Dispersive Properties for the Linear Schrödinger Equation on Star Graphs. Application to NLS

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Outline

Preliminaries on quantum graphs

Definition

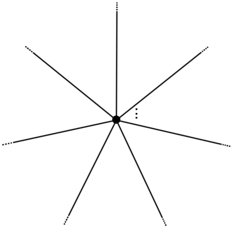
A discrete graph $\mathcal{G} := (V, E, \partial)$ consists of

- $V = \{v_i\}$ a finite or countably infinite set of *vertices*
- $E = \{e_j\}$ a set of adjacent edges at the vertices of length $l_j \in (0, \infty]$ ($l_j < \infty \leftrightarrow e_j internal$, $l_j = \infty \leftrightarrow e_j external$)
- $\partial : E \to V \times V$ an orientation map which associates to each internal e_j edge the pair $(\partial_- e_j, \partial_+ e_j)$ of its initial and terminal vertex, and to an external edge its initial vertex only.

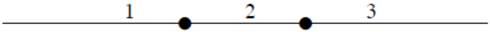
$$E \ni e_j \quad \longleftrightarrow \quad [0, l_j] =: \mathrm{I}_j$$

A metric graph is a *discrete graph* equipped with a natural metric: the distance of two points is the length of the shortest path in $\mathcal{G}$.

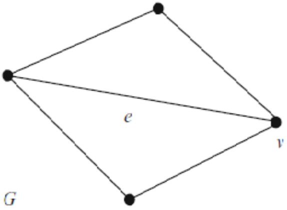
Examples of metric graphs



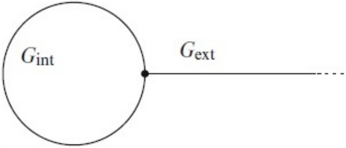
Star-shaped graph, $\mathcal{G}$



Graph with internal and external edges



Compact graph



Graph with cycle

Function spaces

In the sequel, we consider $\mathcal{G}$=star-graph, with $n \in \mathbb{N}^*$ edges of infinite length.

Given $1 \leq p \leq \infty$, one can define $L^p(\mathcal{G})$ as the set of functions $\boldsymbol{f} = (f_j)_{j=\overline{1,n}}$, whose components f_j are elements of $L^p(I_j)$

$$||f||^p_{L^p(\mathcal{G})} = \sum_{j=1}^{n} ||f_j||^p_{L^p(I_j)} \quad \textit{for} \quad 1 \leq p < \infty, \quad ||f||_{L^\infty(\mathcal{G})} = \sup_{1 \leq j \leq n} ||f_j||_{L^\infty(I_j)},$$

and the Sobolev space

$$H^2(\mathcal{G}) = \bigoplus_{j=1}^{n} H^2(I_j), \quad ||f||^2_{H^2(\mathcal{G})} = \sum_{j=1}^{n} ||f_j||^2_{H^2(I_j)},$$

where $I_j = [0, \infty)$.

The LSE on star-graph, $\mathcal{G}$

Let us consider the linear Schrödinger equation

$$\begin{cases} iu_t(t,x) + \Delta_x u(t,x) = 0, & t \neq 0, \quad x \in \mathcal{G} \\ u(0,x) = u_0(x), & \qquad\quad x \in \mathcal{G} \end{cases},$$

where u_t represents the time derivative of u, and the Laplacian $\Delta_x =: \Delta(A,B)$ with domain

$$D(\Delta(A,B)) = \{u \in H^2(\mathcal{G}) : A\underline{u} + B\underline{u'} = 0\},$$

acts as the second derivative along the edges.

A and B are $n \times n$ matrices which express the coupling condition at the common vertex, and $\underline{u} = (u_j(t,0))_{j=\overline{1,n}}$, $\underline{u'} = (u_j'(t,0_+))_{j=\overline{1,n}}$, respectively.

Self-adjoint Laplacians on $\mathcal{G}$

V. Kostrykin and R. Schrader, 2006

Let A, B be $n \times n$ matrices. Are equivalent:

(i) $\Delta(A, B)$ is self-adjoint

(ii) A and B satisfy:

$(H1)$ (A, B) has maximal rank;

$(H2)$ $AB^{\dagger}$ is self-adjoint.

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[1] *Laplacians on metric graphs: Eigenvalues, resolvents and semigroups*, Quantum Graphs and Their Applications, Vol. 415

Examples

Kirchhoff

$\psi_i(0) = \psi_j(0), \quad i,j = \overline{1,n}$ $\qquad \sum_{j=1}^{n} \psi_j'(0_+) = 0$

δ

$\psi_i(0) = \psi_j(0), \quad i,j = \overline{1,n}$ $\qquad \sum_{j=1}^{n} \psi_j'(0_+) = \alpha \psi_k(0), \ \alpha \in \mathbb{R}$

δ'

$\psi_i'(0_+) = \psi_j'(0_+), \quad i,j = \overline{1,n}$ $\qquad \sum_{j=1}^{n} \psi_j(0) = \beta \psi_k'(0_+), \ \beta \in \mathbb{R}$

Quantum graphs = metric graph + self-adjoint differential operator

Modeling phenoma such as:

- nonlinear electromagnetic pulse propagation in optical fibers
- electrical signal propagation in the nervous system, etc.

G. Berkolaiko, P. Kuchment, *Introduction to Quantum Graphs*, Mathematical Surveys and Monographs, Vol. 186, *2013*

The modelization depends on the real network, where each edge has a thickness, but usually idealizations (metric graphs) of these graphs are considered.
The convergence of these so-called *graph-like spaces* to metric graphs (with 0-thickness limit) is analyzed in:

O. Post, *Spectral Analysis on Graph-like Spaces*, Lecture Notes in Mathematics, Springer, Vol. 2039, *2012*

P.Exner, O. Post, *"A general approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds*, Communications in Mathematical Physics 322.1: 207-227, *2013*

Definition 1

A and B are real matrices if there exist $\tilde{A}, \tilde{B} \in M^{n\times n}(\mathbb{R})$ and an invertible matrix C such that $A = C\tilde{A}$ and $B = C\tilde{B}$.

Definition 2

The exponent pair (q, r) is $\sigma-$admissible if $q, r \geq 2$, $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}.$$

If equality holds, we say that (q, r) is sharp $\sigma-$admissible.

(q, r) sharp $\frac{1}{2}-$ admissible

$$r \in [2, \infty], \quad q = \frac{4r}{r-2}$$

Main results

Assume A and B to be real matrices satisfying $(H1)$ and $(H2)$.

Theorem 1. (Dispersive estimates)

$$(1.1)\quad \left\| e^{it\Delta} P_{ac}(-\Delta) \right\|_{L^p(\mathcal{G}) \to L^q(\mathcal{G})} \lesssim |t|^{\frac{1}{2}-\frac{1}{p}}, \quad t \neq 0,\ q \geq 2,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 2. (Strichartz estimates)

$$(2.1)\quad \left\| e^{it\Delta} P_{ac}(-\Delta) u_0 \right\|_{L_t^q(\mathbb{R}) L_x^r(\mathcal{G})} \lesssim \left\| u_0 \right\|_{L^2(\mathcal{G})},$$

$$(2.2)\quad \left\| \int_{\mathbb{R}} e^{is\Delta} P_{ac}(-\Delta) F(s,\cdot) ds \right\|_{L^2(\mathcal{G})} \lesssim \left\| F \right\|_{L_t^{q'}(\mathbb{R}) L_x^{r'}(\mathcal{G})},$$

$$(2.3)\quad \left\| \int_{s<t} e^{i(t-s)\Delta} P_{ac}(-\Delta) F(s,\cdot) ds \right\|_{L_t^q(\mathbb{R}) L_x^r(\mathcal{G})} \lesssim ||F||_{L_t^{\tilde{q}'}(\mathbb{R}) L_x^{\tilde{r}'}(\mathcal{G})},$$

where $P_{ac}(-\Delta)$ denotes the projection onto the absolutely continuous spectral subspace of $L^2(\mathcal{G})$ associated to $-\Delta(A,B)$ and $(q,r), (\tilde{q},\tilde{r})$ are sharp $\frac{1}{2}$−admissible exponent pairs.

Application

$(H3)$ $AB^\dagger$ does not have any positive eigenvalues.

Under $(H1) - (H3)$, we have the following result.

Theorem 3. Well-posedness of the NLS

For every $u_0 \in L^2(\mathcal{G})$, there exists a unique mild solution

$$u \in C(\mathbb{R}, L^2(\mathcal{G})) \bigcap_{(q,r)} L^q_{loc}(\mathbb{R}, L^r(\mathcal{G}))$$

of the nonlinear Schrödinger equation

$$\begin{cases} iu_t(t,x) + \Delta_x u(t,x) + \lambda |u|^{p-1} u = 0, & t \neq 0, \quad x \in \mathcal{G} \\ u(0,x) = u_0(x), & x \in \mathcal{G} \end{cases},$$

where $p \in (1,5)$, $\lambda \in \mathbb{R}$.

Moreover, the $L^2(\mathcal{G})$-norm of u is preserved along time

$$\|u(t)\|_{L^2(\mathcal{G})} = \|u_0\|_{L^2(\mathcal{G})}.$$

Previous work

On star-graphs

R. Adami, C. Cacciapuoti, D. Finco, D. Noja: Kirchhoff, δ, δ' *(2011)*

F.A. Mehmeti, K. Ammari, S. Nicaise: Kirchhoff, real valued potential *(2014)*

Other

V. Bănică and L. Ignat: trees with the last generation of edges infinite, Kirchhoff *(2010,2011)*

F.A. Mehmeti, K. Ammari, S. Nicaise: tadpole graph, Kirchhoff *(2017)*

The Resolvent

V. Kostryin, R. Schrader

The resolvent $R_k := (-\Delta(A,B) - k^2)^{-1}$, for $k^2 \in \mathbb{C} \setminus \sigma(-\Delta(A,B))$ is the integral operator with the $n \times n$ matrix-valued integral kernel $r(x,y,k)$, $\mathfrak{Im}\ k > 0$, admitting the representation

$$r(x,y,k) = r^{(0)}(x,y,k) + \frac{i}{2k}\phi(x,k)G(k,A,B)\phi(y,k),$$

where

- $[r^{(0)}(x,y,k)]_{j,j'} = \dfrac{i}{2k}\delta_{j,j'}e^{ik|x_j - y_{j'}|},\ \ x_j \in I_j, y_{j'} \in I_{j'}$
- $\phi(x,k) = diag\{e^{ikx_j}\}_j, \quad \phi(y,k) = diag\{e^{iky_j}\}_j$
- $G(k,A,B) = -(A+ikB)^{-1}(A-ikB)$.

[2]

[2] *Laplacians on metric graphs: Eigenvalues, resolvents and semigroups*, Quantum Graphs and Their Applications, Vol. 415, *2006*

Sketch of the proof of Thm 1 (Dispersive estimates)

Step 1. $\sigma_{ac}(-\Delta(A,B)) = [0,\infty)$.

Step 2. The solution via the resolvent.

Notation: $H := -\Delta(A,B)$, $P_{ac} := P_{ac}(H)$.

$$R_k u_0(x) = (H-k^2)^{-1} u_0(x) = \int_{\mathcal{G}} \tilde{r}(x,y,k) u_0(y) dy,$$

where

$$\tilde{r}(x,y,k) = \begin{cases} r(x,y,k), & \mathfrak{Im}\ k > 0 \\ \overline{r(x,y,\overline{k})}, & \mathfrak{Im}\ k < 0 \end{cases}.$$

We point out that $k\tilde{r}(x,y,k)$ is well-defined, analytic in k and bounded on a region containing the k-real axis.

Let $u_0 \in C_0^\infty(\mathcal{G})$.

Since $\sigma_{ac}(H) = [0, \infty)$, together with *the Spectral Theorem* [3] of representation of bounded functions of unbounded self-adjoints operators,

$$e^{-itH} P_{ac} u_0(x) = \lim_{\epsilon \to 0} \frac{1}{\pi i} \int_{\mathcal{G}} u_0(y) \int_{\mathbb{R}} e^{-(it+\epsilon)k^2} k \; \tilde{r}(x, y, k) \; dk \; dy.$$

and the limit is in $L^2(\mathcal{G})$.

[3] Dunford, N., Schwartz, J. T., Bade, W. G., Bartle, R. G., *Linear operators. Part II, Spectral theory: Self adjoint operators in Hilbert space*, 1963

Step 3. $L^1 \to L^\infty$ Dispersive estimates.

$$(e^{-itH}P_{ac}u_0(x))_j = \lim_{\epsilon\to 0} \frac{1}{\pi i}\sum_{j'=1}^{n}\int_{I_{j'}}\int_{\mathbb{R}} e^{-(it+\epsilon)k^2}(k\tilde{r}(x,y,k))_{j,j'}\, dk\, u_{0_{j'}}(y)dy,$$

with

$$k\tilde{r}(x,y,k) = \frac{i}{2}diag(e^{ik|x_j-y_j|})_j + \frac{i}{2}\, diag(e^{ikx_j})_j\ (G_{i,j}(k))_{i,j}\ diag(e^{iky_j})_j,$$

where $G_{i,j}(k)$, $i,j=\overline{1,n}$, are elements of

$$G(k,A,B) = -(A+ikB)^{-1}(A-ikB).$$

Lemma (Van Der Corput)

Let $\phi : [a,b] \longrightarrow \mathbb{R}$ be a $C^k -$ *function* and ψ a smooth complex valued function. Suppose that $|\phi^{(k)}| \geq 1$ for some $k \geq 1$ and all $x \in [a,b]$. If $k = 1$, assume in addition that ϕ' is monotone. Then, for every $\lambda \in \mathbb{R}$,

$$\Big| \int_a^b e^{i\lambda\phi(x)}\psi(x)dx \Big| \leq c_k \frac{1}{|\lambda|^{\frac{1}{k}}} \left(||\psi||_{L^\infty} + ||\psi'||_{L^1} \right),$$

where the constant c_k is independent of a, b and ϕ.

Thus,

$$||(e^{-itH}P_{ac}u_0)_j||_{L^\infty(I_j)} \lesssim \frac{1}{\sqrt{|t|}} \sum_{j'=1}^{n} \Big(\underbrace{||G_{j,j'}||_{L^\infty(\mathbb{R})}}_{<\infty} + \underbrace{||G'_{j,j'}||_{L^1(\mathbb{R})}}_{<\infty} \Big) ||u_{0_{j'}}||_{L^1(I_{j'})},$$

uniformly w.r.t. ϵ.

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[4] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals* ,Princeton University Press, *1993*

Step 4. $L^p \to L^q$ Dispersive estimates.

Since

$$||e^{-itH} P_{ac} u_0||_{L^2((\mathcal{G})} \leq ||u_0||_{L^2(\mathcal{G})}.$$

By interpolation, for all $q \geq 2$

$$||e^{-itH} P_{ac} u_0||_{L^q((\mathcal{G})} \lesssim |t|^{\frac{1}{2}-\frac{1}{p}} \; ||u_0||_{L^p(\mathcal{G})}, \quad t \neq 0,\; \frac{1}{p} + \frac{1}{q} = 1$$

.

which implies the estimate (1.1).

□

Sketch of the proof of Theorem 2. (Strichartz estimates)

Recall the genereal result of M. Keel and T. Tao under the following assumptions:

Let (X, dx) be a measure space, $\mathcal{H}$ a Hilbert space. Suppose that for each time $t \in \mathbb{R}$, we have an operator $U(t) : \mathcal{H} \to L^2(X)$ which obeys the following:

- For all t and $f \in \mathcal{H}$, we have
$$||U(t)f||_{L^2(X)} \lesssim ||f||_{\mathcal{H}}$$
- For all $t \neq s$ and all $g \in L^1(X)$,
$$||U(t)U^*(s)g||_{L^\infty(X)} \lesssim |t-s|^{-1/2}||g||_{L^1(X)}.$$

Then,

$$||U(t)f||_{L^q_t L^r_x} \lesssim ||f||_{\mathcal{H}}$$

$$\left\| \int_{\mathbb{R}} U^*(s)F(s,\cdot)ds \right\|_{\mathcal{H}} \lesssim ||F||_{L^{q'}_t L^{r'}_x}$$

$$\left\| \int_{s<t} U(t)U^*(s)F(s,\cdot)ds \right\|_{L^q_t L^r_x} \lesssim ||F||_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

hold for all q, r such that $r \in [2,\infty]$ and $q = \frac{4r}{r-2}$, and $\tilde{q}, \tilde{r}$, likewise.[a]

[a] *Endpoint Strichartz Estimates*, American Journal of Mathematics, Vol. 120, p. 955-980, *1998*

Taking $X = \mathcal{G}$, $\mathcal{H} = L^2(\mathcal{G})$ and $U(t) = e^{-itH}P_{ac}$ for $t \in \mathbb{R}$ in the previous theorem, from the properties of e^{-itH}, P_{ac} and the previously proved results, consequently follow the estimates $(2.1) - (2.3)$. □

Application

$(H3)$ $AB^\dagger$ does not have any positive eigenvalues.

Under $(H1)-(H3)$, we have the following result.

Theorem 3. WELL-POSEDNESS OF THE NLS

For every $u_0 \in L^2(\mathcal{G})$, there exists a unique mild solution

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where $p \in (1,5)$, $\lambda \in \mathbb{R}$.

Moreover, the $L^2(\mathcal{G})$-norm of u is preserved along time

$$\|u(t)\|_{L^2(\mathcal{G})} = \|u_0\|_{L^2(\mathcal{G})}.$$

Sketch of the proof of Theorem 3. (Well-posedness of the NLS)

Standard proof

Step 1. Mild formulation of the solution $\rightarrow$ fixed point problem.

Step 2. Strichartz estimates $\rightarrow$ contraction on a space-time ball $\rightarrow$ local in time existence and uniqueness

Step 3. Conservation of the L^2-norm $\rightarrow$ global solution.

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[5] F. Linares, G.Ponce, *Introduction to Nonlinear Dispersive Equations, 2009*

[6] T. Cazenave, *Semilinear Schrödinger Equations, 2003*

Thank you for your time and kind attention!