}\psi_0\\ \psi_1\end{bmatrix} = (I - L_\tau)^{-1}\begin{bmatrix} f\\ g\end{bmatrix}$, $\;q(t) = w(t) - w_b(t)$ we obtain:

$$\ddot{q}(t) + A_0 q(t) + B\overbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(B^* w(t) + B^* w^b(t)\right)}^{u(t)} = 0, \qquad \begin{bmatrix} q(0)\\ \dot{q}(0)\end{bmatrix} = \begin{bmatrix} f\\ g\end{bmatrix} \qquad \begin{bmatrix} q(\tau)\\ \dot{q}(\tau)\end{bmatrix} = \begin{bmatrix} 0\\ 0\end{bmatrix}.$$

Two properties

- Russell type controls preserve the regularity of the initial data. For instance, if $B \in \mathcal{L}(U, H)$ and $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$ then
$$\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = (I - L_\tau)^{-1} \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1,$$
so that $u \in C^1([0, \tau]; U)$ and $Bu \in C([0, \tau]; H_{\frac{1}{2}})$.
(see Weiss and M.T, Ervedoza and Zuazua, Dehman and Lebeau for related regularity results).

- These controls are (in principle) easy to compute. Indeed,
$$(I - L_\tau)^{-1} = \sum_{n \geqslant 0} L_\tau^n,$$
and computing L_τ^n is just folving n times forward and n times backwards the closed loop system.

From distributed control to boundary control

Neumann boundary control (I)

Consider the problems $(P_\varepsilon)_{\varepsilon>0}$

$$\ddot{q}_\varepsilon(x,t) + \frac{\partial^2 q_\varepsilon}{\partial x^2}(x,t) + \frac{1}{\sqrt{\varepsilon}}\mathbb{1}_{[0,\varepsilon]}(x) u_\varepsilon(x,t) = 0, \qquad ((x,t)\in(0,\pi)\times[0,\tau])$$

$$\frac{\partial q_\varepsilon}{\partial x}(0,t) = q_\varepsilon(\pi,t) = 0 \qquad (t \geqslant 0)$$

$$q_\varepsilon(x,0) = f(x), \quad \dot{q}_\varepsilon(x,0) = g(x) \qquad (x\in(0,\pi)),$$

and the problem P_0

$$\ddot{q}_0(x,t) + \frac{\partial^2 q_0}{\partial x^2}(x,t) = 0, \quad (x,t)\in[0,\pi]\times[0,\tau]$$

$$\frac{\partial q_0}{\partial x}(0,t) = u_0(t), \qquad \frac{\partial q_0}{\partial x}(\pi,t) = 0, \quad (t \geqslant 0)$$

$$q_0(x,0) = f(x), \quad \dot{q}_0(x,0) = g(x), \quad x\in[0,\pi].$$

Neumann boundary control (II)

Theorem 2. (Hansen and M.T., 2017)

Given $\tau \geqslant 2\pi$, $f \in H^1(0,\pi)$, $g \in L^2[0,\pi]$ with $f(\pi) = 0$, there exists a family $(u_\varepsilon)_{\varepsilon\in(0,\pi)}$ in $L^2([0,\tau];L^2[0,\pi])$ and $u_0 \in L^2[0,\tau]$ such that

1. For each $\varepsilon \in (0,\pi)$ the solution of (P_ε) satisfies

$$q_\varepsilon(x,\tau) = 0, \quad \dot{q}_\varepsilon(x,\tau) = 0, \qquad x \in [0,\pi];$$

2. $\lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\varepsilon}} u_\varepsilon \mathbb{1}_{[0,\varepsilon]} = u_0\delta_0$ weakly in $L^2([0,\tau];H^{-1}(\mathbb{R}))$, where δ_0 stands for the Dirac mass concentrated at the origin;

3. $\lim_{\varepsilon\to 0^+} \left(\|q_\varepsilon - q_0\|_{C([0,\tau];H^1(0,\pi))} + \|\dot{q}_\varepsilon - \dot{q}_0\|_{C([0,\tau];L^2[0,\pi])}\right) = 0$, where q_0 is the solution of (P_0).

Dirichlet boundary control (I)

For $\varepsilon \in (0,\pi)$ we consider $\chi_\varepsilon \in \mathcal{D}(\mathbb{R})$ s.t. $\chi_\varepsilon > 0$, $\chi_\varepsilon = 1$ if $|x| \leqslant \frac{3\varepsilon}{4}$ and χ_ε vanishes if $|x| \geqslant \frac{5\varepsilon}{4}$. We introduce the control problems $(P_\varepsilon)_{\varepsilon\in(0,\pi)}$:

$$\begin{aligned}
\ddot{q}_\varepsilon(x,t) + \frac{\partial^2 q_\varepsilon}{\partial x^2}(x,t) + \frac{\chi_\varepsilon(x)}{\sqrt{\varepsilon}}\, u_\varepsilon(x,t) = 0 \qquad & ((x,t)\in(0,\pi)\times[0,\tau]),\\
q_\varepsilon(0,t) = q_\varepsilon(\pi,t) = 0 \qquad & (t \geqslant 0)\\
q_\varepsilon(x,0) = f(x), \quad \dot{q}_\varepsilon(x,0) = g(x) \qquad & (x\in(0,\pi)).
\end{aligned}$$

The problem (P_0) is

$$\begin{aligned}
\ddot{q}_0(x,t) + \frac{\partial^2 q_0}{\partial x^2}(x,t) = 0 \qquad & ((x,t)\in(0,\pi)\times[0,\tau])\\
q_0(0,t) = u_0(t), \qquad q_0(\pi,t) = 0 \qquad & (t \geqslant 0)\\
q_0(x,0) = f(x), \quad \dot{q}_0(x,0) = g(x) \qquad & (x\in(0,\pi)).
\end{aligned}$$

Dirichlet boundary control (II)

Theorem 3. (Hansen and M.T., 2017) Given $\tau \geqslant 2\pi$, $f \in L^2[0,\pi]$, $g \in H^{-1}(0,\pi)$, there exists a family $(u_\varepsilon)_{\varepsilon\in(0,\frac{4\pi}{5})}$ in $L^2([0,\tau];H^{-1}(0,\pi))$ and $u_0 \in L^2[0,\tau]$ such that

1. For each $\varepsilon \in (0,\frac{4\pi}{5})$ the solution of P_ε satisfies $q_\varepsilon(\cdot,\tau)=0, \;\; \dot{q}_\varepsilon(\cdot,\tau)=0.$
2. For every $\psi \in L^2([0,\tau];H^2(0,\pi)\cap H_0^1(0,\pi))$ we have
$$\lim_{\varepsilon\to 0^+}\int_0^\tau\int_0^{\frac{5\varepsilon}{4}}\frac{\chi_\varepsilon(x)}{\sqrt{\varepsilon}}u_\varepsilon(x,t)\overline{\psi}(\eta,t)\,\mathrm{d}t = \int_0^\tau u_0(t)\frac{\partial\overline{\psi}(0,t)}{\partial x}\,\mathrm{d}t;$$
3. $\lim_{\varepsilon\to 0^+}\left(\|q_\varepsilon - q_0\|_{C([0,\tau];L^2[0,\pi])} + \|\dot{q}_\varepsilon - \dot{q}_0\|_{C([0,\tau];H^{-1}(0,\pi))}\right) = 0$, where q_0 is the solution of (P_0).

Moreover, if $f \in H_0^1(0,\pi)$ and $g \in L^2[0,\pi]$ then the family $(u_\varepsilon)_{\varepsilon\in(0,\frac{4\pi}{5})}$ can be chosen in $L^2([0,\tau];L^2[0,\pi])$.

Bibliographical comments

- Only few works on this topic seem to available in the literature.

- For the Neumann control (even in 1D) we are not aware of results similar to those in Theorem 2. The related questions of internal pointwise controllability has been tackled in Fabre (1994) and Joly (2006).

- For the Dirichlet control, the results of Fabre and Puel (1992, 1993) provide convergence in a weak sense (in one or several space dimensions)

An abstract framework

Let $\varepsilon_0 > 0$ and let $(B_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)} \subset \mathcal{L}(\mathcal{U},\mathcal{H})$ be a family of input operators.
We consider the control problems (P_e)

$$\ddot{q}(t) + A_0 q(t) = B_\varepsilon u(t), \qquad q(0) = f, \qquad \dot{q}(0) = g.$$

We will also need to refer to the homogeneous system (HS)

$$\ddot{\varphi}(t) + A_0\varphi(t) = 0, \qquad \varphi(0) = f, \qquad \dot{\varphi}(0) = g.$$

Basic assumptions:

$$K_\tau^2 \int_0^\tau \|B_\varepsilon^*\dot{\varphi}(t)\|_{\mathcal{U}}^2 \mathrm{d}t \geqslant \|f\|_{\frac{1}{2}}^2 + \|g\|^2. \tag{1}$$

$$\|sB_\varepsilon^*(s^2I + A_0)^{-1}B_\varepsilon\|_{\mathcal{L}(\mathcal{U})} \leqslant d_\gamma \qquad (\varepsilon \in (0,\varepsilon_0),\ \mathrm{Re}\, s = \gamma), \tag{2}$$

$$\lim_{\varepsilon\to 0^+} B_\varepsilon B_\varepsilon^* f = B_0 B_0^* f \qquad \text{in } \mathcal{H}_{-\frac{1}{2}} \qquad (f \in \mathcal{H}_{\frac{1}{2}}), \tag{3}$$

for some $B_0 \in \mathcal{L}(\mathcal{U}_0, \mathcal{H}_{-\frac{1}{2}})$.

A singular perturbation result

Theorem 4. With the above notation and assumptions, for every $f \in \mathcal{H}_{\frac{1}{2}}$ and $g \in \mathcal{H}$ there exists a family of controls $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ in $L^2([0,\tau];\mathcal{U})$ such that

1. The corresponding family (q_ε) of solutions of (P_ε) satisfies

$$q_\varepsilon(\tau) = 0, \qquad \dot{q}_\varepsilon(\tau) = 0 \qquad\qquad (\varepsilon \in (0,\varepsilon_0)).$$

2. There exists $u_0 \in L^2([0,\tau];\mathcal{U}_0)$ s. t.

$$\ddot{q}_0(t) + A_0 q_0(t) = B_0 u_0, \quad q_0(0) = f,\ \dot{q}_0(0) = g, \quad q_0(\tau) = 0,\ \dot{q}_0(\tau) = 0$$

$$\lim_{\varepsilon\to 0^+} B_\varepsilon u_\varepsilon = B_0 u_0 \quad \text{weakly in} \quad L^2\left([0,\tau];\mathcal{H}_{-\frac{1}{2}}\right).$$

3. The corresponding controlled trajectories satisfy

$$\lim_{\varepsilon\to 0^+}\left(\|q_\varepsilon - q_0\|_{C([0,\tau];\mathcal{H}_{\frac{1}{2}})} + \|\dot{q}_\varepsilon - \dot{q}_0\|_{C([0,\tau];\mathcal{H})}\right) = 0.$$

Comments on the proofs

- The proof of Theorem 4 relies on Theorem 1.1 and on the Trotter-Kato theorem.

- Checking assumption (1) for 1D problems is relatively easy. In the case of Neuman control, for instance, a basic estimate is

$$\int_0^\varepsilon \phi_n^2(x)\,\mathrm{d}x \geqslant \frac{\varepsilon}{3} \qquad (\varepsilon \in (0,\pi),\ n \in \mathbb{N}).$$

  where $\phi_n(x) = \cos\left[\left(n - \frac{1}{2}\right)x\right]$.

- Checking (2) requires long calculations in 1D and limits the applicability to several space dimensions.

- Checking (3) is untrivial even in 1D. Generalizations of this approximation property to several space dimensions have been discussed by Joly (2006).

Approximation by finite dimensional systems

Standing assumptions

Let $\mathcal{H}$ be a Hilbert space and let $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ be a strictly positive operator. For $\alpha \geqslant 0$, $\mathcal{H}_\alpha$ is $\mathcal{D}(A_0^\alpha)$ with the graph norm. $\mathcal{H}_{-\alpha} = (\mathcal{H}_\alpha)'$.

$\mathcal{U}$ is another Hilbert space and $B_0 \in \mathcal{L}(U, \mathcal{H})$ is control operator. We assume that (A_0, B_0) is exatly controllabl in time τ.

Assume that there exists family $(V_h)_{h>0}$ of finite dimensional subspaces of $H_{\frac{1}{2}}$ and that there exist $\theta > 0$, $h^* > 0$, $C_0 > 0$ such that, for every $h \in (0, h^*)$,

$$\|\pi_h\varphi - \varphi\|_{\frac{1}{2}} \leq C_0\, h^\theta \|\varphi\|_1 \qquad (\varphi \in H_1),$$

$$\|\pi_h\varphi - \varphi\| \leq C_0\, h^\theta \|\varphi\|_{\frac{1}{2}} \qquad (\varphi \in H_{\frac{1}{2}}),$$

where π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h.

A numerical scheme (I)

- $$\ddot{w}_h^n(t) + A_{0h} w_h^n(t) + B_{0h} B_{0h}^* \dot{w}_h^n(t) = 0 \qquad (t \geqslant 0) \tag{1}$$

$$w_h^n(0) = \begin{cases} \pi_h q_0, & \text{if } n = 1 \\ w_{b,h}^{n-1}(0), & \text{if } 1 < n \leq N(h) \end{cases} \tag{2}$$

$$\dot{w}_h^n(0) = \begin{cases} \pi_h q_1, & \text{if } n = 1 \\ \dot{w}_{b,h}^{n-1}(0), & \text{if } 1 < n \leq N(h), \end{cases} \tag{3}$$

- A backward system

$$\ddot{w}_{b,h}^n(t) + A_{0h} w_{b,h}^n(t) - B_{0h} B_{0h}^* \dot{w}_{b,h}^n(t) = 0 \qquad (t \leqslant \tau) \tag{4}$$

$$w_{b,h}^n(\tau) = w_h^n(\tau), \qquad \dot{w}_{b,h}^n(\tau) = \dot{w}_h^n(\tau). \tag{5}$$

A numerical scheme (II)

- Compute $\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix}$ as follows

$$\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix} = \begin{bmatrix} \pi_h q_0 \\ \pi_h q_1 \end{bmatrix} + \sum_{n=1}^{N(h)} \begin{bmatrix} w_{b,h}^n(0) \\ \dot{w}_{b,h}^n(0) \end{bmatrix} = \sum_{n=1}^{N(h)+1} \begin{bmatrix} w_h^n(0) \\ \dot{w}_h^n(0) \end{bmatrix}. \qquad (1)$$

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$$u_h = B_{0h}^* \dot{w}_h + B_{0h}^* \dot{w}_{b,h}, \qquad (2)$$

where w_h and $w_{b,h}$ are the solution of

$$\ddot{w}_h(t) + A_{0h} w_h(t) + B_{0h} B_{0h}^* \dot{w}_h(t) = 0 \qquad (t \geqslant 0) \qquad (3)$$

$$w_h(0) = w_{0h}, \quad \dot{w}_h(0) = w_{1h}, \qquad (4)$$

$$\ddot{w}_{b,h}(t) + A_{0h} w_{b,h}(t) - B_{0h} B_{0h}^* \dot{w}_{b,h}(t) = 0 \qquad (t \leqslant \tau) \qquad (5)$$

$$w_{b,h}(\tau) = w_h(\tau), \quad \dot{w}_{b,h}(\tau) = \dot{w}_h(\tau). \qquad (6)$$

A numerical scheme (III)

Theorem. (Cindea, Micu and MT, 2011)

With the above notation and assumptions, assume furthermore that the system is exactly controllable in some time $\tau > 0$ and that $B_0 B_0^* \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_{\frac{1}{2}})$. Then there exists a constant $m_\tau > 0$ such that the family $(u_h)_{h>0}$ of $C([0,\tau]; U_h)$, with $N(h) = \left[\theta\, m_\tau \ln(h^{-1})\right]$, converges when $h \to 0$ to an exact control in time τ, denoted by u, for every $Q_0 = \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in \mathcal{H}_{\frac{3}{2}} \times \mathcal{H}_1$. Moreover, there exist constants $h^* > 0$ and $C := C_\tau$ such that we have

$$\|u - u_h\|_{C([0,\tau];\mathcal{U})} \leqslant C h^\theta \ln^2(h^{-1}) \|Q_0\|_{\mathcal{H}_{\frac{3}{2}} \times \mathcal{H}_1} \qquad (0 < h < h^*).$$

A numerical scheme (IV)

Main steps of the proof:

- Use standard numerical analysis to get that

$$\|(u-v_h)(t)\|_{\mathcal{U}} \leqslant \frac{C_2+tC_3}{1-\|L_\tau\|_{\mathcal{L}(\mathcal{H}_{\frac{3}{2}}\times\mathcal{H}_1)}} h^\theta \|Q_0\|_{\mathcal{H}_{\frac{3}{2}}\times\mathcal{H}_1} \qquad (t\in[0,\tau]),$$

where

$$v_h(t) = \mathcal{B}_h^* \mathbb{T}_{h,t} \Pi_h W_0 + \mathcal{B}_h^* \mathbb{S}_{h,\tau-t} \mathbb{T}_{h,\tau} \Pi_h W_0 \qquad (t\in[0,\tau]).$$

- Note that

$$\|(v_h-u_h)(t)\|_{\mathcal{U}} \leqslant C \left\| \sum_{n=0}^{\infty} L_\tau^n Q_0 - \sum_{n=0}^{N(h)} L_{h,\tau}^n \Pi_h Q_0 \right\|_X .$$