DIS as a probe of entanglement

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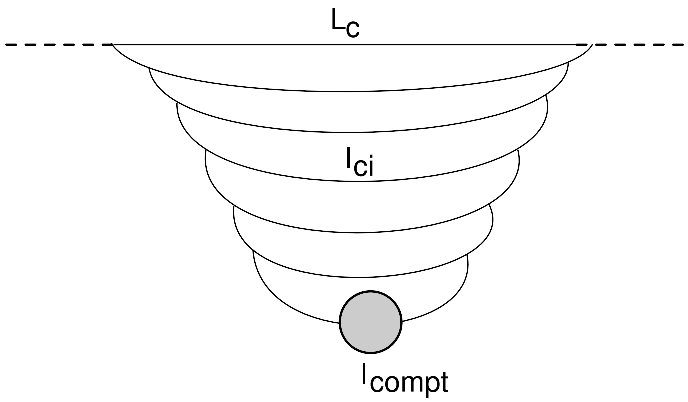
D. E. Kharzeev and E.L.: *"Deep inelastic scattering as a probe of entanglement,"* Phys. Rev. D **95** (2017) 114008, arXiv:1702.03489 [hep-ph].

Motivations and Disclaimers:

- How does the pure state in the r.f. evolves to the set of 'quasi free' patrons in the IMF?

M.Martinelli: " Photons, Bits and Entropy: From Planck to Shannon at the Roots of the Information Age",Entropy,19, 347 (2017)

- What is the rigorous definition of 'quasi free' parton distribution?
- In DIS we measure a tube of radius $1/Q^2$ and longitudinal size $1/(mx)$ (region A):



In DIS we can measure $\rho_A = \mathrm{tr}_B \rho$.

- Is there an EE $S_E = -\mathrm{tr}[\rho_A \ln \rho_A]$ associated with DIS experiment?
- If yes how does it relates to the pdf?
- PDF versus multiplicity in DIS?
- What we need to use instead of patrons deep in the saturation region?

Do not expect:

- **A thorough knowledge of EE**

  C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B 424 (1994) 443, [hep-th/9403108];

  P. Calabrese and J. L. Cardy, Int. J. Quant. Inf. 4 (2006) 429, [quant-ph/0505193].

- **A rigorous answer to every questions.**

- **A list of prediction for DIS deep in the saturation region.**

The paper (and this talk) is an attempt to give the answers to all above questions, based on simple calculations and the observed similarities between CFT and the parton cascade if we discuss it in terms of entropy.

Follow the example of Max Planck, we look at DIS from the point of ENTROPY

Ideas and results (1) :

- In toy (1+1) dimensional model as well as in the full QCD cascade we computed von Neumann entropy $S(x)$;
- We found that $S(x) = \ln\Big(xG\left(x, Q^2\right)\Big)$

  where $xG(x, Q^2)$ is the multiplicity of partons(gluons);
- This equation implies that all microstates of the system are equally probable and S is maximal;
- This equipartitioning of microscopic states that maximizes the von Neumann entropy corresponds to the parton saturation;
- S diverges logarithmically at $x \to 0$; $S(x) = \Delta \ln\left(\frac{1}{x}\right) = \Delta \ln\left(\frac{L}{\epsilon}\right)$

  with $L = 1/(mx)$ and $\epsilon = 1/m \leftarrow$ proton's Compton wave length, Δ is the BFKL intercept $\Delta = 2.8\bar{\alpha}_S$;

Ideas and results (2) :

- Reminds the expression for EE in (1+1) CFT: $S(x) = \frac{c}{3} \ln \frac{L}{\epsilon}$

- We argue that this agreement is not coincidental, and propose that the parton distributions, and the entropy associated with them, arise from the entanglement between the spatial domain probed by DIS and the rest of the target;

- The maximal value of the entanglement entropy attained at small x implies that the corresponding partonic state is *maximally entangled*;

- Unlike the parton distribution, the EE is an appropriate observable even at strong coupling when the description in terms of quasi-free partons fails.

QM of parton entanglement, as I understood it

A is the region that we measure in DIS, The physical sates are in $H^A_{\text{Hilbert space}}(n_A)$.

B is a complementary region, unobserved state $\in H^B(n_B)$

the entire space: $A \cap B$. the composite system in $H^A \otimes H^B$

$$|\Psi_{AB}\rangle = \sum_{ij} c_{ij} |\phi^A_i\rangle \otimes |\phi^B_j\rangle;$$ matrix C has $n_A \times n_B$ dimension

If $y \in A$ and $z \in B$ the density matrix:

$\rho\,(y, z, y', z') = \Psi_{AB}\,(y, z)\,\Psi^*_{AB}\,(y, z') \leftarrow$ pure state with S=0.

$$\rho_A\,(y, y') = \int dz\;\rho\,(y, z, y', z) \equiv \mathrm{tr}_B \rho_{AB}$$

Schmidt decomposition theorem:

$$|\Psi_{AB}\rangle = \sum_n \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

where $\alpha_n = \sqrt{CC^\dagger}$.

• • • $$\rho_A = \mathrm{tr}_{\mathrm{B}}\ \rho_{\mathrm{AB}} = \sum_{\mathrm{n}} \alpha_{\mathrm{n}}^2\ |\Psi_{\mathrm{n}}^{\mathrm{A}}\rangle\langle\Psi_{\mathrm{n}}^{\mathrm{A}}|$$ • • •

where $\alpha_n^2 \equiv p_n \leftarrow$ the probability of a state with n partons.

• • • $$S_{\text{von Neumann}} = -\sum_n p_n \ln p_n$$ • • •

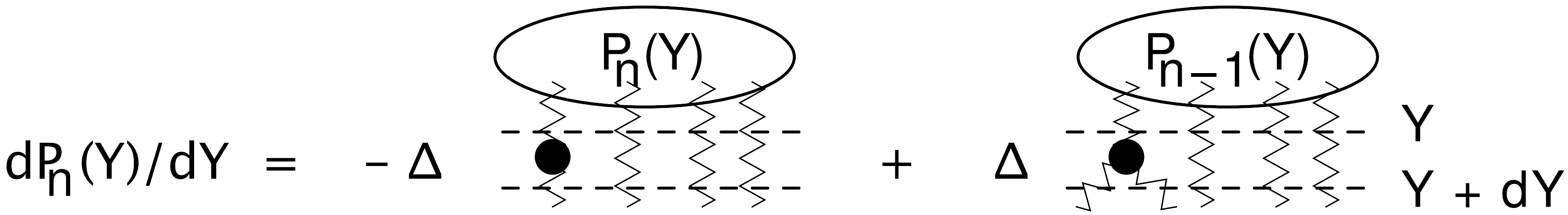
S $\equiv$ Shannon entropy (EE) for probability distribution $\{p_1, p_2, \dots, p_n\}$

Good Old Parton model in new framework:

Balitsky-Kovchegov cascade for dipoles with fixed sizes (1+1 toy model)

BFKL Pomeron:

- $$\frac{d\sigma\,(Y)}{dY} \;=\; \Delta\,\sigma\,(Y) \qquad \text{where} \qquad \Delta = 2.8\,\bar{\alpha}_S$$



- $$\frac{dP_n\,(Y)}{dY} \;=\; \underbrace{-\Delta\,n\,P_n\,(Y)}_{\text{depletion of the probability}} \;+\; \underbrace{(n-1)\,\Delta P_{n-1}\,(Y)}_{\text{growth due to splitting}}$$

Generating function:

$Z\,(Y,u) \;=\; \sum_n\, P_n\,(Y)\;u^n$, with $Z\,(Y=0,u) \;=\; u;\;\; Z\,(Y,u=1) \;=\; 1$

Equation:

- $$\frac{\partial Z(Y,u)}{\partial Y} = -\Delta\, u\,(1-u)\,\frac{\partial Z(Y,u)}{\partial u} \xrightarrow{Z(u(Y))} \frac{\partial\, Z}{\partial\, Y} = -\Delta\left(Z - Z^2\right)$$

For scattering amplitude

$$N\,(Y;\gamma) = 1 - Z\,(Y, 1-\gamma) \longrightarrow dN\,(Y)\,/dY = \Delta\left(N - N^2\right)$$

Solution:

- $$Z\,(Y,\,u) = \frac{u\,e^{-\Delta\,Y}}{1 + u\left(e^{-\Delta\,Y} - 1\right)} = u\,e^{-\Delta\,Y}\sum_{n=1}^{\infty} u^n\left(1 - e^{-\Delta\,Y}\right)^n$$

- $$P_n\,(Y) = e^{-\Delta\,Y}\left(1 - e^{-\Delta\,Y}\right)^{n-1}$$

Gluon sructure function:

$$xG\,(x) = \langle n\rangle = \sum_n n P_n\,(Y) = u\frac{dZ\,(Y,u)}{du}\bigg|_{u=1} = e^{\Delta\,Y} = \left(\frac{1}{x}\right)^{\Delta}$$

Entropy: $S_{\text{von Neumann}} = -\sum_n p_n \ln(p_n) \; = \; -\sum_n P_n(Y) \ln(P_n(Y))$

$$S = -\sum_n e^{-\Delta Y}\left(1 - e^{-\Delta Y}\right)^{n-1}\Big(-\ln\left(e^{\Delta Y} - 1\right) + n \ln\left(1 - e^{-\Delta Y}\right)\Big)$$

$$= \ln\left(e^{\Delta Y} - 1\right) \; Z\left(Y, \, u = 1\right) \; + \; \ln\left(\frac{1}{1 - e^{-\Delta Y}}\right) \, u \frac{\partial Z(Y, u)}{\partial u}\Big|_{u=1}$$

$$= \ln\left(e^{\Delta Y} - 1\right) \; + \; e^{\Delta Y} \ln\left(\frac{1}{1 - e^{-\Delta Y}}\right) \xrightarrow{\Delta Y \gg 1} \Delta Y = \Delta \ln\left(\frac{1}{x}\right)$$

$$S_{\text{v. N.}} \longrightarrow \begin{cases} \ln\left(xG(x)\right) & \text{if } \Delta Y \gg 1 \\ -\ln\left[\frac{xG(x) - xG(x=x_0)}{xG(x=x_0)}\right] \left[\frac{xG(x) - xG(x=x_0)}{xG(x=x_0)}\right] & \text{if } \Delta Y \ll 1 \end{cases}$$

Multiplicity distributions: $(\tilde{N} = \tilde{n} - 1)$

$$P_n(Y) \;=\; e^{-\Delta Y}\left(1 - e^{-\Delta Y}\right)^{n-1} \;=\; \frac{1}{\bar{n}}\left(\frac{\bar{n}-1}{\bar{n}}\right)^{n-1} \;=\; \frac{1}{\bar{N}}\left(\frac{\bar{N}}{\bar{N}+1}\right)^{n}$$

Negative binomial distribution:

$$\frac{\sigma_n}{\sigma_{in}} = P^{\rm NBD}(r, \bar{n}, n) = \left(\frac{r}{r+\langle n\rangle}\right)^r \frac{\Gamma(n+r)}{n!\,\Gamma(r)}\left(\frac{\langle n\rangle}{r+\langle n\rangle}\right)^n$$

- $$\frac{\sigma_n}{\sigma_{in}} = \frac{\bar{n}-1}{\bar{n}} P^{\rm NBD}(1, \bar{n}-1, n)$$

with r = 1(number of failures) and $p = \bar{N}/\left(\bar{N}+1\right) = 1 - 1/\bar{n}$(probability of success)

Cumulants: $C_q = \,< n^q > / < n >^q \,= \left(u\frac{d}{du}\right)^q Z(Y, u)\Big|_{u=1}$

$$C_2 = 2 - 1/\bar{n}; \quad C_3 = \frac{6(\bar{n}-1)\bar{n}+1}{\bar{n}^2};$$

$$C_4 = \frac{(12\bar{n}(\bar{n}-1)+1)(2\bar{n}-1)}{\bar{n}^3}; \quad C_5 = \frac{(\bar{n}-1)(120\bar{n}^2(\bar{n}-1)+30\bar{n})+1}{\bar{n}^4}$$

Predictions: $C_2 \simeq 1.83$, $C_3 \simeq 5.0$, $C_4 \simeq 18.2$ and $C_5 \simeq 83$.

Experiment: $C_2^{\rm exp} = 2.0 \pm 0.05$, $C_3^{\rm exp} = 5.9 \pm 0.6$, $C_4^{\rm exp} = 21 \pm 2$, and $C_5^{\rm exp} = 90 \pm 19$

EE from the $(3+1)$ dimensional Balitsky-Kovchegov equation:

$$\frac{\partial\, P_n\left(Y;\, r_1,\, r_2 \dots r_i \dots r_n\right)}{\bar{\alpha}_s\, \partial\, \left(Y\right)} \quad = \quad - \sum_{i=1}^{n} \omega(r_i)\, P_n\left(Y;\, r_1,\, r_2 \dots r_i \dots r_n\right)$$

$$+ \quad \sum_{i=1}^{n-1} \frac{(\vec{r}_i \,+\, \vec{r}_n)^2}{(2\,\pi)\, r_i^2\, r_n^2}\, P_{n-1}\left(Y;\, r_1,\, r_2 \dots (\vec{r}_i \,+\, \vec{r}_n) \dots r_{n-1}\right)$$

Probability for one dipole to survive depends on the dipole size:

- $\bar{\alpha}_S\ \omega(r_i) \equiv \bar{\alpha}_S \omega_i = \frac{\bar{\alpha}_S}{2\,\pi} \int_\rho \frac{r_i^2}{(\vec{r}_i \,-\, \vec{r}')^2\, r'^2}\, d^2 r' = \bar{\alpha}_S\ \ln(r_i^2/\rho^2)$

The probability for a decay $|\vec{r}_1 \,+\, \vec{r}_2| \to r_1 + r_2$: • $K\left(r_1, r_2 | \vec{r}_1 + \vec{r}_2\right) = \frac{\bar{\alpha}_S}{2\,\pi}\, \frac{(\vec{r}_1 \,+\, \vec{r}_2)^2}{r_1^2\, r_2^2}$

- **For $n = 1$ equation has the solution** $P_1\left(Y; r_1\right) \,=\, \delta\left(\vec{r} - \vec{r}_1\right)\, e^{-\bar{\alpha}_S \omega(r_1)\, Y}$

- $$\sum_{n=1}^{\infty} \int \prod_{i=1}^{n} d^2 r_i\, P_n\left(Y; \{r_i\}\right) \,=\, 1$$

i.e. the sum of all probabilities is equal to 1.

$P_n(Y;\{r_i\}) \Longrightarrow P_n(\omega;\{r_i\})$(Mellin image)

- $P_n(Y;\{r_i\}) \;=\; \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\omega}{2\pi}e^{\omega\,\bar{\alpha}_S\,Y}\,P_n(\omega;\{r_i\})$

  - $P_n(\omega;\{r_i\}) =$

$$-\quad \sum_{i=1}^{n}\omega_i\,P_n(\omega;\{r_i\}) + \frac{1}{2\pi}\sum_{j=1}^{n-1}\frac{(\vec{r}_j+\vec{r}_n)^2}{r_j^2\,r_n^2}P_{n-1}\left(\omega;\,\{r_i,\vec{r}_j\to(\vec{r}_j+\vec{r}_n)\}\right)$$

  - $P_n(\omega;\{r_i\}) = 2\pi\,r^2\,\delta(\vec{r}-\vec{r}_1)\left(\frac{1}{2\pi}\right)^n\prod_{i=1}^{n}\frac{1}{r_i^2}\,\Omega_n(\omega,\{\omega_i\})$

- $\Omega_n(\omega,\{\omega_i\}) = -\left(\sum_{i=1}^{n}\omega_i\right)\Omega_n(\omega,\{\omega_i\}) + \sum_{j=1}^{n-1}\Omega_{n-1}\left(\omega,\{\omega_i,\omega_{jn}\}\right)$

Recurrent equation:

- $\Omega_n(\omega,\{\omega_i\}) \;=\; (n-1)\,\Omega_{n-1}\left(\omega,\{\omega_i,\omega_{n-1,n}\}\right)\frac{1}{\omega+\sum_{j=1}^{n}\omega_j};$

$\omega_i = \omega(\vec{r}_i)$ and $\omega_{ij} = \omega\left(\vec{r}_i+\vec{r}_j\right)$.

Failed to solve in a general case but

Large dipole $\longrightarrow$ one large + one small dipoles. $|\vec{r}_i + \vec{r}_n| \to r_i$ while $r_n \ll r_i$.

Summation $\ln^n\left(r_i^2 Q_s^2\right)$ for $r_i^2 Q_s^2 \gg 1$;

- $$\Omega_n\left(\omega,\{\omega_i\}\right) = (n-1)!\prod_{j=1}^{n}\frac{1}{\omega + \sum_{l=1}^{j}\omega_l} = (n-1)!\prod_{j=1}^{n}\frac{1}{\omega + \sum_{l=1}^{j} z_l}$$
- $$P_n\left(Y;\{r_i\}\right) = 2\,\pi\, r^2\,\delta\left(\vec{r} - \vec{r}_1\right)\left(\frac{1}{2\,\pi}\right)^n\prod_{i=1}^{n}\frac{1}{r_i^2}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\omega}{2\,\pi}e^{\omega\,\bar{\alpha}_S\,Y}\,\Omega_n\left(\omega,\{\omega_i\}\right)$$
- $$\int\prod_{i=1}^{n} d^2 r_i\,P_n\left(Y;\{r_i\}\right) = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\omega}{2\,\pi}e^{\omega\,\bar{\alpha}_S\,Y}\int\prod_{i=1}^{n} dz_i\,\Omega_n\left(\omega,\{z_i\}\right)$$

- • $\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\omega}{2\,\pi} e^{\omega\,\bar{\alpha}_S\,Y}\,\Omega_n\left(\omega, \{z_i\}\right)$

$$= \left(\bar{\alpha}_S Y\right)^n \int_0^1 \prod_{i=2}^n d\alpha_i \exp\left\{ - \left(z_1 + z_2 \sum_{i=2}^n \alpha_i + z_3 \sum_{i=3}^n \alpha_i + \dots + z_l \sum_{i=l}^n \alpha_i + \dots + z_n \alpha_n \right) \bar{\alpha}_S\,Y \right\}$$

$$= \left(\bar{\alpha}_S Y\right)^n \int_0^1 \prod_{i=2}^n d\alpha_i \exp\left\{ - \left(z_1 + \alpha_n \sum_{i=2}^n z_i + \alpha_{n-1} \sum_{i=3}^n z_i + \dots + \alpha_l \sum_{i=l}^n z_i + \dots + z_n \alpha_n \right) \bar{\alpha}_S\,Y \right\}$$

$$= e^{-\bar{\alpha}_S\,z_1\,Y} \prod_{i=2}^n \left(\frac{1 - e^{-\left(\sum_{l=i}^n z_l\right)\bar{\alpha}_S\,Y}}{\sum_{l=i}^n z_l} \right) \equiv \left(\bar{\alpha}_S\,Y\right)^n e^{-\bar{\alpha}_S\,z_1\,Y} \prod_{i=2}^n \Phi\left(\bar{\alpha}_S Y \sum_{l=i}^n z_l \right)$$

- $\int \prod_{i=1}^n dz_i\,P_n\left(Y; \{z_i\}\right) =$

$$e^{-\bar{\alpha}_S\,z_1\,Y} \int_0^{\bar{\alpha}_S\,z_1\,Y} \Phi\left(t_n\right)\,dt_n \int_0^{t_n} dt_{n-1}\,\Phi\left(t_{n-1}\right) \dots \int_0^{t_3} dt_2\,\Phi\left(t_2\right) = \frac{1}{n!}\Xi^n\left(\bar{\alpha}_S\,z_1\,Y\right)\,e^{-\bar{\alpha}_S\,z_1\,Y}$$

where $t_i = \bar{\alpha}_S Y\ \sum_{l=i}^n z_l$ and

$$\Xi\left(t\right) = \int_0^t \Phi\left(t'\right) dt' = C + \Gamma\left(0, t\right) + \ln t$$

$$\Xi\left(t\right) = \begin{cases} t & \text{if } t \ll 1; \\ \ln\left(1/t\right) & \text{if } t \gg 1. \end{cases}$$

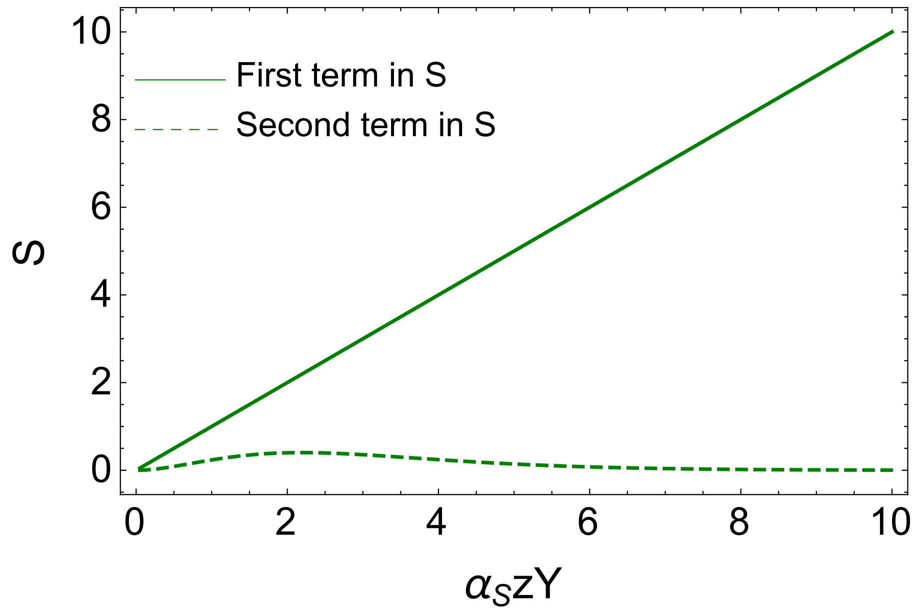
$$z_1 \gg z_2 \gg \dots \gg z_i \gg z_{i-1} \gg \dots \gg 0$$

Gibbs formula:

$$S = -\sum_{n=1}^{\infty}\prod_{i=1}^{n} d^2 r_i\, P_n\left(Y;\{r_i\}\right)\,\ln\Big(P_n\left(Y;\{r_i\}\right)\Big)$$

For $Y \gg 1$:

$$
\begin{aligned}
S &= \omega(r)\bar{\alpha}_S Y \underbrace{\sum_{n=1}^{\infty}\int\prod_{i=1}^{n} d^2 r_i\, P_n\left(Y-y;\{r_i\}\right)}_{=1} \\
&- \underbrace{e^{-\bar{\alpha}_S z_1 Y}\sum_{n=1}^{\infty}\int\prod_{i=2}^{n} dz_i \left\{\sum_{i=2}^{n} z_i - \sum_{i=2}^{n}\ln\left(\sum_{l=i}^{n} z_l\right)\right\}\prod_{i=2}^{n}\left(\frac{1-e^{-\left(\sum_{l=i}^{n} z_l\right)\bar{\alpha}_S Y}}{\sum_{l=i}^{n} z_l}\right)}_{\ll 1}
\end{aligned}
$$



$$S = \underbrace{\bar{\alpha}_S\, z}_{\text{Zamolodchikov, JETP letters, 43,565(1986)}}\; Y$$

Once more: results (discussion)

A1: The entropy originates from the entanglement between the spatial domain probed by DIS and the rest of the target, whereas the entire proton is in a pure quantum state with zero entropy.

A2: Parton distributions have a well-defined meaning only for weakly coupled partons at large momentum transfer Q^2 – but the entanglement entropy is a universal concept that applies to states at any value of the coupling constant.

A3: Unlike the parton distributions, the entanglement entropy is subject to strict bounds – for example, if the small x regime is described by a CFT, the growth of parton distributions should be bounded by $xG(x) \leq \text{const}\ x^{-1/3}$.

A4: If the second law of thermodynamics applies to entanglement entropy then the entropy of a final hadronic state cannot be smaller than the entropy $S(x)$ accessed at a given Bjorken x. The correspondence between the number of partons in the initial state and the number of hadrons in the final state is in accord with the "parton liberation" and "local parton-hadron duality" pictures.

1. **The entropy is a useful measure of information that can be obtained in an experiment.**
2. **The entropic approach underlines the importance of measuring the hadronic final state of DIS.**
3. **We encourage experimentalists to combine the measurements of the DIS cross sections with the determination of hadronic final state at the future facilities.**
4. **The determination of the Shannon entropy of hadrons in the final state of DIS can be done using the event-by-event multiplicity measurements.**
5. **The "asymptotic" small x regime in which our formula begins at $x \leq 10^{-3}$. It is accessible to the current and planned experiments, and can be investigated at the future Electron-Ion Collider (EIC).**