Entire spectrum of fully many-body localized systems using tensor networks

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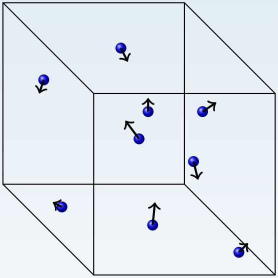
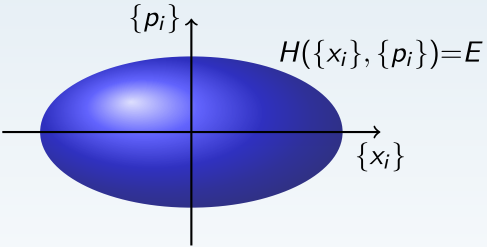
in collaboration with
Arijeet Pal, Steve Simon (Oxford)

T. B. Wahl, A. Pal, and S. H. Simon, arXiv:1609.01552

Ergodicity in classical systems



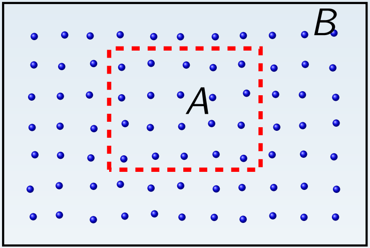
$$H(\{\mathbf{x}_i, \mathbf{p}_i\}) = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \sum_{i<j} V(\mathbf{x}_i - \mathbf{x}_j) = E$$



equiprobability

all $\{x_i\}, \{p_i\}$ consistent with $H(\{x_i\}, \{p_i\}) = E$ are reached with equal probabiltiy

$$E = E_A + E_B + E_{AB}$$



$$|A| \ll |B|$$

microcanonical $\rightarrow$ canonical

$$p(\{x_{Ai}\},\{p_{Ai}\})\mathrm{d}E_A = \exp\left(-\frac{E_A(\{x_{Ai}\},\{p_{Ai}\})}{T}\right)\mathrm{d}E_A$$

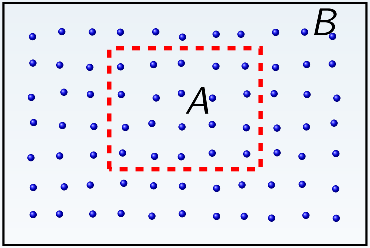
ergodicity

the system acts as its own heat bath

Ergodicity in quantum systems

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$
$$H = H_A + H_B + H_{AB}$$

$$\langle\psi_n|\hat{O}_A|\psi_n\rangle = \frac{\mathrm{tr}\left(\hat{O}_A \exp\left(-\beta_n H_A\right)\right)}{\mathrm{tr}\left(\exp\left(-\beta_n H_A\right)\right)}$$



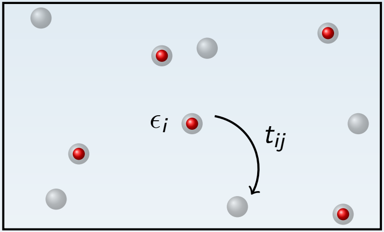
Eigenstate Thermalization Hypothesis (ETH)

J. M. Deutsch, Phys. Rev. A. **43**, 2046 (1991)
M. Srednicki, Phys. Rev. E **50**, 888 (1994)

Violation of ETH

Anderson impurity model:

$$H = \sum_i \epsilon_i a_i^\dagger a_i - \sum_{i<j} \left(t_{ij} a_i^\dagger a_j + \mathrm{h.c.} \right)$$



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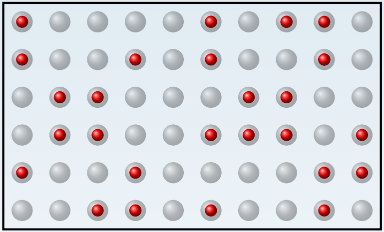


Simplifications:

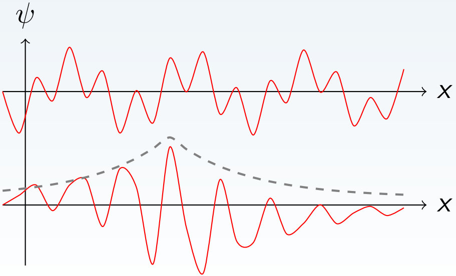
1. impurities lie on a lattice
2. $t_{\langle i,j \rangle} = t$, zero otherwise
3. $\epsilon_i \in [-W, W]$

delocalized

$W = 0$

localized

$W > 0$ in 1D and 2D,
$W > W_{\mathrm{3D}}$ in 3D



Switching on interactions

1D chain: $H = \sum_i \epsilon_i a_i^\dagger a_i - \sum_i t \left(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i \right) + \sum_i n_i n_{i+1}$

localization survives for $W > W_c$:

Many-body localization

D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126 (2006).
I. Gornyi, A. Mirlin, and D. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).
M. Schreiber, *et. al,* Science **349**, 842 (2015).

- no heat or electrical conductivity
- system retains memory of initial state
- topological protection at all energy scales (quantum memory)

Y. Bahri, R. Vosk, E. Altman, and A. Vishwanath, Nat. Comm. **6**, 7341 (2015).

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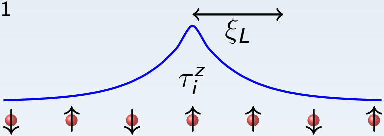
Many-body localization (MBL)

Disordered Heisenberg antiferromagnet: MBL for $W > W_c \approx 3.5$

$$H = \sum_{i=1}^{N} (J\mathbf{S}_i \cdot \mathbf{S}_{i+1} + h_i S_i^z), \qquad h_i \in [-W, W]$$



$\tau_i^z = U\sigma_i^z U^\dagger$
$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$

Many-body localization (MBL)

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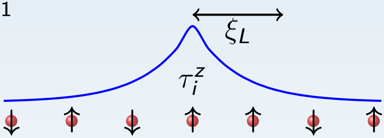

$\tau_i^z = U \sigma_i^z U^\dagger$
$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$

$$H|\psi_{i_1 \dots i_N}\rangle = E_{i_1 \dots i_N} |\psi_{i_1 \dots i_N}\rangle$$

$$\tau_1^z |\psi_{\uparrow i_2 \dots i_N}\rangle = |\psi_{\uparrow i_2 \dots i_N}\rangle$$
$$\tau_1^z |\psi_{\downarrow i_2 \dots i_N}\rangle = -|\psi_{\downarrow i_2 \dots i_N}\rangle \quad \text{etc.}$$

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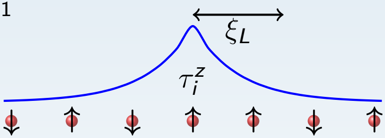

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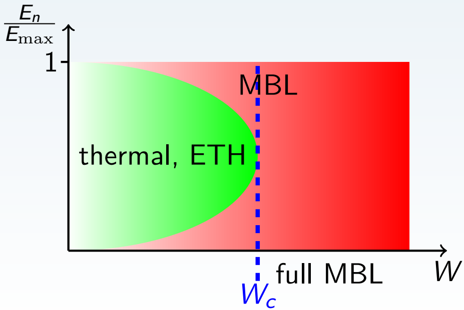

$|\psi_{i_1 \dots i_N}\rangle$:

$$\rho_A = \mathrm{tr}_{\overline{A}} \left(|\psi_{i_1 \dots i_N}\rangle\langle\psi_{i_1 \dots i_N}| \right)$$

entanglement entropy $S(\rho_A) = -\mathrm{tr}(\rho_A \ln(\rho_A)) < \mathrm{const.}$ as $|A| \to \infty$

M. Friesdorf, A. H. Werner, W. Brown, V. B. Scholz, and J. Eisert, Phys. Rev. Lett. **114**, 170505 (2015).

approximation by Matrix Product States

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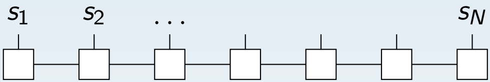
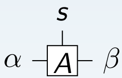
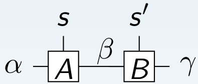
Matrix Product States

$|\psi_{i_1\ldots i_N}\rangle \approx$

$|\psi_{\mathrm{MPS}}\rangle = \sum_{s_1\ldots s_N=\uparrow,\downarrow}$ 

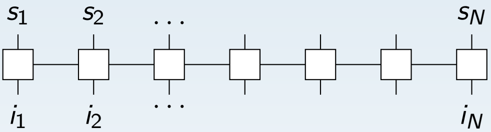
$|s_1,\ldots,s_N\rangle$

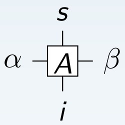
$A^s_{\alpha\beta} =$ 

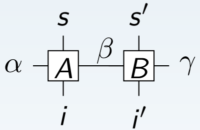
$\sum_\beta A^s_{\alpha\beta} B^{s'}_{\beta\gamma} =$ 

Matrix Product States

$$|\psi_{i_1 \dots i_N}\rangle \approx$$

$$|\tilde{\psi}_{i_1 \dots i_N}\rangle = \sum_{s_1 \dots s_N = \uparrow,\downarrow}$$  $$|s_1, \dots, s_N\rangle$$

$$A^{si}_{\alpha\beta} =$$ 

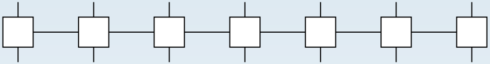
$$\sum_{\beta} A^{si}_{\alpha\beta} B^{s'i'}_{\beta\gamma} =$$ 

Matrix Product Operator

D. Pekker, B. K. Clark, Phys. Rev. B **95**, 035116 (2017).

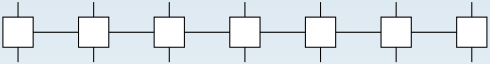
How to make $|\tilde{\psi}_{i_1 \dots i_N}\rangle$ orthonormal?

Spectral Tensor Networks

$\tilde{U} =$ 

we want: $\tilde{U}\tilde{U}^\dagger \overset{!}{=} \mathbb{1}$ and $\tilde{U} H \tilde{U}^\dagger \approx$ diagonal matrix

Spectral Tensor Networks

$\tilde{U} =$ 

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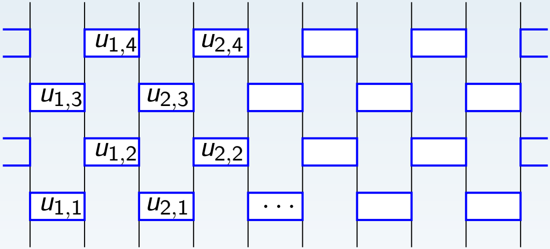
$\tilde{U} =$ 

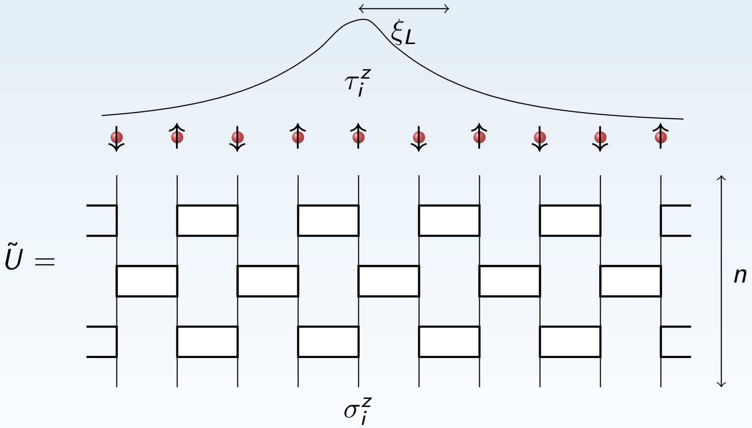
F. Pollmann, V. Khemani, J. I. Cirac, and S. L. Sondhi, Phys. Rev. B **94**, 041116 (2016)

Figure of merit: variance

$$\overline{\Delta H^2} = \frac{1}{2^N} \sum_{i_1 \ldots i_N = \downarrow, \uparrow} \left(\langle \tilde{\psi}_{i_1 \ldots i_N} | H^2 | \tilde{\psi}_{i_1 \ldots i_N} \rangle - \langle \tilde{\psi}_{i_1 \ldots i_N} | H | \tilde{\psi}_{i_1 \ldots i_N} \rangle^2 \right)$$

Scaling

approximate integrals of motion: $\tilde{\tau}_i^z = \tilde{U} \sigma_i^z \tilde{U}^\dagger$



$$t_{\mathrm{CPU}} \sim \exp(n)$$

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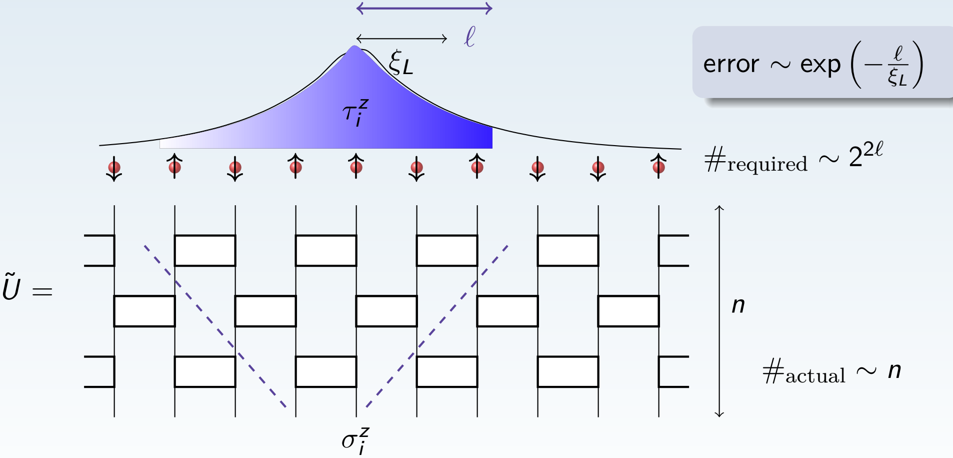
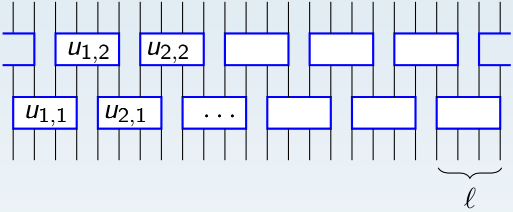


$$t_{\text{CPU}} \sim \exp(n) \sim \exp(\exp(\ell))$$

Alternative tensor network



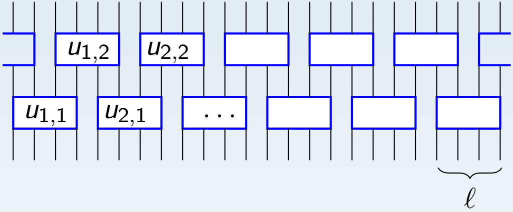
$\#_{\text{parameters}} \sim 2^{\ell}$

sufficient to capture τ_i^z correctly on length scale ℓ

error $\sim \exp\left(-\frac{\ell}{\xi_L}\right)$

$t_{\text{CPU}} \sim \exp(\ell) \;\Rightarrow$ error $\sim \frac{1}{\text{poly}(t_{\text{CPU}})}$

Alternative tensor network



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Variance as a figure of merit

$$t_{\text{CPU}} \sim N \frac{2^{7\ell}}{\ell}$$

Our figure of merit

Recap

$$\tau_i^z = U\sigma_i^z U^\dagger$$
$$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$$
$$\tilde{\tau}_i^z = \tilde{U}\sigma_i^z \tilde{U}^\dagger$$

Our figure of merit

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$\tau_i^z = U\sigma_i^z U^\dagger$

$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$

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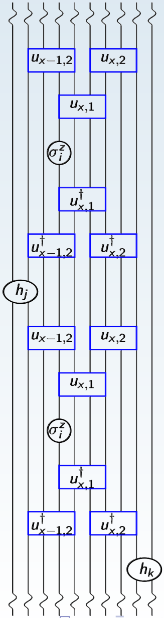
Define figure of merit as:

$$f(\{u_{x,y}\}) = \frac{1}{2^N}\sum_{i=1}^{N} \mathrm{tr}\left([H, \tilde{\tau}_i^z][H, \tilde{\tau}_i^z]^\dagger\right)$$

$$= \mathrm{const.} - \sum_{x=1}^{N/\ell} f_x(u_{x,1}, u_{x-1,2}, u_{x,2})$$

scaling: $t_{\mathrm{CPU}} \sim N 2^{3\ell}\ell^2$



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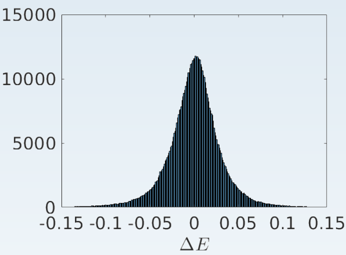
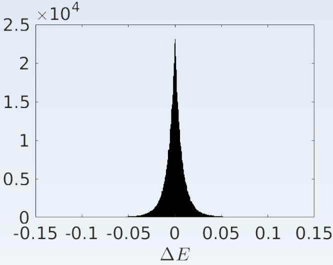
Comparison to exact diagonalization

- antiferromagnetic Heisenberg chain: H is real and S^z-symmetric
- $h_i \in [-W, W]$, $W_c \approx 3.5$
- $N = 16, \;\; W = 6$

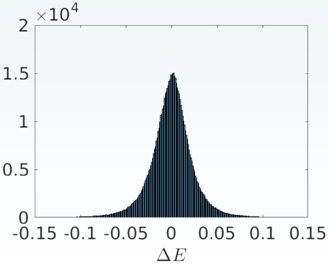
$\ell = 2$



$\ell = 4$



4 layers



Comparison to exact diagonalization

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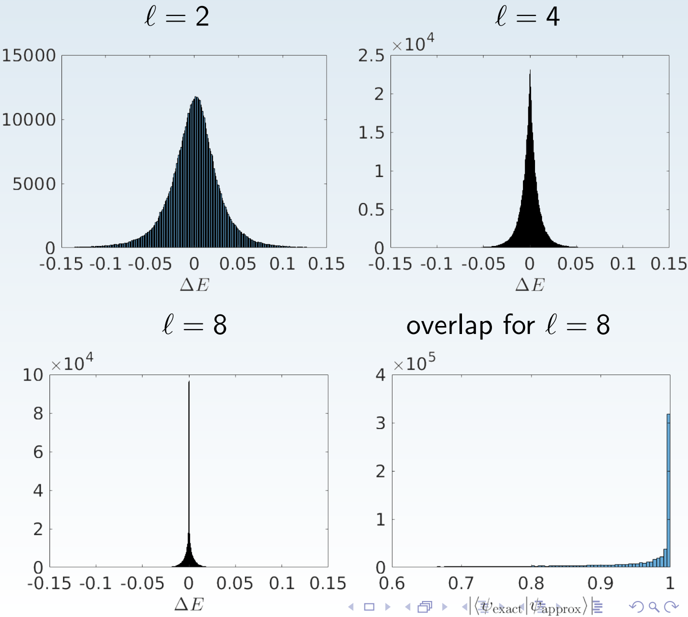
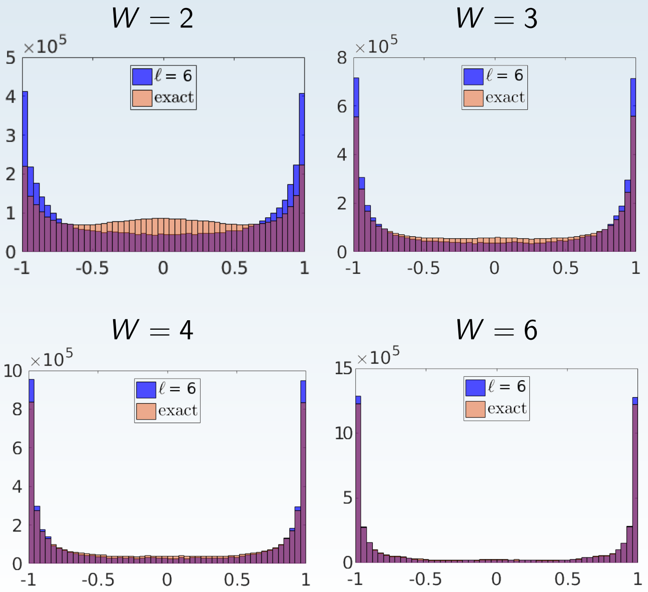
$$dim_{\mathrm{Hilbert}} = 2^{16} = 65,536$$



Approximation of local observables

plotting:
$|\langle\tilde{\psi}_{i_1\ldots i_N}|\sigma_j^z|\tilde{\psi}_{i_1\ldots i_N}\rangle|$

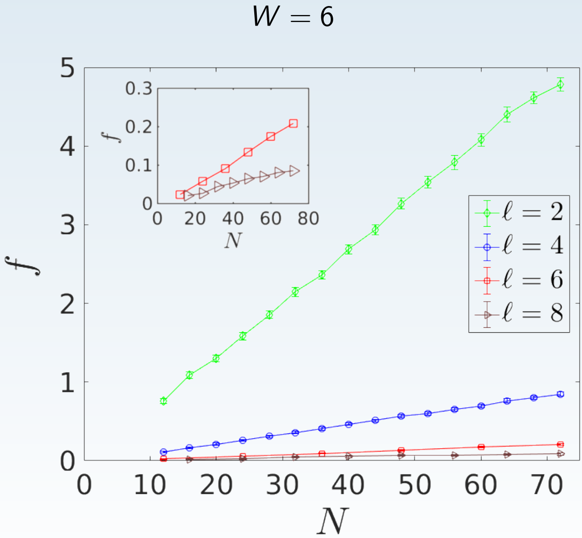
$N = 12$, use $\ell = 6$



Scaling with system size

remember:

$$f(\{u_{x,y}\}) = \text{const.} - \sum_x f_x$$

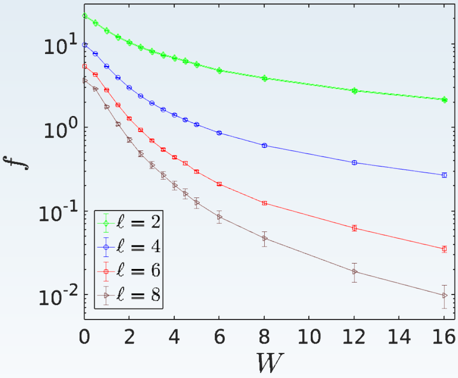


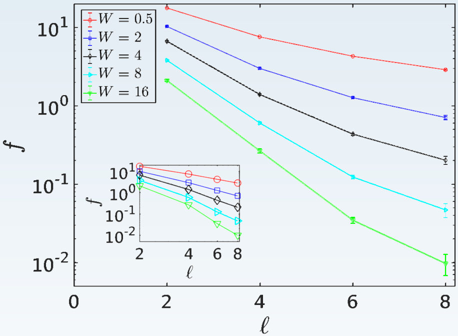
Summary benchmark results

- very high precisions for $\ell = 6, 8$
- local observables approximated accurately at $t_{\mathrm{CPU}} \sim N$

$\Rightarrow$ simulation of large MBL systems with high accuracies

Approaching the phase transition for $N = 72$

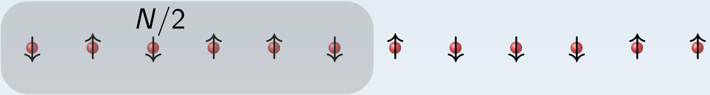




$$\#_{\text{param}}(\ell = 8) = 6307$$

Detection of the phase transition

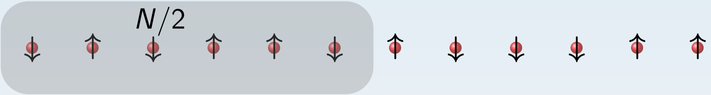
$|\psi_{i_1 \ldots i_N}\rangle$



$$\# \; 1: \quad |\psi_{i_1 \ldots i_N}\rangle \; \rightarrow \; \rho_{i_1 \ldots i_N} \; \rightarrow S(\rho_{i_1 \ldots i_N}) \; \rightarrow \quad \text{average } S(1)$$

Detection of the phase transition

$|\psi_{i_1 \dots i_N}\rangle$

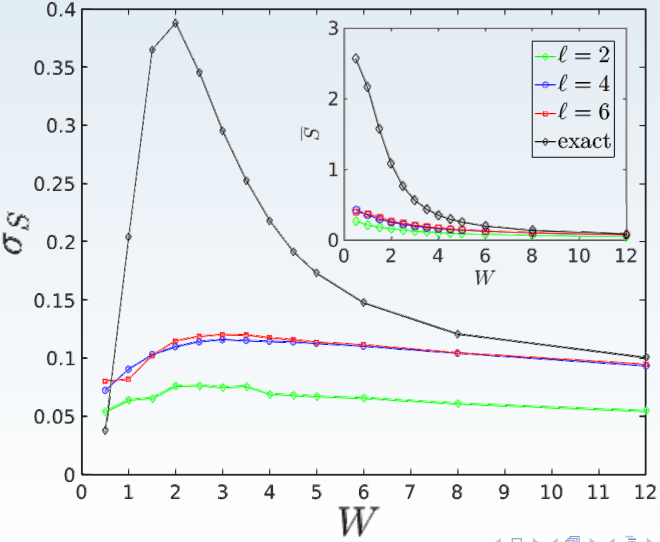


$$
\begin{array}{rl}
\#\ 1: & |\psi_{i_1 \dots i_N}\rangle \rightarrow \rho_{i_1 \dots i_N} \rightarrow S(\rho_{i_1 \dots i_N}) \rightarrow \quad \text{average } S(1) \\
\#\ 2: & \qquad \text{average } S(2) \\
\dots & \qquad \dots \\
\#\ 100: & \qquad \underline{\text{average } S(100)}
\end{array}
$$

$$\overline{S},\ \sigma_S$$

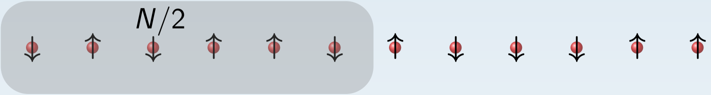
Comparison to exact diagonalization

$N = 12$



Now again for a large system of $N = 72$

$|\tilde{\psi}_{i_1 \dots i_N}\rangle$



$$
\begin{array}{lccccccc}
\#\ 1: & |\tilde{\psi}_{i_1 \dots i_N}\rangle & \rightarrow & \tilde{\rho}_{i_1 \dots i_N} & \rightarrow & S(\tilde{\rho}_{i_1 \dots i_N}) & \rightarrow & \text{average } S(1) \\
\#\ 2: & & & & & & & \text{average } S(2) \\
\dots & & & & & & & \dots \\
\#\ 100: & & & & & & & \underline{\text{average } S(100)}
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$$\overline{S}, \ \sigma_S$$

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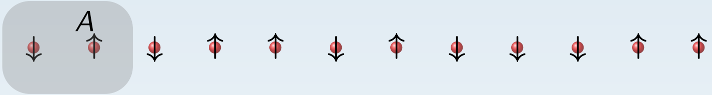


$\# \, 1: \quad |\tilde{\psi}_{i_1 \dots i_N}\rangle \rightarrow \tilde{\rho}_{i_1 \dots i_N} \rightarrow S(\tilde{\rho}_{i_1 \dots i_N}) \rightarrow$ average $S(1)$

$\# \, 2:$ average $S(2)$

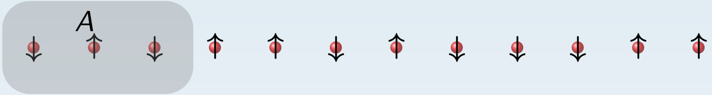
$\dots \qquad \dots$

$\# \, 100:$ average $S(100)$

$\overline{S}, \ \sigma_S$

Now again for a large system of $N = 72$

$|\tilde{\psi}_{i_1 \ldots i_N}\rangle$

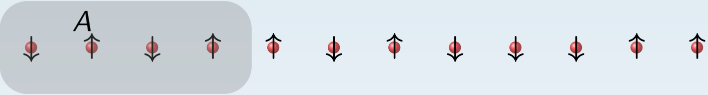


$$
\begin{array}{lllll}
\#\,1: & |\tilde{\psi}_{i_1 \ldots i_N}\rangle \rightarrow \tilde{\rho}_{i_1 \ldots i_N} & \rightarrow S(\tilde{\rho}_{i_1 \ldots i_N}) & \rightarrow & \text{average } S(1) \\
\#\,2: & & & & \text{average } S(2) \\
\ldots & & & & \ldots \\
\#\,100: & & & & \underline{\text{average } S(100)}
\end{array}
$$

$$\overline{S}, \ \sigma_S$$

Now again for a large system of $N = 72$

$|\tilde{\psi}_{i_1 \dots i_N}\rangle$

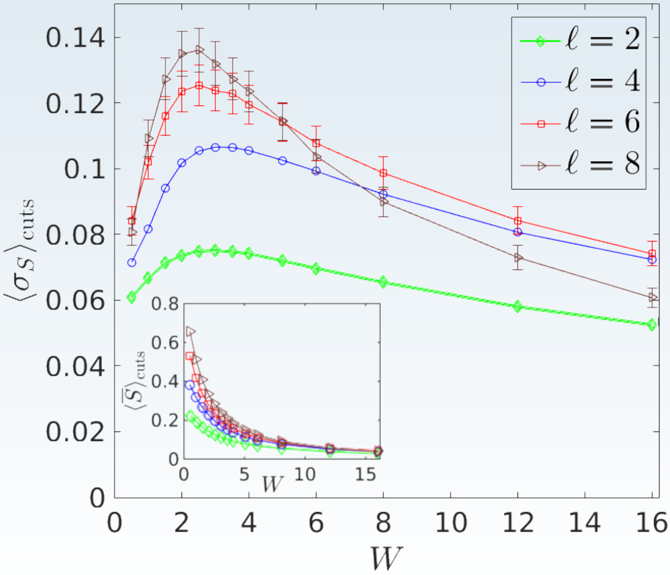


$$
\begin{array}{llllll}
\#\ 1: & |\tilde{\psi}_{i_1 \dots i_N}\rangle \rightarrow & \tilde{\rho}_{i_1 \dots i_N} & \rightarrow S(\tilde{\rho}_{i_1 \dots i_N}) \rightarrow & & \text{average } S(1) \\
\#\ 2: & & & & & \text{average } S(2) \\
\dots & & & & & \dots \\
\#\ 100: & & & & & \underline{\text{average } S(100)}
\end{array}
$$

$$\overline{S}, \ \sigma_S$$

$$\langle \overline{S} \rangle_{\mathrm{cuts}}, \ \langle \sigma_S \rangle_{\mathrm{cuts}}$$

$N = 72$



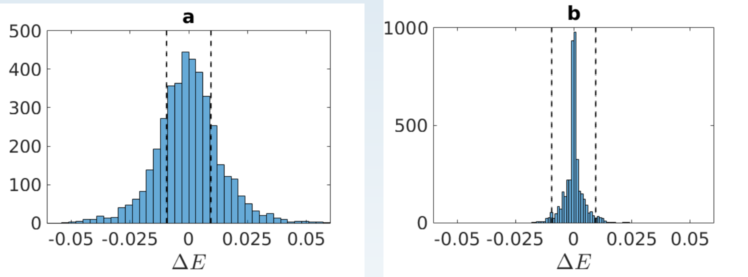
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
$\langle \sigma_S \rangle_{\text{cuts}}$
0 2 4 6 8 10 12 14 16
W
$\ell = 2$
$\ell = 4$
$\ell = 6$
$\ell = 8$
0.8
0.6
0.4
0.2
0
$\langle \overline{S} \rangle_{\text{cuts}}$
0 5 10 15
W

Conclusions

- tensor network ansatz for fully MBL systems
- computational complexity $t_{\mathrm{CPU}} \propto N$
- scalable: error $\sim 1/\mathrm{poly}\,(t_{\mathrm{CPU}})$ for given N
- figure of merit decomposes into local parts (improved scaling $2^{7\ell} \to 2^{3\ell}$)
- very high accracies, even in vicinity of MBL-to-thermal transition

Comparison to previous scheme for $N = 12$

$\ell = 2$ $\ell = 4$



$n = 4$ layers same without imposing symmetries