

Emergence of hydrodynamics in expanding quark-gluon plasmas

**Collectivity and correlations in high-energy
hadron and nuclear collision**

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August 9, 2018

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Collective flow

Matter produced in HI collisions appears to flow like a fluid and is well described by relativistic hydrodynamics

$$\partial_{\mu} T^{\mu\nu} = 0 \quad \partial_{\mu} j^{\mu} = 0$$

Description in terms of a few local fields (energy density, pressure, baryon density, etc)

Effective theory for long wavelength modes

This description works amazingly well !

even in situations where, a priori, it should not ...

Fluid behavior requires (some degree of) local equilibration.
How is this achieved?

- microscopic degrees of freedom relax quickly towards local equilibrium
- long wavelength modes, associated to conservation laws, relax on longer time scales

Thermalization

Assume some form of kinetic description

$$f(\mathbf{x}, \mathbf{p}, t) \longrightarrow f_{\text{eq}}(\mathbf{x}, \mathbf{p}, t)$$

Two main issues

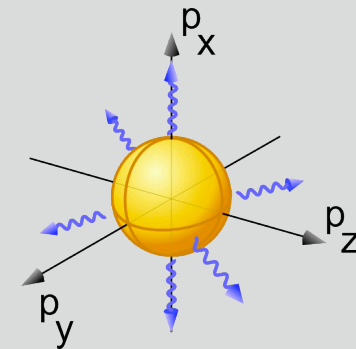
i) relative populations of different momentum modes

ii) isotropy of momentum distribution



Main topic for the rest of this talk

"Isotropization"

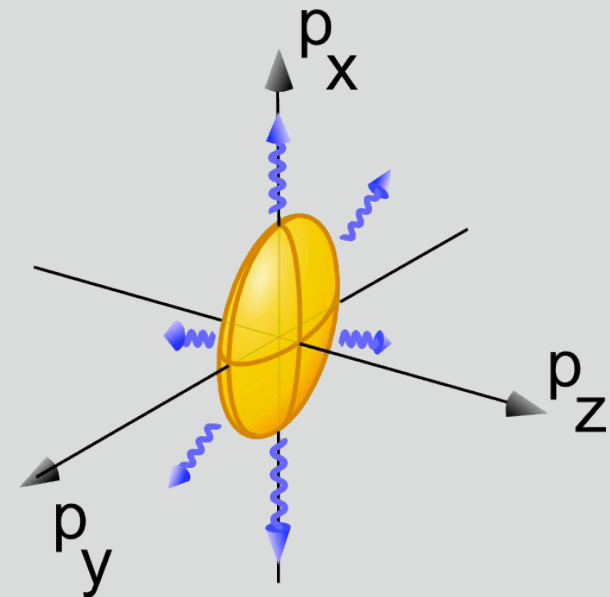


Longitudinal expansion hinders isotropization

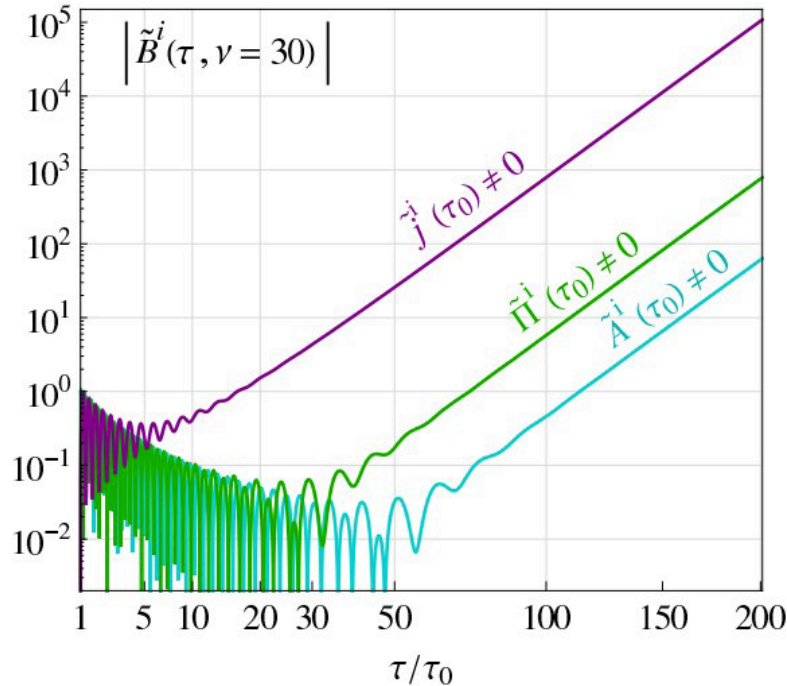
The fast expansion of the matter along the collision axis tends to drive the momentum distribution to a very flat distribution along the z direction

Translates into the existence of two different pressures

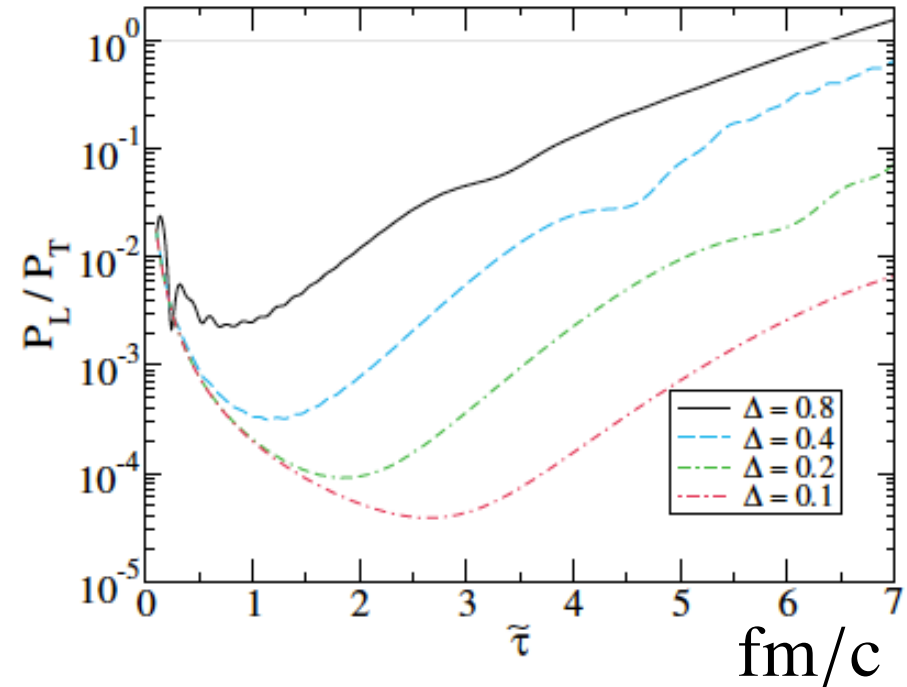
$$\begin{array}{cc} P_L & P_T \\ \text{(longitudinal)} & \text{(transverse)} \end{array}$$



Mean field instabilities may contribute to restore/maintain isotropy



(Rebhan, Steineder, 2011)

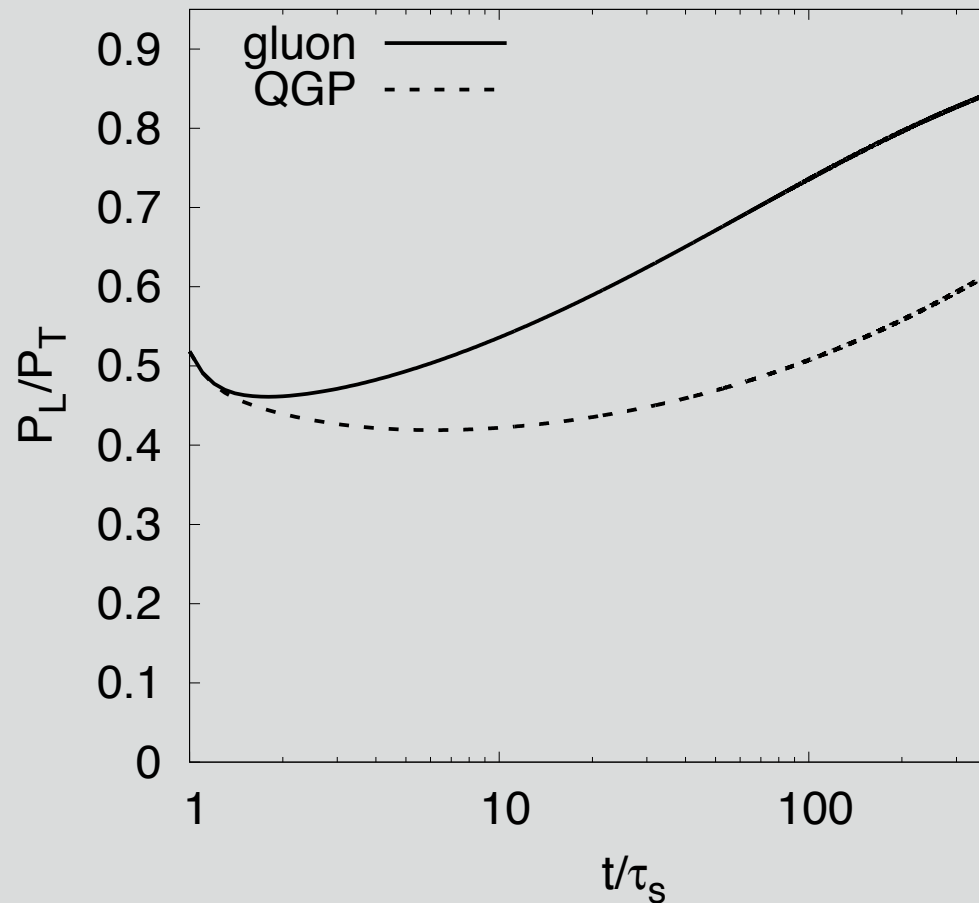


(Attems, Rebhan, Strickland, 2012)

Growth of instability is sensitive to initial fluctuations
(hard particles versus fields)

Collisions can restore/maintain isotropy, but not on small time scale

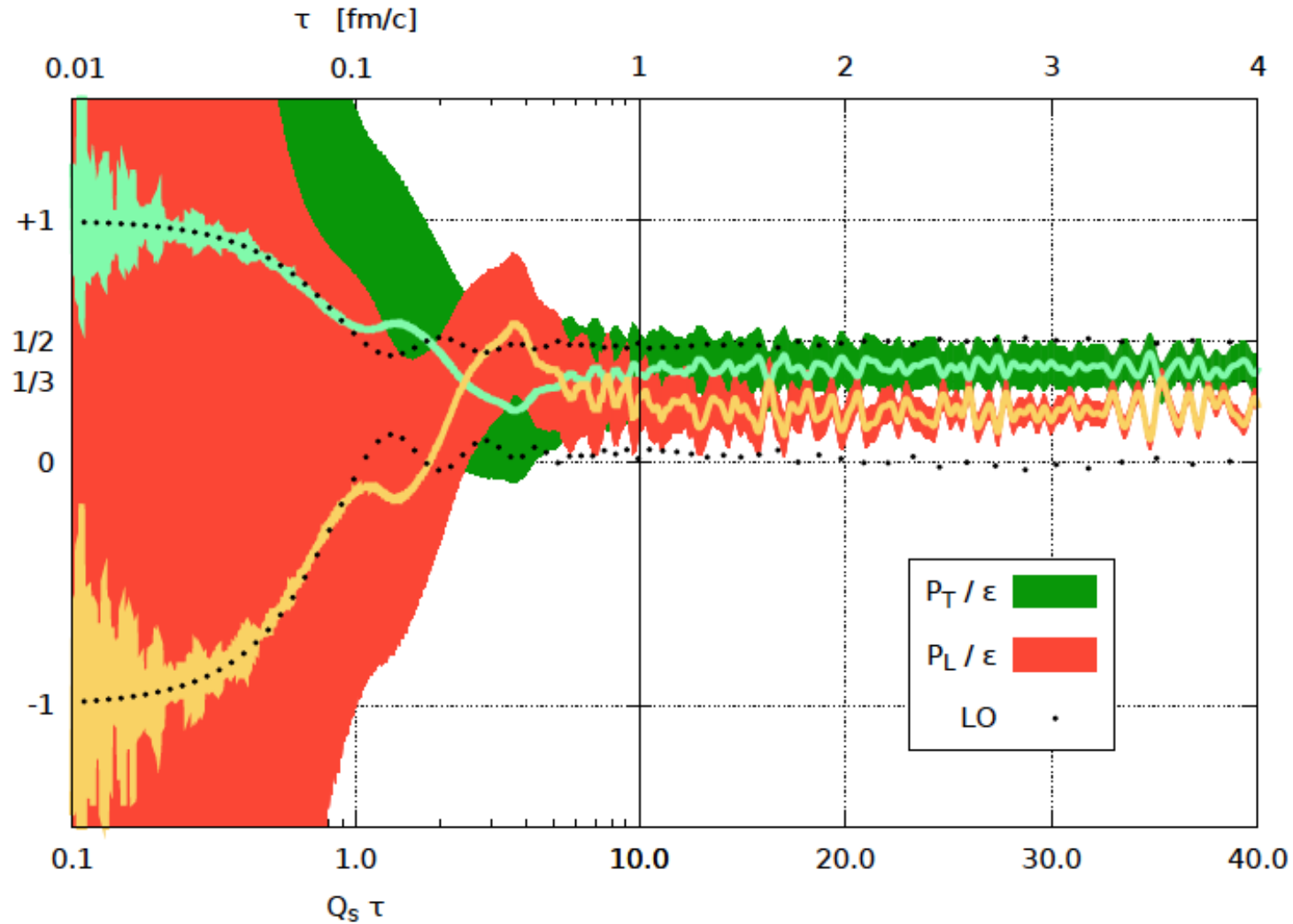
Solution of Boltzmann equation in small
scattering angle approximation



(JPB, Yan, 2017)

Statistical-classical field simulations

(Solving classical Yang-Mills equations and averaging over initial conditions)



T. Epelbaum and F. Gelis, PRL (2013)

"Hydrodynamization"

Hydrodynamic behavior may emerge
before local isotropization is achieved

Anisotropy relaxes slowly,
like a 'collective' variable

Ideal hydrodynamics of boost invariant systems

(Bjorken flow)

$$t = \tau \cosh y$$

$$z = \tau \sinh y$$

$$\tau = \sqrt{t^2 - z^2}$$

Energy-momentum tensor

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu}$$

In local rest frame

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_\perp & 0 & 0 \\ 0 & 0 & P_\perp & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix}$$

Equation of motion $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\tau(\tau\epsilon) = -P_L \quad \frac{\partial\epsilon}{\partial\tau} = -\frac{\epsilon + P_L}{\tau}$$

Three independent components ϵ, P_\perp, P_L

but: $\epsilon = 2P_\perp + P_L$
conformal symmetry

In local equilibrium $P_\perp = P_L = \epsilon/3$ **(equation of state)**

Then $\epsilon \sim \tau^{-4/3}$ $T \sim \tau^{-1/3}$ $(\epsilon \sim T^4)$
energy density **temperature**

viscous hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0 \quad T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P(\epsilon) g^{\mu\nu} + \pi^{\mu\nu}$$

(viscous corrections)

Ideal Hydro $\pi^{\mu\nu} = 0$

Navier-Stokes $\pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$ $P_\perp - P_L = \frac{\eta}{\tau}$ (gradient expansion)

(shear viscosity)

In boost invariant systems, the gradient expansion is an expansion in inverse powers of $w = \tau T$ ($T \sim \epsilon^{1/4}$)

$$\frac{1}{w} = \frac{1/T}{\tau} \sim \text{Knudsen number} \sim \frac{\text{micro}}{\text{macro}}$$

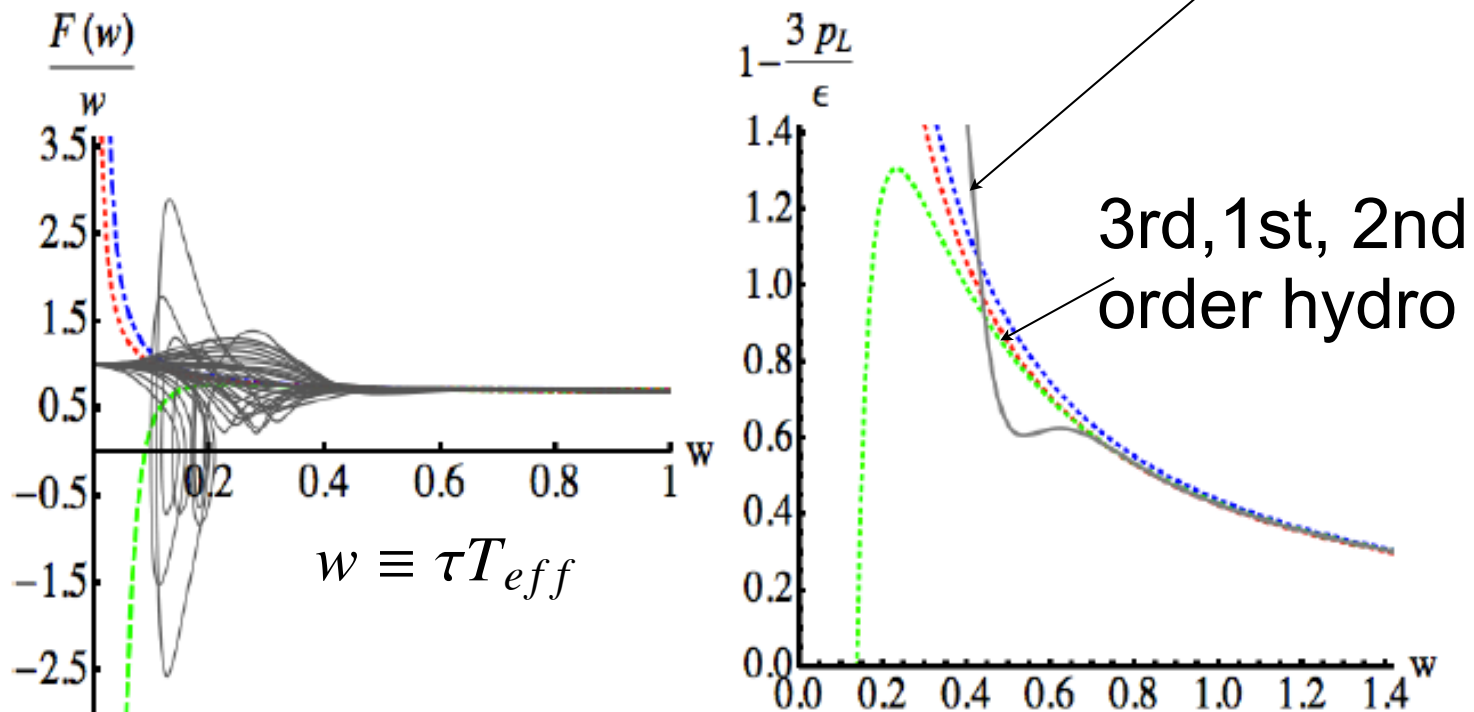
Coefficients of gradients = transport coefficients (e.g. shear viscosity)

Holographic description of a boost invariant plasma

(Heller, Janik, Witaszczyk, [1103.3452])

Define

$$\mathcal{R} = \frac{\mathcal{P}_T - \mathcal{P}_L}{\epsilon} \quad f \equiv \frac{2}{3} + \frac{\mathcal{R}}{6} = \sum_{n=0}^{\infty} f_n w^{-n}$$



Viscous hydro can cope with partial thermalization, and large differences between longitudinal and transverse pressures

The gradient expansion is divergent

$$f \equiv \frac{2}{3} + \frac{\mathcal{R}}{6} = \sum_{n=0}^{\infty} f_n w^{-n} \quad f_n \sim n!$$

f_n has been calculated up to $n=240$ (!) (Heller, Janik, Witaszczyk, 2013)

Sophisticated resummation yields a 'transseries'

(Heller, Spalinski, 2015)

$$f = \sum_{n=0}^{\infty} f_n w^{-n} + c e^{-\frac{3}{2C_{\tau\Pi}} w} \left(w^{\frac{C_{\eta} - 2C_{\lambda_1}}{C_{\tau\Pi}}} \sum_{n=0}^{\infty} f_n^{(1)} w^{-n} \right) + \dots$$

Similar features are observed in kinetic theory

(Heller, Kurkela, Spalinski, Svensson, 2016)

Simple kinetic equation

- Relaxation time approximation

$$\left[\partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] f(\mathbf{p}/T) = - \frac{f(\mathbf{p}, \tau) - f_{\text{eq}}(\mathbf{p}, \tau)}{\tau_R}$$

(derivative at constant $p_z \tau$)

- Free streaming (e.g. in absence of collisions)

$$f(t, \mathbf{p}) = f_0(\mathbf{p}_\perp, p_z t/t_0)$$

- Solved long ago by Baym [PLB 138 (1984) 18]

$$\epsilon(\tau) = e^{-(\tau-\tau_0)/\theta} \epsilon^{(0)}(\tau) + e^{-\tau/\theta} \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_R} e^{\tau'/\tau_R} \frac{\tau'}{\tau} \epsilon(\tau') h(\tau'/\tau)$$

(free streaming)

$$h(x) \equiv \int_0^1 d\mu \sqrt{1 - \mu^2 + \mu^2 x^2}$$

(angular integral)

New insight from a simpler approach

(JPB, Li Yan, 2017, 2018)

Special moments

$$p_z = p \cos \theta$$

$$\mathcal{L}_n \equiv \int_p p^2 P_{2n}(\cos \theta) f(\mathbf{p})$$

(Legendre polynomial)

$$P_0(z) = 1 \quad P_2(z) = \frac{1}{2}(3z^2 - 1)$$

Why moments ?

- There is too much information in the distribution function
- We want to focus on the angular degrees of freedom

The energy momentum tensor is described by first two moments

$$T^{\mu\nu} = \int_p f(\mathbf{p}) p^\mu p^\nu$$

$$\mathcal{L}_0 = \varepsilon$$

$$\mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

Isotropization

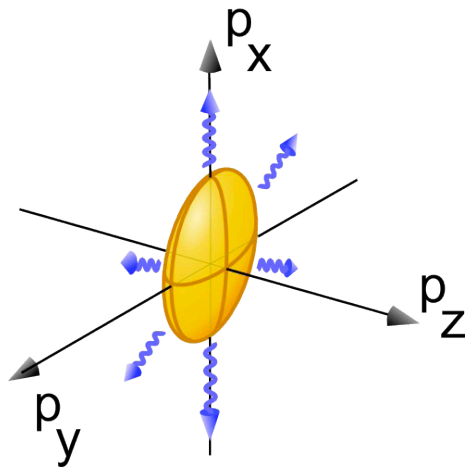
Two antagonistic agents

free streaming

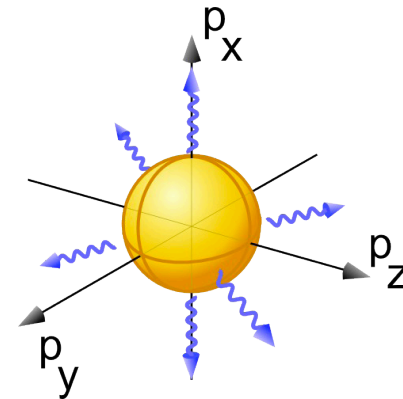
and

collisions

drives the momentum distribution
toward a flat distribution in the longitudinal direction



drive the momentum distribution
toward an isotropic distribution



Coupled equations for the moments

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = - \frac{1}{\tau} \left[\underbrace{a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}}_{\text{(Free streaming)}} \right] - \frac{\mathcal{L}_n}{\tau_R} \quad (n \geq 1)$$

(collisions)

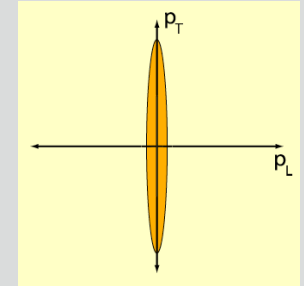
$$\frac{\partial \mathcal{L}_0}{\partial \tau} = - \frac{1}{\tau} [a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1] \quad a_0 = 4/3, \quad c_0 = 2/3$$

$$\mathcal{L}_0 = \varepsilon, \quad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

- The coefficients a_n, b_n, c_n are pure numbers
- The competition between expansion and collisions is made obvious
- Interesting system of coupled linear equations
- Emergence of hydrodynamics is transparent: equations for the lowest moments
- Provides much insight on various versions of viscous hydrodynamics

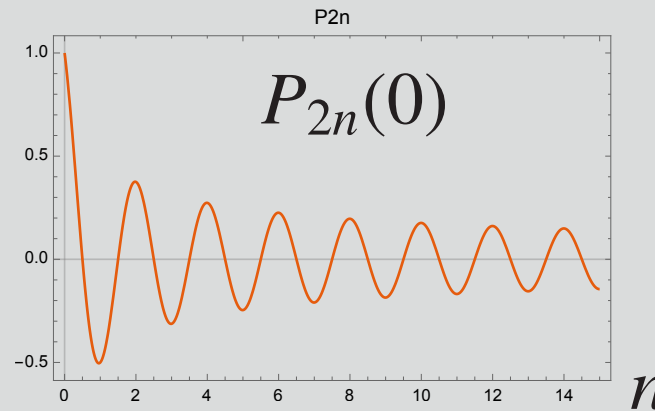
Free streaming solution

$$\mathcal{L}_n^{(0)}(t) = \frac{t_0}{t} \epsilon_0 \mathcal{F}_n \left(\frac{t_0}{t} \right)$$



Poor convergence: all moments are important at late time

$$\mathcal{F}_n(x \rightarrow 0) \rightarrow \frac{\pi}{4} P_{2n}(0)$$



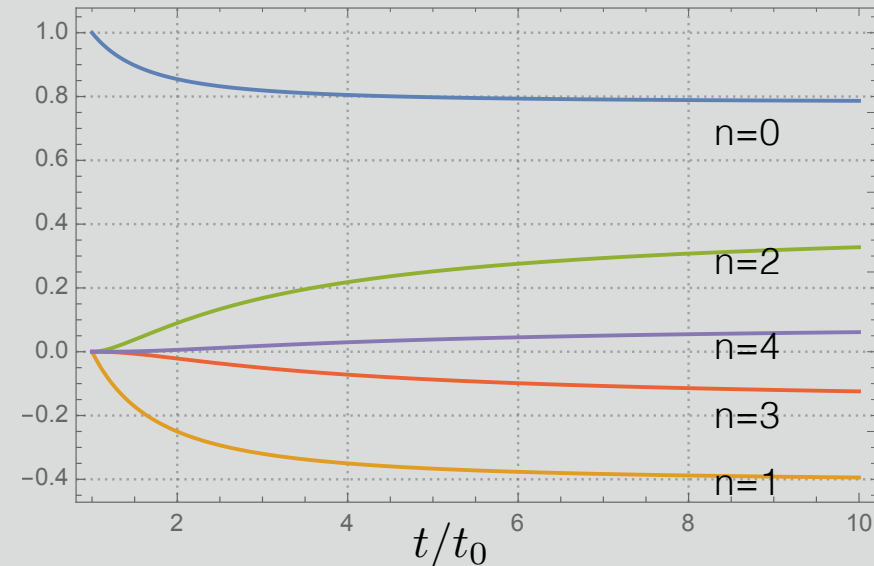
NB

$$g_n(\tau) \equiv \tau \partial_\tau \ln \mathcal{L}_n$$

$$g_n(\tau \rightarrow \infty) \rightarrow -1$$

But higher moments take time to grow

$$\mathcal{F}_n(t_0/t)$$



Free streaming

Simple truncations work

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau} [a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}]$$

$$\partial_t \vec{\mathcal{L}} = -M \vec{\mathcal{L}}$$

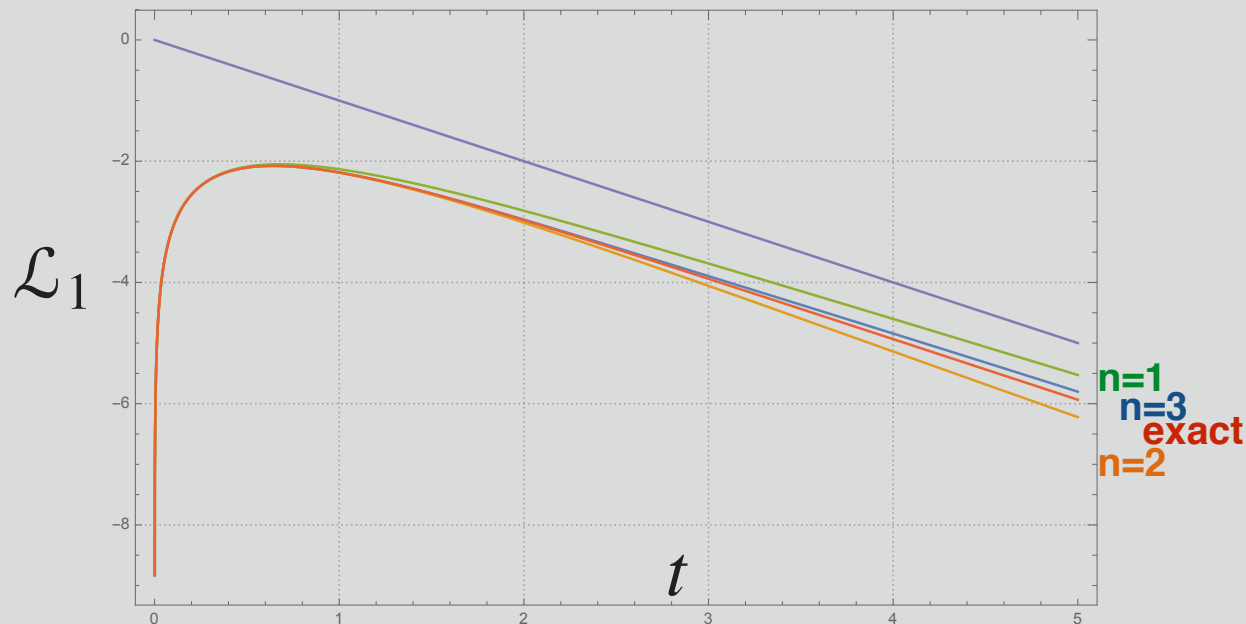
$(t = \ln(\tau/\tau_0))$

Keeping only the first two moments one gets

$$\frac{\partial}{\partial t} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \end{pmatrix} = - \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{8}{15} & \frac{38}{21} \end{pmatrix} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \end{pmatrix}$$

Two eigenmodes $\lambda_0 = 0.929366$ $\lambda_1 = 2.21349$

Truncations are reasonably accurate



Free streaming fixed point

One can transform the coupled linear equations into a single non linear differential equation

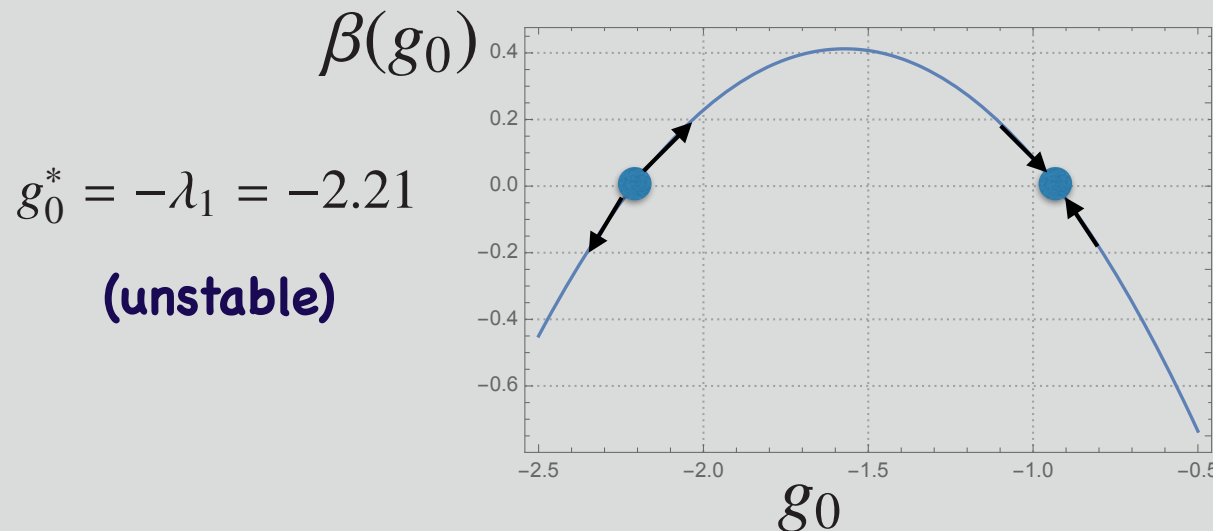
$$\tau \frac{dg_0}{d\tau} + g_0^2 + (a_0 + a_1)g_0 + a_0a_1 - c_0b_1 = 0,$$

$$g_n(\tau) \equiv \tau \partial_\tau \ln \mathcal{L}_n$$

Write this as

$$\tau \frac{dg_0}{d\tau} = \beta(g_0)$$

$$\beta(g_0) = -g_0^2 - (a_0 + a_1)g_0 - a_0a_1 + c_0b_1$$



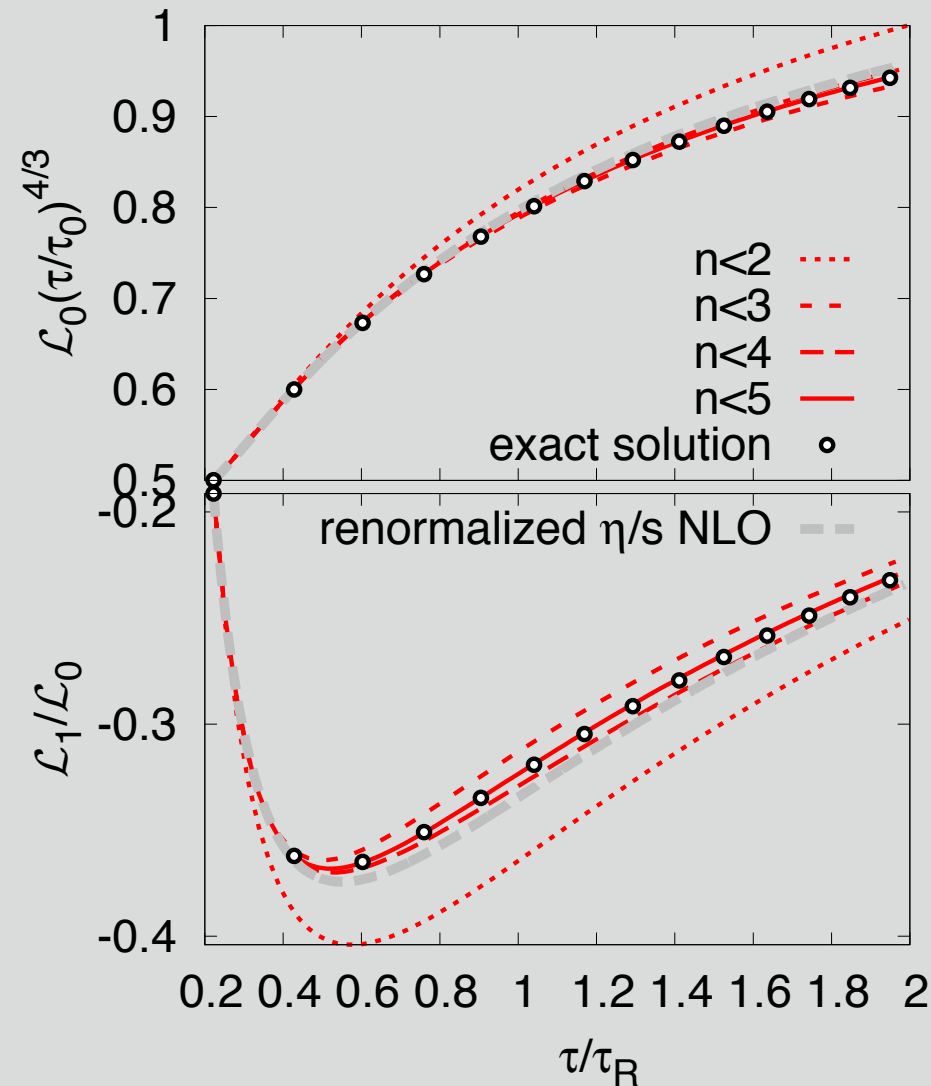
$$g_0^* = -\lambda_0 = -0.929$$

(stable)

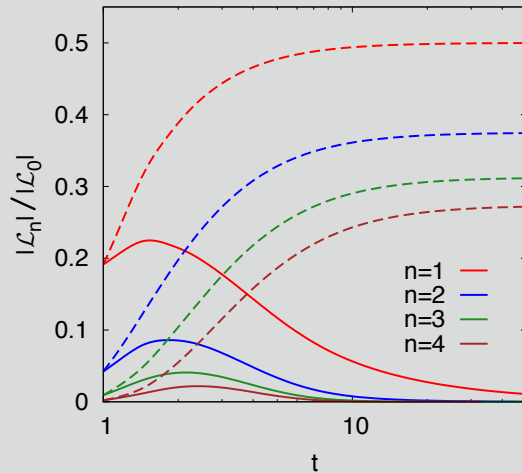
NB exact fixed point

$$g_n(\tau \rightarrow \infty) \rightarrow -1$$

Including collisions
Simple truncations work well
(drop all moments beyond a certain n)



First few moments are relevant



moments of free
streaming solution

damping of higher
moments by collisions

A simple model can be constructed

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau} (a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1)$$

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau} (b_1 \mathcal{L}_0 + a_1 \mathcal{L}_1) - \frac{1}{\tau_R} \mathcal{L}_1$$

**("Almost" viscous hydrodynamics...
in fact, better!)**

This simple equations capture much of the physics and illustrates the analytic features of the transition to viscous hydrodynamics

The hydrodynamic fixed point

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau} (a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1) \quad \partial_\tau \mathcal{L}_1 = -\frac{1}{\tau} (b_1 \mathcal{L}_0 + a_1 \mathcal{L}_1) - \frac{1}{\tau_R} \mathcal{L}_1$$

Complete damping of first moment

$$\tau \partial_\tau \mathcal{L}_0 = -a_0 \mathcal{L}_0 = -(4/3) \mathcal{L}_0 \quad \text{(hydro fixed point)}$$

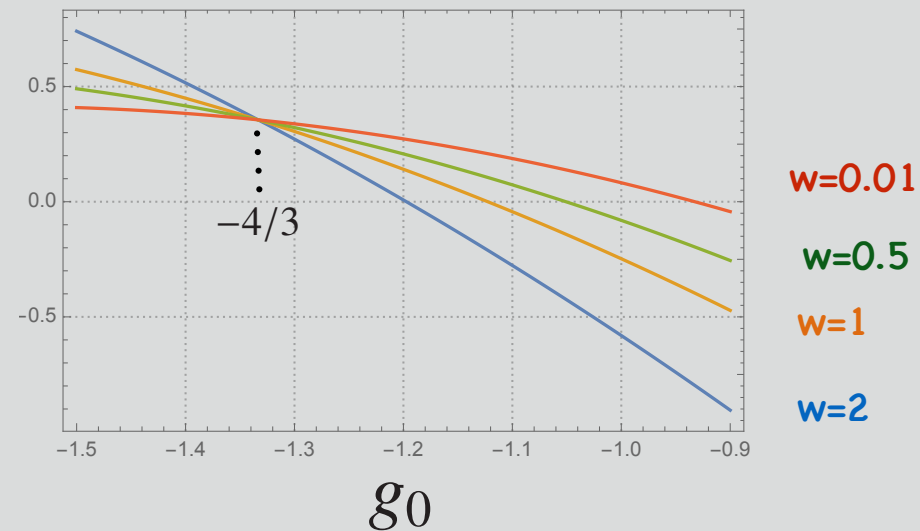
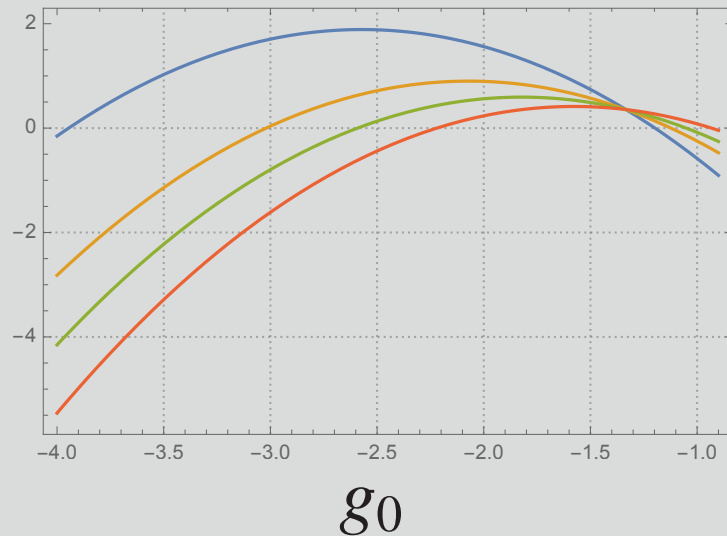
Modified equation of motion

$$w g_0' + g_0^2 + (a_0 + a_1 + w) g_0 + w a_0 + a_0 a_1 - b_1 c_0 = 0, \quad \tau_R = \text{Cste}$$

$$\frac{d g_0}{d \ln w} = \beta(g_0, w)$$

Pseudo fixed points

$$\beta(g_0, w)$$



The transition from free streaming to hydrodynamics

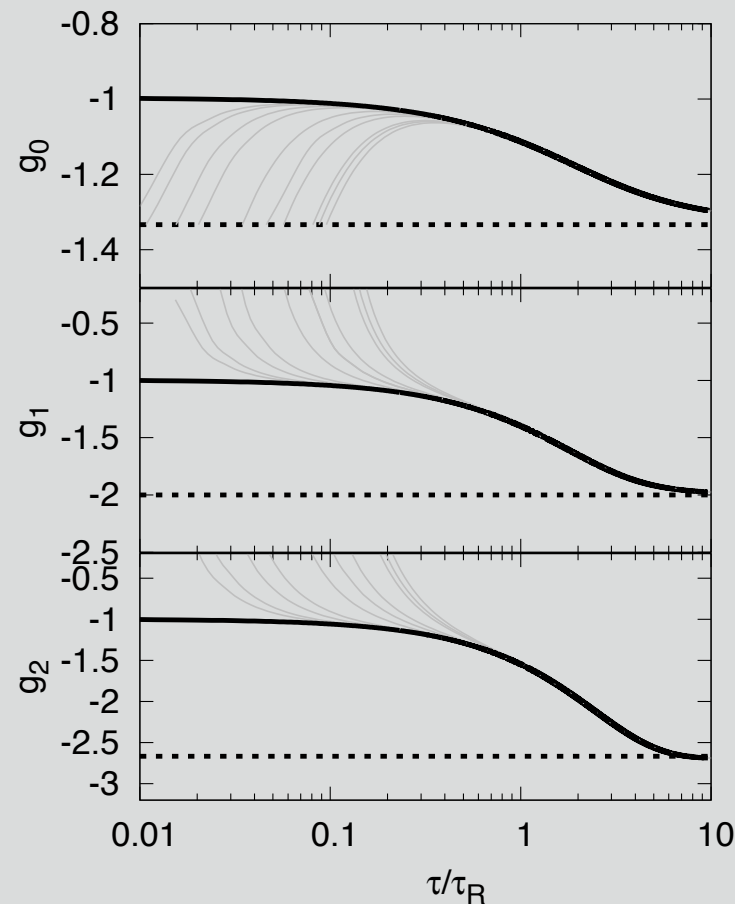
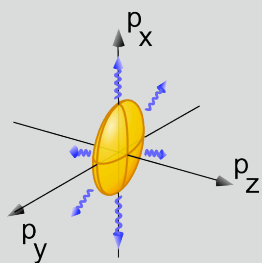
(Attractor solution)

Early and late times are controlled by the free streaming and the hydrodynamic fixed points, respectively

$$g_n(\tau) = \tau \partial_\tau \ln \mathcal{L}_n$$

Free streaming
fixed point

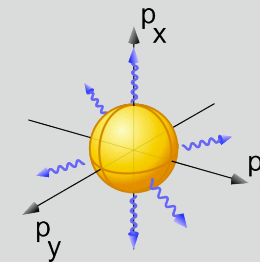
$$g_n = -1$$



Hydro fixed point

$$g_n = -\frac{4 + 2n}{3}$$

(Universal!)



Renormalization of the viscosity

The effect of higher moments can be viewed as a renormalisation of the equation for the lowest moments, i.e. of the viscous hydrodynamical equations

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau} (a_1 \mathcal{L}_1 + b_1 \mathcal{L}_0) - \left[1 + \frac{c_1 \tau_R}{\tau} \frac{\mathcal{L}_2}{\mathcal{L}_1} \right] \frac{\mathcal{L}_1}{\tau_R}$$

renormalisation of relaxation time,

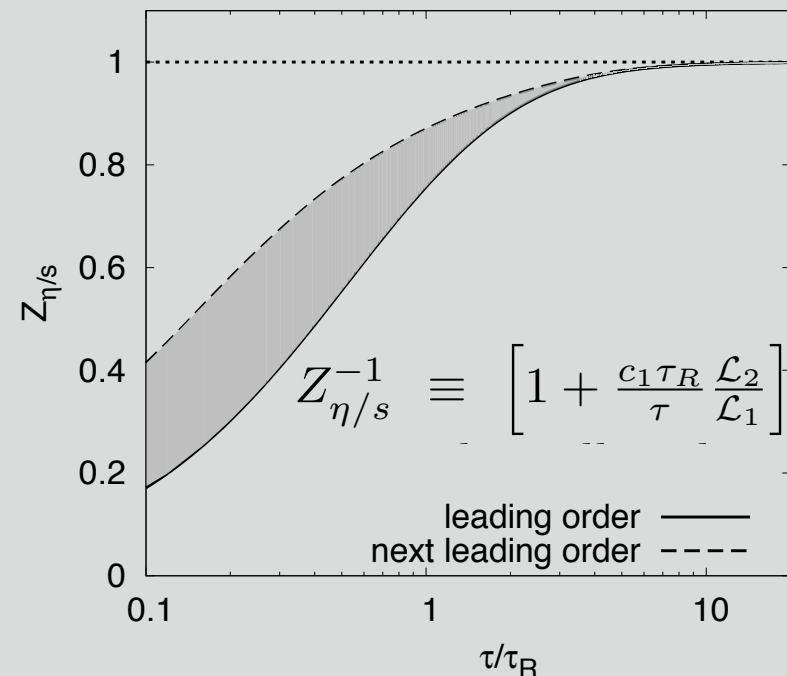
$$\tau_R \rightarrow Z_{\eta/s} \tau_R$$

or, equivalently, of the viscosity

$$\eta = \frac{4}{15} \tau_R \epsilon$$

[For an early suggestion of such an effect:
Lublinsky-Shuryak (2007)]

Sizeable reduction of the effective viscosity at early time



variants of viscous hydrodynamics

Different approximate ways to solve the equations for the first two moments

Navier Stokes

$$\partial_\tau \epsilon = -\frac{4}{3} \frac{\epsilon}{\tau} + \frac{\Pi}{\tau} \quad \Pi = \frac{4\eta}{3\tau} \quad \partial_\tau \epsilon = -\frac{4}{3} \frac{\epsilon}{\tau} + \frac{4\eta}{3\tau^2} \quad \Pi = -c_0 \mathcal{L}_1$$

Mueller-Israel-Steward

$$\partial_\tau \Pi = -\frac{\Pi}{\tau_\pi} + \frac{4\eta}{3\tau\tau_\pi} = -\frac{1}{\tau_\pi} \left(\Pi - \frac{4\eta}{3\tau} \right)$$

Second order hydro (DNMR)

$$\partial_\tau \Pi = \frac{4}{3} \frac{\eta}{\tau\tau_\pi} - \beta_{\pi\pi} \frac{\Pi}{\tau} - \frac{\Pi}{\tau_\pi} \quad \beta_{\pi\pi} = \frac{38}{21} = a_1 \quad \tau_\pi = \tau_R$$

same as $\frac{\partial \mathcal{L}_1}{\partial \tau} = -b_1 \frac{\epsilon}{\tau} - a_1 \frac{\mathcal{L}_1}{\tau} - \frac{\mathcal{L}_1}{\tau_R}$ provided $\frac{4}{3} \frac{\eta}{\tau\tau_\pi} = c_0 b_1 \frac{\epsilon}{\tau}$

which holds in leading order if $\tau_\pi = \tau_R$

Similar analysis can be made for BRSSS hydro (full second order, conformal), or third order (Jaiswal).

[DNMR= Denicol, Niemi, Molnar, Rischke (2012)]

[BRSSS= Baier, Romatschke, Son, Starinets, Stephanov (2008)]

Conclusions

In high energy collisions, the longitudinal expansion prevents the system to reach full isotropy in a short time (expansion plays a role somewhat similar to a conservation law...)

However strong anisotropy does not hinder the emergence of (viscous) hydrodynamic behavior

A simple picture based on special set of moments of the distribution functions provides much insight into the mathematical structure of viscous hydrodynamics of expanding (boost invariant) systems

Strong reduction of the viscosity at early times due to out of equilibrium effects (coupling to higher moments)

Coupled equations for the first few (two) moments could be a convenient alternative to viscous hydrodynamics