Dynamics of entanglement in expanding quantum fields

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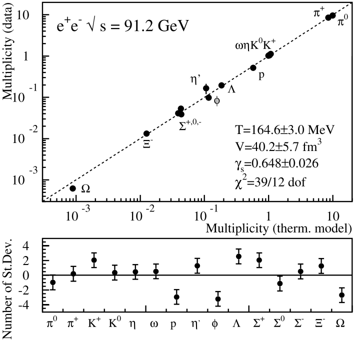
Collectivity and correlations in high-energy hadron and nuclear collisions, Benasque, 15/08/2018.

Entanglement and QCD physics

- how strongly entangled is the nuclear wave function?
- what is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]
- does saturation at small Bjorken-x have an entropic meaning?
- entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015); Kovner, Lublinsky, Serino (2018)]
- could entanglement entropy help for a non-perturbative extension of the parton model?
- entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]

The thermal model puzzle

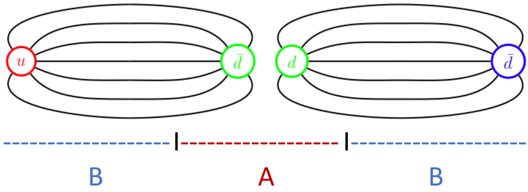
- elementary particle collision experiments such as $e^+ \, e^-$ collisions show some thermal-like features
- particle multiplicities well described by thermal model



[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by collisions unlikely
- more thermal-like features difficult to understand in PYTHIA [Fischer, Sjöstrand (2017)]
- alternative explanations needed

QCD strings



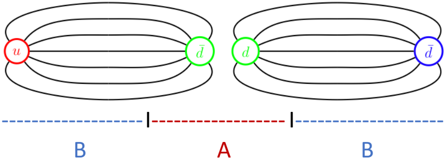
- particle production from QCD strings
- Lund string model (e. g. PYTHIA)
- different regions in a string are entangled
- subinterval A is described by reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho$$

- reduced density matrix is of mixed state form
- could this lead to thermal-like effects?

Entropy and entanglement

- consider a split of a quantum system into two $A + B$



- reduced density operator for system A

$$\rho_A = \mathsf{Tr}_B\{\rho\}$$

- entropy associated with subsystem A: **entanglement entropy**

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$
- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive**

Microscopic model

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2} \mathrm{tr}\, \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}$$

- fermionic fields ψ_i with sums over flavor species $i = 1, \ldots, N_f$
- SU(N_c) gauge fields $\mathbf{A}_\mu$ with field strength tensor $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling g has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for $N_c \to \infty$ with $g^2 N_c$ fixed
  ['t Hooft (1974)]

Schwinger model

- QED in 1+1 dimension

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly
  [Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \Big\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 - \frac{m\, q\, e^\gamma}{2\pi^{3/2}} \cos\left(2\sqrt{\pi}\phi + \theta\right) \Big\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- scalar mass related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model $m = 0$ leads to free bosonic theory

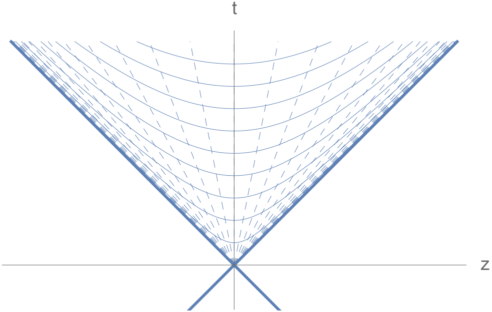
Transverse coordinates

- so far dynamics strictly confined to 1+1 dimensions
- transverse coordinates may fluctuate, can be described by Nambu-Goto action ($h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$)

$$\begin{aligned} S_{\mathsf{NG}} &= \int d^2x \sqrt{-\det h_{\mu\nu}}\, \{-\sigma + \ldots\} \\ &\approx \int d^2x \sqrt{g} \left\{-\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \ldots\right\} \end{aligned}$$

- two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates X^i with $i = 1, 2$

Expanding string solution 1



- external quark-anti-quark pair on trajectories $z = \pm t$
- coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \mathrm{arctanh}(z/t)$
- metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- symmetry with respect to longitudinal boosts $\eta \to \eta + \Delta\eta$

Expanding string solution 2

- Schwinger boson field depends only on τ

$$\bar{\phi} = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau}\partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field $E = q\phi/\sqrt{\pi}$ must approach the U(1) charge of the external quarks $E \to q_\mathsf{e}$ for $\tau \to 0_+$

$$\bar{\phi}(\tau) \to \frac{\sqrt{\pi} q_\mathsf{e}}{q} \qquad (\tau \to 0_+)$$

- solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \frac{\sqrt{\pi} q_\mathsf{e}}{q} J_0(M\tau)$$

Gaussian states

- theories with quadratic action often have Gaussian density matrix
- fully characterized by field expectation values

$$\bar{\phi}(x) = \langle \phi(x) \rangle, \qquad \bar{\pi}(x) = \langle \pi(x) \rangle$$

  and connected two-point correlation functions, e. g.

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if ρ is Gaussian, also reduced density matrix ρ_A is Gaussian

Functional representation

- Schrödinger functional representation of quantum field theory
- pure state $|\Psi\rangle$ has functional

$$\Psi[\phi] = \langle\phi|\Psi\rangle$$

  with field "positions" ϕ_n

- density matrix

$$\rho[\phi_+, \phi_-] = \langle\phi_+|\rho|\phi_-\rangle$$

- fields and conjugate momenta

$$\phi_m, \qquad \pi_m = -i\frac{\delta}{\delta\phi_m}$$

- canonical commutation relation

$$[\phi_m, \pi_n] = i\delta_{mn}$$

Symplectic transformations

- combined field

$$\chi = \begin{pmatrix} \phi \\ \pi^* \end{pmatrix}, \qquad \chi^* = \begin{pmatrix} \phi^* \\ \pi \end{pmatrix}$$

- commutation relation as symplectic metric

$$[\chi_m, \chi_n^*] = \Omega_{mn}, \qquad \Omega = \Omega^\dagger = \begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix},$$

- symplectic transformations S_{mn}

$$\chi_m \to S_{mn}\chi_n, \qquad \chi_m^* \to \chi_n^* (S^\dagger)_{nm}, \qquad S\Omega S^\dagger = \Omega,$$

have unitary representations on Gaussian states

Williamson's theorem and entropy

- Covariance matrix

$$\Delta_{mn} = \frac{1}{2} \langle \chi_m \chi_n^* + \chi_n^* \chi_m \rangle_c$$

  transforms as

$$\Delta \to S \Delta S^\dagger \neq S \Delta S^{-1}$$

- Williamson's theorem: can find S_{mn} such that

$$\Delta \to \mathsf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_1, \lambda_2, \ldots),$$

- symplectic eigenvalues $\lambda_j > 0$
- Heisenbergs uncertainty principle: $\lambda_j \geq 1/2$
- von Neumann entropy

$$S = \sum_j \left\{ \left(\lambda_j + \frac{1}{2}\right) \ln \left(\lambda_j + \frac{1}{2}\right) - \left(\lambda_j - \frac{1}{2}\right) \ln \left(\lambda_j - \frac{1}{2}\right) \right\}$$

- pure state: $\lambda_j = 1/2$, $S = 0$

Entanglement entropy for Gaussian state

- entanglement entropy of Gaussian state in region A
  [Berges, Floerchinger, Venugopalan, JHEP 1804 (2018) 145]

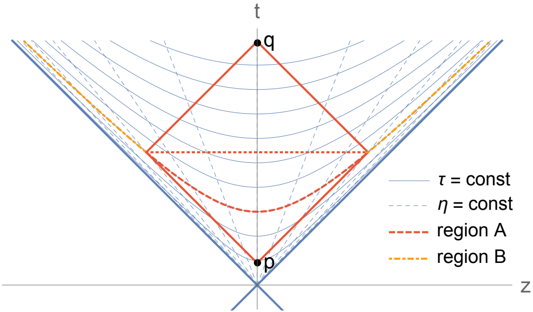
$$S_A = \frac{1}{2} \mathsf{Tr}_A \left\{ D \ln(D^2) \right\}$$

- operator trace over region A only
- matrix of correlation functions

$$D(x,y) = \begin{pmatrix} -i\langle\phi(x)\pi(y)\rangle_c & i\langle\phi(x)\phi(y)\rangle_c \\ -i\langle\pi(x)\pi(y)\rangle_c & i\langle\pi(x)\phi(y)\rangle_c \end{pmatrix}$$

- involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- expectation value $\bar{\phi}$ does not appear explicitly
- coherent states and vacuum have equal entanglement entropy S_A

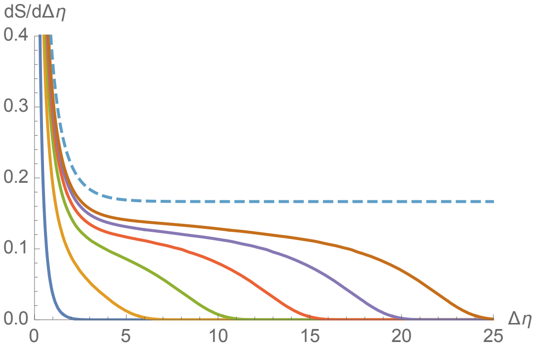
Rapidity interval



- consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time τ
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

Bosonized massless Schwinger model

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{q}{\sqrt{\pi}}$)



$M\tau = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$, and 10^{-5}

Conformal limit

- For $M\tau \to 0$ one has conformal field theory limit
  [Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln \left(\Delta z / \epsilon\right) + \text{constant}$$

  with small length ϵ acting as UV cutoff.
- Here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left(2\tau \, \text{sinh}(\Delta\eta/2)/\epsilon\right) + \text{constant}$$

- Conformal charge $c = 1$ for free massless scalars or Dirac fermions.
- Additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) =& \frac{c}{6} \text{coth}(\Delta\eta/2) \\ \to& \frac{c}{6} \qquad\qquad (\Delta\eta \gg 1) \end{aligned}$$

- Entropy becomes extensive in $\Delta\eta$!

Universal entanglement entropy density

- for very early times "Hubble" expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

with conformal charge c

- for QCD in 1+1 D (gluons not dynamical, no transverse excitations)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

Temperature and entanglement entropy

- for conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- for static interval of length L [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T,l) = \frac{c}{3}\ln\left(\frac{1}{\pi T\epsilon}\sinh(\pi L T)\right) + \mathsf{const}$$

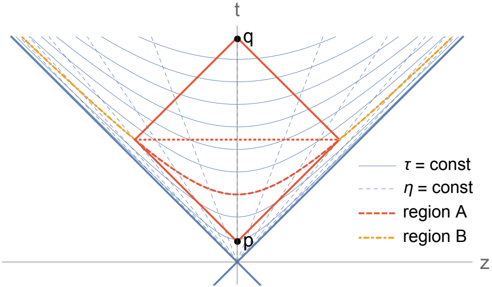
- compare this to our result in expanding geometry

$$S(\tau,\Delta\eta) = \frac{c}{3}\ln\left(\frac{2\tau}{\epsilon}\,\mathsf{sinh}(\Delta\eta/2)\right) + \mathsf{const}$$

- expressions agree for $L = \tau\Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

Modular or entanglement Hamiltonian 1



- conformal field theory
- hypersurface Σ with boundary on the intersection of two light cones
- reduced density matrix [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \qquad Z_A = \mathrm{Tr}\, e^{-K}$$

- modular or entanglement Hamiltonian K

Modular or entanglement Hamiltonian 2

- modular or entanglement Hamiltonian is **local expression**

$$K = \int_\Sigma d\Sigma_\mu \, \xi_\nu(x) \, T^{\mu\nu}(x).$$

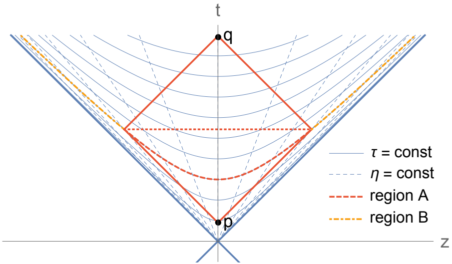
- energy-momentum tensor $T^{\mu\nu}(x)$ of excitations
- vector field

$$\begin{aligned}\xi^\mu(x) = \tfrac{2\pi}{(q-p)^2} [&(q-x)^\mu (x-p)(q-p) \\ &+ (x-p)^\mu (q-x)(q-p) - (q-p)^\mu (x-p)(q-x)]\end{aligned}$$

  end point of future light cone q, starting point of past light cone p
- inverse temperature and fluid velocity

$$\xi^\mu(x) = \beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

Modular or entanglement Hamiltonian 3



- for $\Delta\eta \to \infty$: fluid velocity in τ-direction, τ-dependent temperature

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \hbar c/(2\pi x)$

Alternative derivation: mode functions

- fluctuation field $\varphi = \phi - \bar{\phi}$ has equation of motion

$$\partial_\tau^2 \varphi(\tau,\eta) + \frac{1}{\tau}\partial_\tau \varphi(\tau,\eta) + \left(M^2 - \frac{1}{\tau^2}\frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau,\eta) = 0$$

- solution in terms of plane waves

$$\varphi(\tau,\eta) = \int \frac{dk}{2\pi} \left\{ a(k) f(\tau,|k|) e^{ik\eta} + a^\dagger(k)\, f^*(\tau,|k|) e^{-ik\eta} \right\}$$

- mode functions as Hankel functions

$$f(\tau,k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau,k) = \frac{\sqrt{\pi}}{\sqrt{2\sinh(\pi k)}} J_{-ik}(M\tau)$$

Bogoliubov transformation

- mode functions are related

$$\begin{aligned}\bar{f}(\tau,k) =&\alpha(k) f(\tau,k) + \beta(k) f^*(\tau,k)\\ f(\tau,k) =&\alpha^*(k)\bar{f}(\tau,k) - \beta(k)\bar{f}^*(\tau,k)\end{aligned}$$

- creation and annihilation operators are related by

$$\begin{aligned}\bar{a}(k) =&\alpha^*(k) a(k) - \beta^*(k) a^\dagger(k)\\ a(k) =&\alpha(k)\bar{a}(k) + \beta(k)\bar{a}^\dagger(k)\end{aligned}$$

- Bogoliubov coefficients

$$\alpha(k) = \sqrt{\frac{e^{\pi k}}{2\sinh(\pi k)}} \qquad \beta(k) = \sqrt{\frac{e^{-\pi k}}{2\sinh(\pi k)}}$$

- vacuum $|\Omega\rangle$ with respect to $a(k)$ such that $a(k)|\Omega\rangle = 0$ contains excitations with respect to $\bar{a}(k)$ such that $\bar{a}(k)|\Omega\rangle \neq 0$ and *vice versa*

Role of different mode functions

- Hankel functions $f(\tau, k)$ are superpositions of *positive* frequency modes with respect to Minkowski time t
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to Minkowski time t
- at very early time $1/\tau \gg M, m$ conformal symmetry

$$ds^2 = \tau^2 \left[-d\ln(\tau)^2 + d\eta^2 \right]$$

- Hankel functions $f(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to conformal time $\ln(\tau)$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive* frequency modes with respect to conformal time $\ln(\tau)$

Occupation numbers

- Minkowski space coherent states have two-point functions

$$\langle \bar{a}^\dagger(k)\bar{a}(k')\rangle_c = \bar{n}(k)\, 2\pi\, \delta(k-k') = |\beta(k)|^2\, 2\pi\, \delta(k-k')$$
$$\langle \bar{a}(k)\bar{a}(k')\rangle_c = \bar{u}(k)\, 2\pi\, \delta(k+k') = -\alpha^*(k)\beta^*(k)\, 2\pi\, \delta(k+k')$$
$$\langle \bar{a}^\dagger(k)\bar{a}^\dagger(k')\rangle_c = \bar{u}^*(k)\, 2\pi\, \delta(k+k') = -\alpha(k)\beta(k)\, 2\pi\, \delta(k+k')$$

- occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k}-1}$$

- Bose-Einstein distribution with excitation energy $E = |k|/\tau$ and temperature

$$T = \frac{1}{2\pi\tau}$$

- off-diagonal occupation number $\bar{u}(k) = -1/(2\sinh(\pi k))$ make sure we still have pure state

Local description

- consider now rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n \, e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \, \varphi(\eta) \, \frac{1}{2} \left[e^{-in\pi\frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi\frac{\eta}{\Delta\eta}} \right]$$

- relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin(\tfrac{k\Delta\eta}{2} - \tfrac{n\pi}{2}) \left[\frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \qquad \langle \pi_n \rangle, \qquad \langle \varphi_n \varphi_m \rangle_c, \qquad \text{etc.}$$

Emergence of locally thermal state

- mode functions at early time

$$\bar{f}(\tau, k) = \frac{1}{\sqrt{2k}} e^{-ik\ln(\tau) - i\theta(k,M)}$$

- phase varies strongly with k for $M \to 0$

$$\theta(k, M) = k\ln(M/2) + \arg(\Gamma(1 - ik))$$

- off-diagonal term $\bar{u}(k)$ have factors strongly oscillating with k

$$\begin{aligned}\langle\varphi(\tau, k)\varphi^*(\tau, k')\rangle_c = \;& 2\pi\delta(k - k')\frac{1}{|k|} \\ & \times \left\{\left[\tfrac{1}{2} + \bar{n}(k)\right] + \cos\left[2k\ln(\tau) + 2\theta(k, M)\right]\, \bar{u}(k)\right\}\end{aligned}$$

  cancel out when going to finite interval !

- only Bose-Einstein occupation numbers $\bar{n}(k)$ remain

Physics picture

- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically limits $\Delta\eta \to \infty$ and $M\tau \to 0$ do not commute
    - $\Delta\eta \to \infty$ for any finite $M\tau$ gives pure state
    - $M\tau \to 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

Conclusions

- rapidity intervals in an expanding string are entangled
- at very early times theory effectively conformal

$$\frac{1}{\tau} \gg m, q$$

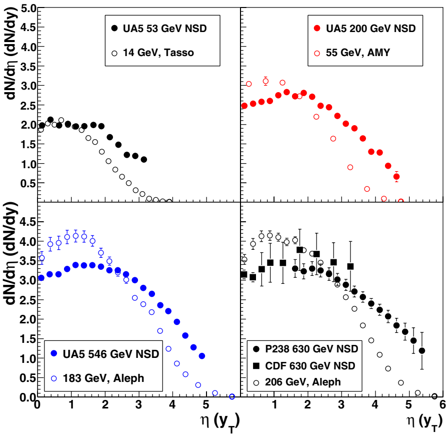
- entanglement entropy extensive in rapidity $\frac{dS}{d\Delta\eta} = \frac{c}{6}$
- determined by conformal charge $c = N_c \times N_f + 2$
- reduced density matrix for conformal field theory is of locally thermal form with temperature

$$T = \frac{\hbar}{2\pi\tau}$$

- entanglement could be important ingredient to understand apparent "thermal effects" in e^+e^- and other collider experiments

Backup

Rapidity distribution



[open (filled) symbols: e^+e^- (pp), Grosse-Oetringhaus & Reygers (2010)]

- rapidity distribution $dN/d\eta$ has plateau around midrapidity
- only logarithmic dependence on collision energy

Experimental access to entanglement ?

- could longitudinal entanglement be tested experimentally?
- unfortunately entropy density $dS/d\eta$ not straight-forward to access
- measured in e^+e^- is the number of charged particles per unit rapidity $dN_{\mathsf{ch}}/d\eta$ (rapidity defined with respect to the thrust axis)
- typical values for collision energies $\sqrt{s} = 14 - 206$ GeV in the range

$$dN_{\mathsf{ch}}/d\eta \approx 2 - 4$$

- entropy per particle S/N can be estimated for a hadron resonance gas in thermal equilibrium $S/N_{\mathsf{ch}} = 7.2$ would give

$$dS/d\eta \approx 14 - 28$$

- this is an upper bound: correlations beyond one-particle functions would lead to reduced entropy