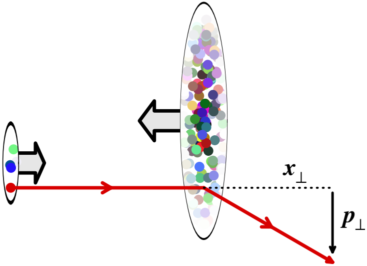
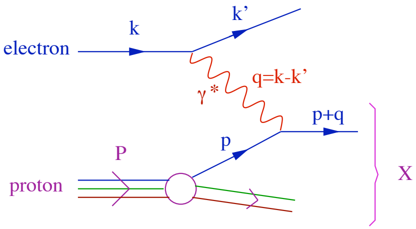
On the proper formulation of the high-energy QCD evolution beyond leading order

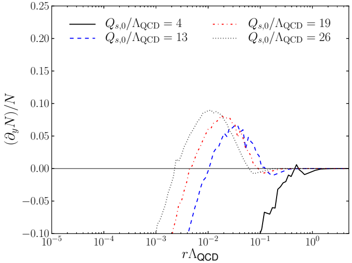
Edmond Iancu
IPhT, Université Paris-Saclay
w/ B. Ducloué, A.H. Mueller, G.Soyez, and D.N. Triantafyllopoulos



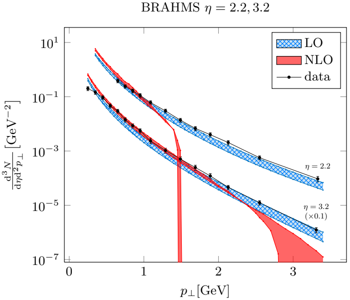


Introduction

- The CGC formalism is now being promoted to NLO
  - NLO versions for the BK and B-JIMWLK equations
    (Balitsky and Chirilli, 2008, 2013; Kovner, Lublinsky, and Mulian, 2013)
  - NLO impact factor for particle production in pA collisions
    (Chirilli, Xiao, and Yuan, 2012; Mueller and Munier, 2012)
- The strict NLO approximations turned out to be problematic



Lappi, Mäntysaari, arXiv:1502.02400



Stasto, Xiao, and Zaslavsky, arXiv:1307.4057

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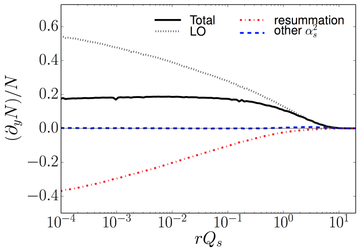
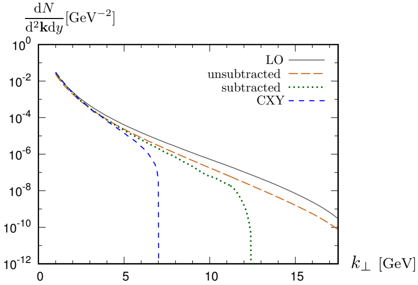


Lappi, Mäntysaari, arXiv:1601.06598



Ducloué, Lappi, and Zhu, arXiv:1703.04962

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  (G. Beuf, '14; E.I., J. Madrigal, A. Mueller, G. Soyez, D. Triantafyllo..., '15)
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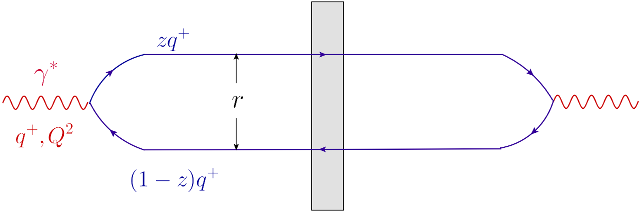
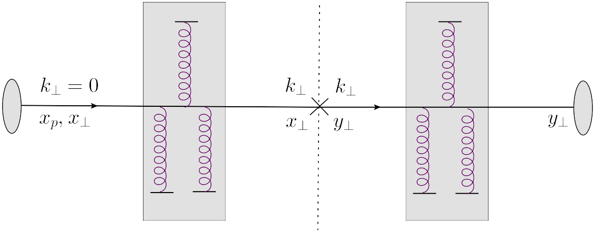
- All-order resummed kernel, full NLO accuracy, stable evolution ☺
- Incomplete resummation of the initial condition: no full access to physics ☹
- A new approach which avoids the need for resumming the initial condition
  - non-local evolution in the target rapidity variable
- Recovering full NLO accuracy looks (still) problematic ☹

Motivation: Dilute-Dense Scattering

- Deep inelastic scattering in the dipole picture

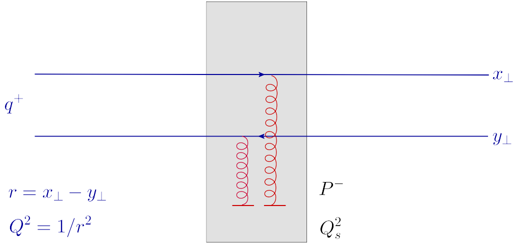


- Forward particle production in proton-nucleus collisions at the LHC



Dipole–hadron scattering

- A single scattering (leading-twist) $\Longrightarrow$ a measure of the gluon density



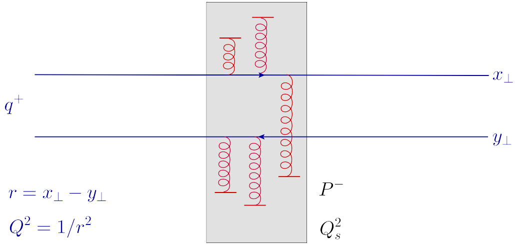
$$T(r,x) \simeq \alpha_s C_F\, r^2\, \frac{xG(x,1/r^2)}{\pi R^2} \simeq r^2\, Q_s^2(x)$$

  - correct so long as $T(r,x) \leq 1$: unitarity constraint
  - for large dipoles, $r \gtrsim 1/Q_s(x)$, multiple scattering becomes important

- 'Duality': gluon saturation $n \sim 1/\alpha_s \longleftrightarrow$ unitarization $T \sim 1$

Multiple scattering (all-twist)

- Transverse coordinate is a "good quantum number": $v_\perp = k_\perp/q^+ \ll 1$



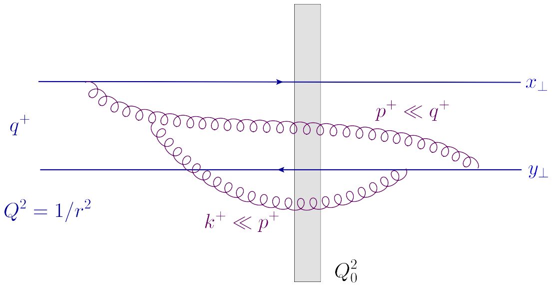
- The only effect of the scattering: a color rotation (Wilson line)

$$V_{\boldsymbol{x}} = \mathrm{T}\exp\left\{\mathrm{i}g\int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x})t^a\right\} \qquad S_{\boldsymbol{xy}}(Y) \equiv \frac{1}{N_c}\left\langle \mathrm{tr}\left(V_{\boldsymbol{x}}V_{\boldsymbol{y}}^\dagger\right)\right\rangle_Y$$

- $\langle \dots \rangle_Y$: average over the target gluon distribution evolved to $Y = \ln(1/x)$
- Dipole scattering amplitude: $T(r, x) = 1 - S_{\boldsymbol{xy}}(Y)$ with $r = |\boldsymbol{x} - \boldsymbol{y}|$

High energy evolution at LO

- The evolution can be associated with either the wavefunction of the dilute projectile ("BK"), or the dense gluon distribution of the target ("JIMWLK")
- The analysis is conceptually simpler for the dilute projectile

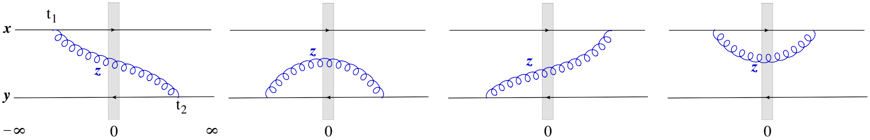


$$Y = \ln \frac{q^+}{k^+} = \ln \frac{1}{x}\,, \qquad x_{\min} = \frac{Q_0^2}{2P^- q^+} = \frac{Q_0^2}{s} \ll 1$$

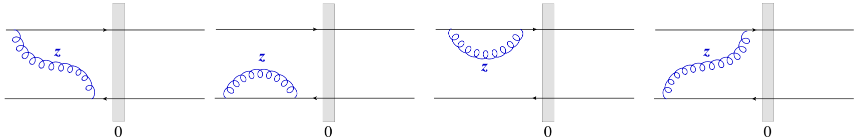
- Leading logarithmic approximation: powers of $(\alpha_s N_c/\pi)\ln(s/Q_0^2)$

One step in the high energy evolution

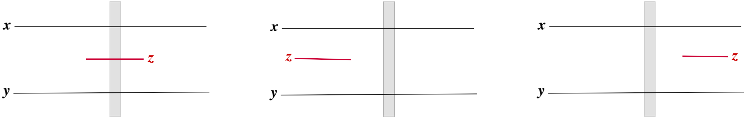
- 'Real corrections' : the soft gluon crosses the shockwave



- 'Virtual corrections' : evolution in the initial/final state



- Large N_c : the original dipole splits into two new dipoles



The BK equation *(Balitsky, '96; Kovchegov, '99)*

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \big[S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}} \big]$$

- Dipole kernel: BFKL kernel in the dipole picture *(Al Mueller, '90)*

$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2} \right]^2$$

- Color transparency: $\mathcal{M}_{\boldsymbol{xyz}} \propto r^2$, hence $S_{\boldsymbol{xy}} \to 1$ when $r \to 0$
- Ultraviolet-safe: one very small daughter dipole
  - the singularities in the kernel cancel between 'real' and 'virtual'

$$S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} \;\to\; S_{\boldsymbol{xy}} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \to 0 \text{ or } |\boldsymbol{z}-\boldsymbol{y}| \to 0$$

- Unitarity bound: non-linear equation for $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$
  - fixed point $T = 1$, saturation scale: $T(Y, r) \to 1$ when $r \gtrsim 1/Q_s(Y)$

Double logarithmic approximation

- Infrared behavior: very large daughter dipoles ("hard-to-soft")

$$\mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

  - the kernel is rapidly decreasing: good convergence ?

- Not necessarily ! For weak scattering $(T \ll 1)$, this can be compensated by the rapid growth of the amplitude with the dipole size: $T(Y, r) \propto r^2$

$$T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} \simeq 2T_{\boldsymbol{xz}} \equiv 2(\boldsymbol{z}-\boldsymbol{x})^2 Q_0^2 \, \mathcal{A}_{\boldsymbol{xz}}$$

- Logarithmic phase-space for $z_\perp$, at $r^2 \ll (\boldsymbol{z}-\boldsymbol{x})^2 \ll 1/Q_0^2$

$$\frac{\partial \mathcal{A}(Y, r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2} \, \mathcal{A}(Y, z^2)$$

- Large transverse separation: $Q^2 = 1/r^2 \gg Q_0^2$, or $\rho \equiv \ln(Q^2/Q_0^2) > 1$

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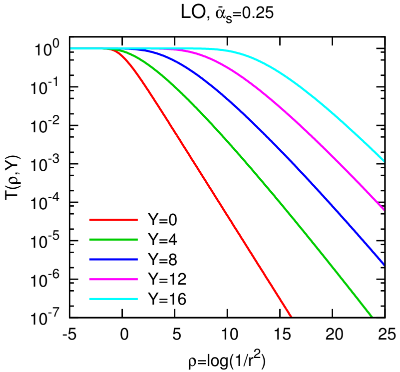
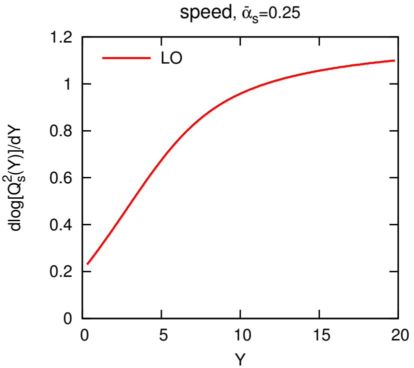
- Double-logarithmic correction $\sim \mathcal{O}(\bar{\alpha}_s Y \rho)$: energy log $\times$ collinear log

The saturation front

- $T(Y,r)$ as a function of $\rho \equiv \ln(1/r^2Q_0^2)$ with increasing Y





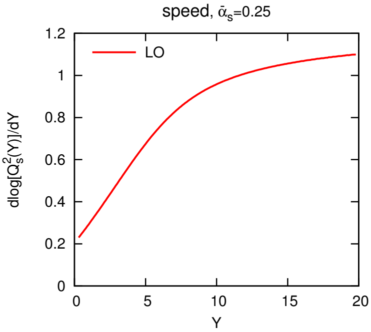
$$T(Y,r) = \begin{cases} 1 & \text{for } rQ_s \gtrsim 1 \\ \left(r^2Q_s^2(Y)\right)^{\gamma_s} & \text{for } rQ_s \ll 1 \end{cases},$$

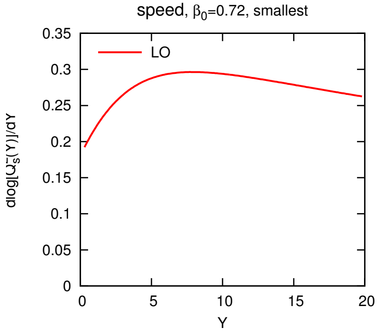
$$Q_s^2(Y) \simeq Q_0^2\, \mathrm{e}^{\lambda_s Y}$$
$$\lambda_s \simeq 4.88\bar{\alpha}_s$$
$$\gamma_s \simeq 0.63$$

- DLA regime unimportant, due to "anomalous dimension" $1-\gamma_s \simeq 0.37$

Adding running coupling: rcBK

- Saturation exponent: $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$ for $Y \gtrsim 5$: much too large
  - phenomenology requires a much smaller valuer $\lambda_s \simeq 0.2 \div 0.3$
- Including running coupling dramatically slows down the evolution

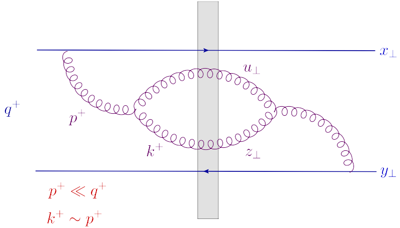


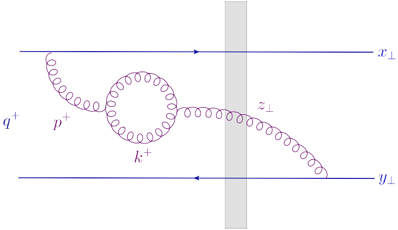


- Rather successful phenomenology based on rcBK
- ... but what about the other NLO corrections ?

Next-to-leading order

- Any effect of $\mathcal{O}(\bar{\alpha}_s^2 Y) \Longrightarrow \mathcal{O}(\bar{\alpha}_s)$ correction to the r.h.s. of BK eq.





- The prototype: two successive, soft, emissions, with similar longitudinal momentum fractions: $p^+ \sim k^+ \ll q^+$
- Exact kinematics (full QCD vertices, as opposed to eikonal)
- Typically: two transverse momentum convolutions: $\boldsymbol{u}_\perp, \boldsymbol{z}_\perp$
- New color structures, up to 3 dipoles at large N_c
- NLO BFKL: *Fadin, Lipatov, Camici, Ciafaloni ... 95-98*

BK equation at NLO *Balitsky, Chirilli (arXiv:0710.4330)*

$$
\begin{aligned}
\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar\alpha_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, &(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})\Bigg\{1+ \\
&+ \bar\alpha_s\Bigg[\bar b\, \ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar b\, \frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2} \\
&+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\Bigg]\Bigg\} \\
+ \frac{\bar\alpha_s^2}{8\pi^2}\int \frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\, &(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}) \\
&\Bigg\{-2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \\
&+ \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}
\end{aligned}
$$

- green : leading-order (LO) terms
- violet : running coupling corrections
- blue : single collinear logarithm (DGLAP)
- red : double collinear logarithm : troublesome !

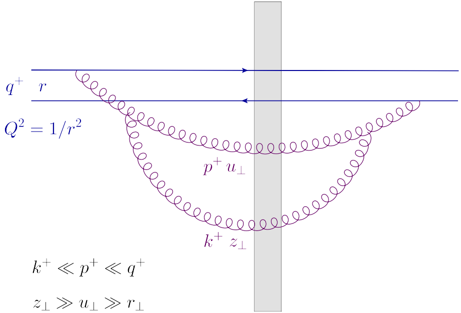
Collinear logarithms

- Important in the "hard-to-soft" evolution (large daughter dipoles, $T \ll 1$)

$$-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \simeq -\frac{1}{2}\ln^2\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2} \quad \text{if} \quad |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

- The single logs are still hidden: needs to perform the integral over $\boldsymbol{u}$

$$1/Q_s \gg |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \simeq |\boldsymbol{z}-\boldsymbol{u}| \gg |\boldsymbol{u}-\boldsymbol{x}| \simeq |\boldsymbol{u}-\boldsymbol{y}| \gg r$$



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- Keeping just the collinear logarithms ($|\boldsymbol{z}-\boldsymbol{x}| \to z$):

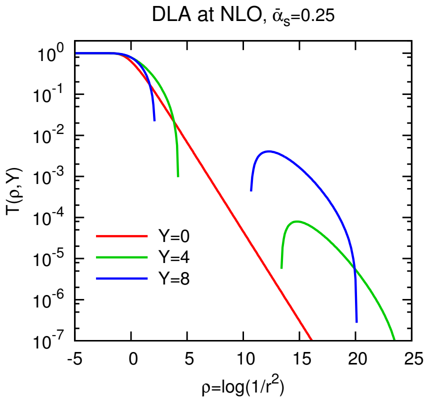
$$\frac{\partial T(Y,r)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \mathrm{d}z^2\, \frac{r^2}{z^4}\left\{1-\bar{\alpha}_s\left(\frac{1}{2}\ln^2\frac{z^2}{r^2}+\frac{11}{12}\ln\frac{z^2}{r^2}\right)\right\}T(Y,z)$$

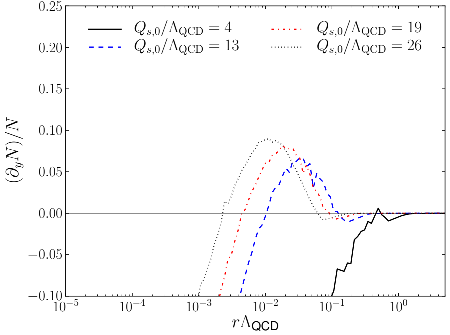
- Write $T(Y,z) \equiv z^2 Q_0^2\, \mathcal{A}(Y,z^2)$ and chose $\mathcal{A}(Y=0,z^2) \to 1$ (GBW)

$$\Delta\mathcal{A}(Y,r^2) \;=\; \bar{\alpha}_s Y \rho \left(1-\frac{\bar{\alpha}_s}{6}\rho^2\right)$$

- This can be negative if $\rho = \ln(Q^2/Q_0^2)$ is large enough.

Unstable numerical solution

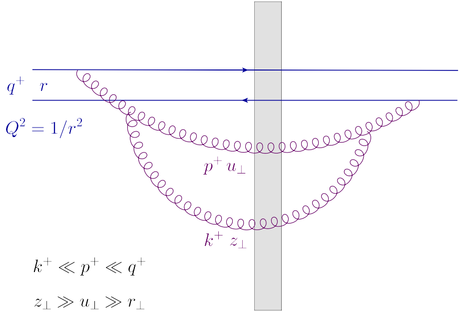




- Left: LO BK + the double collinear logarithm
- Right: full NLO BK *(Lappi, Mäntysaari, arXiv:1502.02400)*
- The main source of instability: the double collinear logarithm

Time ordering & Double collinear logs

- Successive emissions must be ordered in lifetimes
- This condition can be violated by the LO evolution "hard-to-soft"



- lifetime $\approx$ energy denominator

$$\Delta t \simeq \frac{1}{\Delta E} \simeq \frac{1}{p^- + k^-}$$

- light-cone energies

$$p^- = \frac{p_\perp^2}{2p^+} \simeq \frac{1}{p^+ u_\perp^2}$$

- Integrate out the harder gluon $(p^+, u_\perp)$ to double-log accuracy:

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \frac{p^+ u^2}{p^+ u^2 + k^+ z^2}$$

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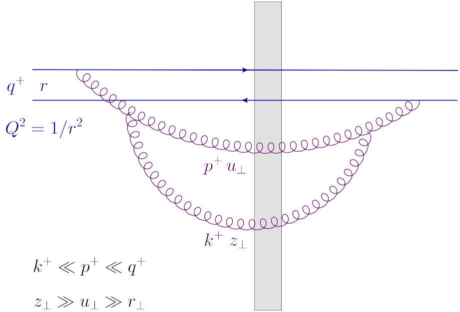

- lifetime ≈ energy denominator

$$\Delta t \simeq \frac{1}{\Delta E} \simeq \frac{1}{p^- + k^-}$$

- to have double logs, one needs

$$\tau_p \simeq p^+ u_\perp^2 \gg \tau_k \simeq k^+ z_\perp^2$$

- Integrate out the harder gluon $(p^+, u_\perp)$ to double-log accuracy:

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \Theta(p^+u^2 - k^+z^2) = \bar{\alpha}_s Y \ln\frac{z^2}{r^2} - \frac{\bar{\alpha}_s}{2} \ln^2 \frac{z^2}{r^2}$$

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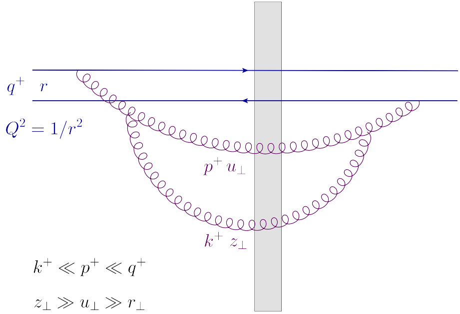

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- Naive LO: one writes $\Theta(p^+u^2 - k^+z^2) = 1 - \Theta(k^+z^2 - p^+u^2)$

  ... and one expands $\Theta(k^+z^2 - p^+u^2)$ in perturbation theory

Enforcing time-ordering

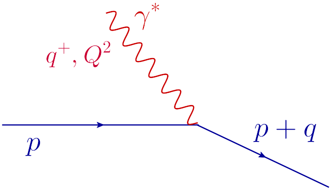
- A genuine instability: bad organization of the perturbation theory
- Can be cured by enforcing time-ordering in the evolution equation
  - systematic resummation of the double-collinear logs to all orders
  - straightforward/unique at DLA level, more subtle for BFKL/BK
- This amounts to reducing the phase-space for the projectile evolution
- $Y = \ln \frac{q^+}{q_0^+}$ with q_0^+ determined by a lifetime condition ("Ioffe's time"):

$$\Delta x^+ = \frac{2k^+}{k_\perp^2} \gtrsim \frac{1}{P^-} \quad \& \quad k_\perp^2 > Q_0^2 \quad \Longrightarrow \quad \frac{2q_0^+}{Q_0^2} \sim \frac{1}{P^-} \quad \Longrightarrow \quad Y_{\max} = \ln \frac{2q^+P^-}{Q_0^2}$$

- But the harder gluons with say $k^+ \sim q_0^+$ but $k_\perp^2 \gg Q_0^2$ can violate this condition $\Delta x^+ > 1/P^- \implies$ need for explicit time-ordering

Enforcing time-ordering

- A genuine instability: bad organization of the perturbation theory
- Can be cured by enforcing time-ordering in the evolution equation
  - systematic resummation of the double-collinear logs to all orders
  - straightforward/unique at DLA level, more subtle for BFKL/BK
- This amounts to reducing the phase-space for the projectile evolution
- The correct rapidity phase-space follows from the DIS kinematics



- final quark must be on-shell

$$0 = (p+q)^2 = 2p\cdot q - Q^2$$

- incoming quark collinear with the proton

$$p^\mu = \xi P^- \delta^{\mu -} \;\Rightarrow\; \xi = \frac{Q^2}{s} \equiv x_{\rm Bj}$$

$$\eta \equiv \ln\frac{P^-}{p^-} = \ln\frac{1}{x_{\rm Bj}} = Y - \ln\frac{Q^2}{Q_0^2}$$

- Time-ordering (x^-) is automatic for the soft-to-hard evolution in p^-

Resummation of double logs in DLA

- Start with the "naive" DLA equation in integral form

$$\mathcal{A}(q^+,Q^2) \,=\, \mathcal{A}_0(Q^2) + \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \int_{q_0^+}^{q^+} \frac{\mathrm{d}k^+}{k^+}\, \mathcal{A}(k^+,k_\perp^2)$$

- Enforce time-ordering for the intermediate gluon $(k^+,\boldsymbol{k})$:

$$\frac{2q_0^+}{Q_0^2} \ll \frac{2k^+}{k_\perp^2} \ll \frac{2q^+}{Q^2} \quad\Longrightarrow\quad q_0^+\,\frac{k_\perp^2}{Q_0^2} \ll k^+ \ll q^+\,\frac{k_\perp^2}{Q^2}$$

Resummation of double logs in DLA

- Time-ordered (correct) version of the DLA equation

$$\mathcal{A}(q^+, Q^2) \,=\, \mathcal{A}_0(Q^2) + \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \int_{q_0^+ \frac{k_\perp^2}{Q^2}}^{q^+ \frac{k_\perp^2}{Q_0^2}} \frac{\mathrm{d}k^+}{k^+}\, \mathcal{A}(k^+, k_\perp^2)$$

- Enforce time-ordering for the intermediate gluon $(k^+, \boldsymbol{k})$:

$$\frac{2q_0^+}{Q_0^2} \ll \frac{2k^+}{k_\perp^2} \ll \frac{2q^+}{Q^2} \quad \Longrightarrow \quad q_0^+ \frac{k_\perp^2}{Q_0^2} \ll k^+ \ll q^+ \frac{k_\perp^2}{Q^2}$$

Resummation of double logs in DLA

- Time-ordered (correct) version of the DLA equation

$$\mathcal{A}(Y,\rho) = \mathcal{A}_0(\rho) + \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \int_{\rho_1}^{Y-\rho+\rho_1} \mathrm{d}Y_1\, \mathcal{A}(Y_1,\rho_1)$$

- Logarithmic variables : $Y = \ln(q^+/q_0^+)$, $\rho = \ln(Q^2/Q_0^2)$
- $Y \geq \rho$ and $\mathcal{A}_0(\rho) = \mathcal{A}(Y=\rho,\rho) \Longrightarrow$ a boundary value problem
- Non-local in Y ...

Resummation of double logs in DLA

- Time-ordered (correct) version of the DLA equation

$$\mathcal{A}(Y,\rho) = \mathcal{A}_0(\rho) + \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \int_{\rho_1}^{Y-\rho+\rho_1} \mathrm{d}Y_1\, \mathcal{A}(Y_1,\rho_1)$$

- Logarithmic variables : $Y = \ln(q^+/q_0^+),\ \rho = \ln(Q^2/Q_0^2)$
- $Y \geq \rho$ and $\mathcal{A}_0(\rho) = \mathcal{A}(Y=\rho,\rho) \Longrightarrow$ a boundary value problem
- Non-local in Y ... but local in the target rapidity $\eta \equiv Y - \rho$:

$$\bar{\mathcal{A}}(\eta,\rho) = \mathcal{A}_0(\rho) + \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \int_0^{\eta} \mathrm{d}\eta_1\, \bar{\mathcal{A}}(\eta_1,\rho_1)$$

- A local, initial-value problem for $\bar{\mathcal{A}}(\eta,\rho) \equiv \mathcal{A}(Y=\eta+\rho,\rho)$
- This is of course the standard DLA evolution of the target
- Why bother to work with projectile evolution ? ... Because of NLO !

Getting local

- Two strategies for going from time-ordered DLA to BK
  - time-ordered BK equation $\Rightarrow$ non-local in Y *(G. Beuf, 14; see below)*
  - local equation, but with resummed kernel and initial condition *(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, 2015)*
- DLA: the non-local boundary value problem can be equivalently rewritten as a local, initial-value problem for an analytic continuation down to $Y = 0$:

$$\tilde{\mathcal{A}}(Y,\rho) = \tilde{\mathcal{A}}(0,\rho) + \bar{\alpha}_s \int_0^Y \mathrm{d}Y_1 \int_0^\rho \mathrm{d}\rho_1\, \mathcal{K}_{\rm DLA}(\rho-\rho_1)\tilde{\mathcal{A}}(Y_1,\rho_1)$$

- ... where $\mathcal{K}_{\rm DLA}(\rho)$ resums powers of $\bar{\alpha}_s\rho^2$ to all orders:

$$\mathcal{K}_{\rm DLA}(\rho) \equiv \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)}{\sqrt{\bar{\alpha}_s\rho^2}} = 1 - \frac{\bar{\alpha}_s\rho^2}{2} + \frac{(\bar{\alpha}_s\rho^2)^2}{12} + \cdots$$

- The physical amplitude at DLA: $\mathcal{A}(Y,\rho) = \tilde{\mathcal{A}}(Y,\rho)$ when $Y \geq \rho$

Collinearly improved BK

- The extension of the local equation to BK happens to be straightforward:

$$\frac{\partial \tilde{S}_{\boldsymbol{xy}}}{\partial Y} = \bar{\alpha}_s \int \frac{\mathrm{d}^2\boldsymbol{z}}{2\pi} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \mathcal{K}_{\rm DLA}\big(\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big) \big(\tilde{S}_{\boldsymbol{xz}}\tilde{S}_{\boldsymbol{zy}} - \tilde{S}_{\boldsymbol{xy}}\big)$$

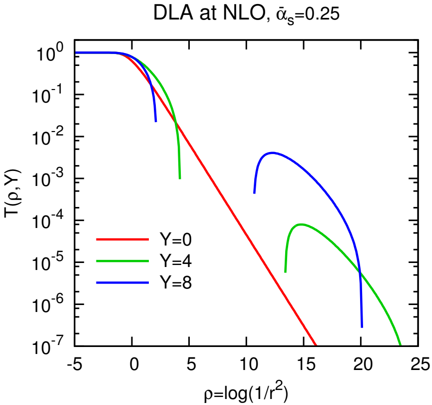
- The argument $\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ of $\mathcal{K}_{\rm DLA}$ is well tuned:

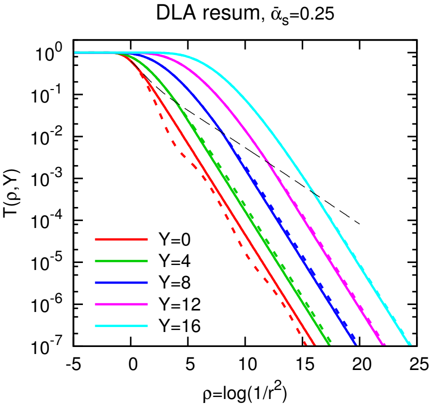
$$\rho^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

- $\mathcal{O}(\bar{\alpha}_s\rho^2)$ contribution to $\mathcal{K}_{\rm DLA}$ exactly matches the NLO double collinear log
- Adding all the other NLO corrections is (in principle) straightforward
- The solution gives the physical dipole S-matrix only for $Y \geq \rho$
- It yields the physical saturation fronts when expressed in terms of $\eta \equiv Y - \rho$

Numerical solutions: collBK

- The resummation stabilizes and slows down the evolution

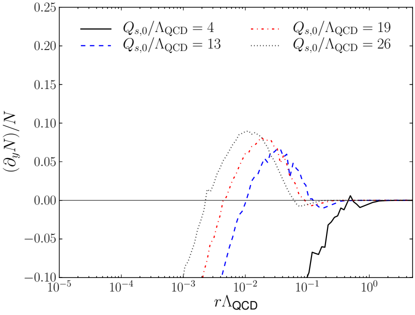




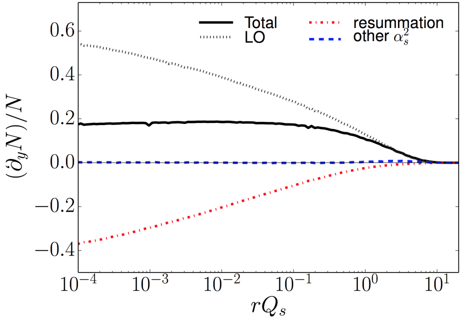
- left: the NLO double-log alone
- right: double collinear logs resummed to all orders

Numerical solutions: collBK + NLO corrections

- The resummation stabilizes and slows down the evolution



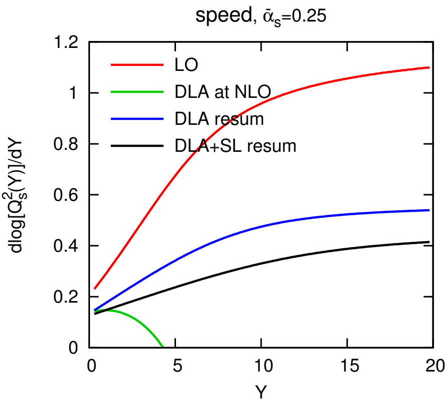
Lappi, Mäntysaari, arXiv:1502.02400



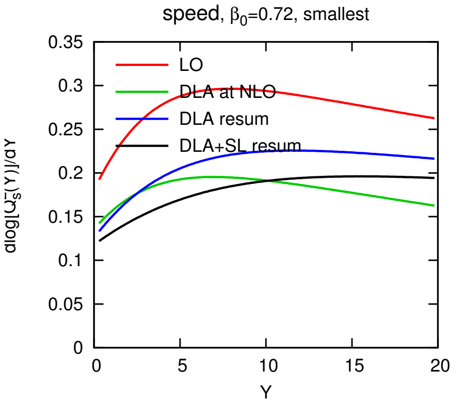
Lappi, Mäntysaari, arXiv:1601.06598

- left: BK equation at strict NLO
- right: NLO BK + collinear improvement

Saturation exponent $\lambda_s \equiv \mathrm{d}\ln Q_s^2/\mathrm{d}Y$



Fixed coupling $\bar{\alpha}_s = 0.25$



Running coupling

- Further slowing down when also resumming single logs (part of DGLAP)
- Altogether: DL + SL +RC : $\lambda_s \simeq 0.2$
- But what about the initial condition ?

The problem of the initial condition

- Recall: the physical I.C. must act as a boundary condition at $Y = \rho$

$$\tilde{S}(Y=\rho,\rho) \;=\; S_0(\rho) \;\equiv\; S(\eta=0,\rho) \;=\; \mathrm{e}^{-\frac{1}{4}r^2Q_0^2\ln\frac{1}{r^2\Lambda^2}}$$

- The unphysical I.C. $\tilde{S}(Y=0,\rho)$ must be chosen in such a way to construct, via the collBK evolution, the physical I.C. at $Y = \rho$.

- At DLA, this is a simple task:

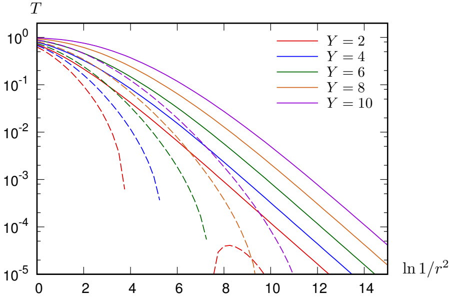
$$\tilde{\mathcal{A}}(0,\rho) = \bar{\mathcal{A}}(\eta=-\rho,\rho) = \begin{cases} \mathrm{J}_0\big(2\sqrt{\bar{\alpha}_s\rho^2}\big) & \text{for} \quad \bar{\mathcal{A}}(0,\rho)=1, \\ \dfrac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)}{\sqrt{\bar{\alpha}_s}} & \text{for} \quad \bar{\mathcal{A}}(0,\rho)=\rho. \end{cases}$$

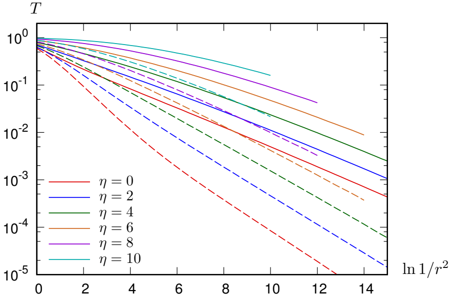
- Beyond DLA, we don't know how to do that (perhaps, via a fit to an Ansatz with many parameters ?)

- The simple guess: "exponentiate the DLA result" ... does not really work !

Numerical solutions: IC at DLA

- The DLA-like I.C. does not reproduce the physical B.C. at $Y = \rho$



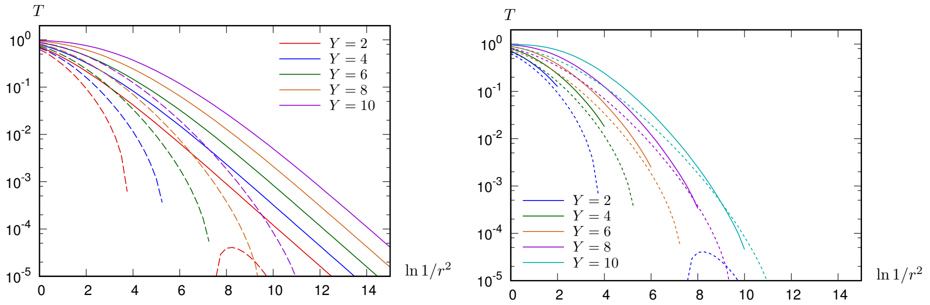


  - L: collBK with 2 I.C.s: GBW (cont.) and resummed DLA (J_0; dashed)
  - R: collBK + resummed DLA (J_0) replotted in terms of η

- Oscillations disappear with increasing $Y \Longrightarrow$ well defined fronts in Y
- ... which however are irrelevant: The fronts in η are not that clear

Numerical solutions: IC at DLA

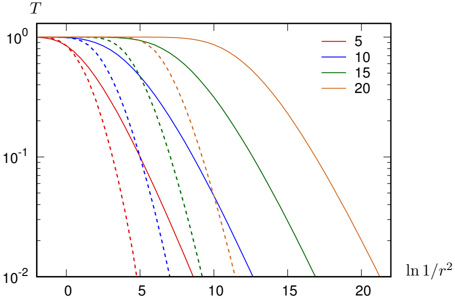
- The DLA-like I.C. does not reproduce the physical B.C. at $Y = \rho$



  - L: collBK with 2 I.C.s: GBW (cont.) and resummed DLA (J_0; dashed)
  - R: collBK + resummed DLA (J_0) vs. LO BK in η replotted in Y
- Oscillations disappear with increasing $Y \Longrightarrow$ well defined fronts in Y
- The effect of the resummation $\approx$ solving LO BK in η

Saturation fronts: η vs. Y

- The physical fronts at LO are obtained by solving LO BK equation in η
- The solution can be replotted in terms of Y to compare with collBK



- saturation exponent λ_s

$$Q_s^2(Y) \simeq Q_0^2\, \mathrm{e}^{\lambda_s Y}$$

- anomalous dimensions γ_s

$$T(Y,r) \simeq \left(r^2 Q_s^2(Y)\right)^{\gamma_s}$$

- similarly for η-evolution: $\bar{\lambda}_s$, $\bar{\gamma}_s$

$$\bar{\gamma}_s = \gamma_s(1-\lambda_s), \qquad \bar{\lambda}_s = \frac{\lambda_s}{1-\lambda_s} \qquad \text{(asymptotically)}$$

- Physical fronts in η are less step & faster

Non-local BK evolution in Y

- Recall: at DLA, one has a non-local (in Y) boundary value problem

$$\mathcal{A}(Y,\rho) = \mathcal{A}_0(\rho) + \bar{\alpha}_s \int_\rho^Y \mathrm{d}Y_1 \int_0^\rho \mathrm{d}\rho_1\, \mathcal{A}(Y_1 - \rho + \rho_1, \rho_1)$$

  - $\rho = \ln \frac{1}{r^2 Q_0^2} > \rho_1 = \ln \frac{1}{z^2 Q_0^2}$ since $z^2 \gg r^2$: hard-to-soft

- For full BFKL dynamics, soft-to-hard ($\rho_1 > \rho$) is possible as well, even in DIS
  - non-locality of the kernel in the transverse plane ("BFKL diffusion")

- Soft-to-hard evolution in Y is correctly time-ordered: no new constraint

$$S_{\boldsymbol{xy}}(Y) = S^0_{\boldsymbol{xy}} + \frac{\bar{\alpha}_s}{2\pi} \int_\rho^Y \mathrm{d}Y_1 \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \big[S_{\boldsymbol{xz}}(Y_1 - \Delta_{\boldsymbol{xz}}) S_{\boldsymbol{zy}}(Y_1 - \Delta_{\boldsymbol{zy}}) - S_{\boldsymbol{xy}}(Y_1)\big]$$

  - $S^0_{\boldsymbol{xy}}$: the physical "initial condition", here a boundary value at $Y = \rho$

$$\Delta_{\boldsymbol{xz}} \equiv \Theta\left(\ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2}\right) \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2}$$

  - slight extension of the equation proposed by Guillaume Beuf (2014)

Non-local BK evolution in η

$$S_{\boldsymbol{xy}}(Y) = S^0_{\boldsymbol{xy}} + \frac{\bar{\alpha}_s}{2\pi}\int_\rho^Y \mathrm{d}Y_1 \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}}\big[S_{\boldsymbol{xz}}(Y_1-\Delta_{\boldsymbol{xz}})S_{\boldsymbol{zy}}(Y_1-\Delta_{\boldsymbol{zy}})-S_{\boldsymbol{xy}}(Y_1)\big]$$

- Some obvious and less obvious complications ...
  - a boundary value problem
  - potentially large non-locality in rapidity: $\bar{\alpha}_s\Delta^2 \gtrsim 1$
  - solution must be replotted in terms of $\eta = Y - \rho$ to study saturation
  - not clear how to add the full NLO corrections (despite contrary claims)
- The first 3 problems can be avoided by changing rapidity variable $Y \to \eta$

$$\bar{S}_{\boldsymbol{xy}}(\eta) = S^0_{\boldsymbol{xy}} + \frac{\bar{\alpha}_s}{2\pi}\int_0^\eta \mathrm{d}\eta_1 \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}}\big[\bar{S}_{\boldsymbol{xz}}(\eta_1-\bar{\Delta}_{\boldsymbol{xz}})\bar{S}_{\boldsymbol{zy}}(\eta_1-\bar{\Delta}_{\boldsymbol{zy}})-\bar{S}_{\boldsymbol{xy}}(\eta_1)\big]$$

  - A product of Θ-functions $\Theta(\eta_1-\bar{\Delta}_{\boldsymbol{xz}})\Theta(\eta_1-\bar{\Delta}_{\boldsymbol{zy}})$ is understood

$$\bar{\Delta}_{\boldsymbol{xz}} \equiv \Theta\left(\ln\frac{r^2}{(\boldsymbol{x}-\boldsymbol{z})^2}\right)\ln\frac{r^2}{(\boldsymbol{x}-\boldsymbol{z})^2}$$

The role of the non-locality

- The rapidity shift in η avoids violations of time-ordering in soft-to-hard
  - N.B. η corresponds to k^-, so the corresponding "time" is x^-
- In DIS, the typical evolution is hard-to-soft $\Longrightarrow$ the shift in η is small
  - its effect is a pure α_s–correction, like many others that are forgotten
- Why can't we just ignore this shift $\bar{\Delta}$? Why not simply use rcBK in η ?
- To answer this, we expanded the non-locality to $\mathcal{O}(\bar{\alpha}_s)$, i.e. to NLO
- The "pure NLO" effect is numerically large !
  - it triggers an instability for all but unphyscally tiny values of $\bar{\alpha}_s$
- The mathematics is quite clear, but a physical understanding is still lacking

Expanding the non-locality to NLO

$$\bar{S}_{\boldsymbol{xy}}(\eta) = S^0_{\boldsymbol{xy}} + \frac{\bar{\alpha}_s}{2\pi}\int_0^{\eta}\mathrm{d}\eta_1\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[\bar{S}_{\boldsymbol{xz}}(\eta_1-\bar{\Delta}_{\boldsymbol{xz}})\bar{S}_{\boldsymbol{zy}}(\eta_1-\bar{\Delta}_{\boldsymbol{zy}})-\bar{S}_{\boldsymbol{xy}}(\eta_1)\big]$$

- Expand e.g. $\bar{S}_{\boldsymbol{xz}}(\eta_1-\bar{\Delta}_{\boldsymbol{xz}})$ to linear order in $\bar{\Delta}_{\boldsymbol{xz}}$:

$$\bar{S}_{\boldsymbol{xz}}(\eta_1-\bar{\Delta}_{\boldsymbol{xz}}) \simeq \bar{S}_{\boldsymbol{xz}}(\eta_1) - \bar{\Delta}_{\boldsymbol{xz}}\frac{\partial \bar{S}_{\boldsymbol{xz}}}{\partial \eta_1}$$

- Use the (usual, local) LO BK equation for $\partial\bar{S}_{\boldsymbol{xz}}/\partial\eta_1$

$$\frac{\partial \bar{S}_{\boldsymbol{xz}}}{\partial \eta_1} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{u}\,\mathcal{M}_{\boldsymbol{xzu}}\big[\bar{S}_{\boldsymbol{xu}}\bar{S}_{\boldsymbol{uz}} - \bar{S}_{\boldsymbol{xz}}\big]$$

- $\mathcal{O}(\bar{\alpha}_s\bar{\Delta})$ corrections which involve up to 3 dipole S-matrices: $\bar{S}_{\boldsymbol{xu}}\bar{S}_{\boldsymbol{uz}}\bar{S}_{\boldsymbol{zy}}$ ...

  ... as expected for 2 successive gluon emissions

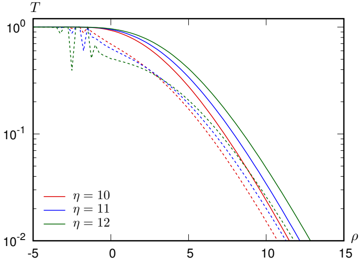
- Since $\bar{\Delta} \sim 1$ should be quite small, why should this bring any problem ?

Expanding the non-locality to NLO

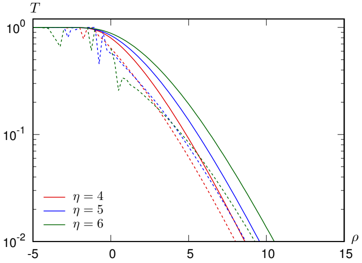
- Problems could be expected in the linear regime, where one can integrate out one of the 2 gluons ...

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \left(1 - \frac{\bar{\alpha}_s}{2} \ln^2 \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{z}-\boldsymbol{y})^2}\right) \left[\bar{S}_{\boldsymbol{xz}}\bar{S}_{\boldsymbol{zy}} - \bar{S}_{\boldsymbol{xy}}\right]$$

- The double-log is important when one daughter dipole is much smaller than the other one ... but in practice this is rarely the case !



dashed: NLO; solid: res; $\bar{\alpha}_s = 0.2$



dashed: NLO; solid: res; $\bar{\alpha}_s = 0.3$

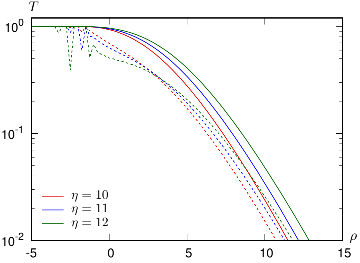
- "NLO": oscillations leading to instability for $\bar{\alpha}_s > 0.03$

Expanding the non-locality to NLO

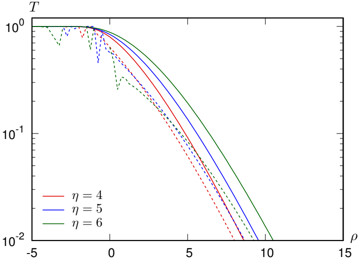
- Problems could be expected in the linear regime, where one can integrate out one of the 2 gluons ...

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}}\, \mathcal{K}_{\mathrm{DLA}} \left(\frac{\bar{\alpha}_s}{2} \ln^2 \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{z}-\boldsymbol{y})^2} \right) [\bar{S}_{\boldsymbol{xz}} \bar{S}_{\boldsymbol{zy}} - \bar{S}_{\boldsymbol{xy}}]$$

- The double-log is important when one daughter dipole is much smaller than the other one ... but in practice this is rarely the case !



dashed: NLO; solid: res; $\bar{\alpha}_s = 0.2$

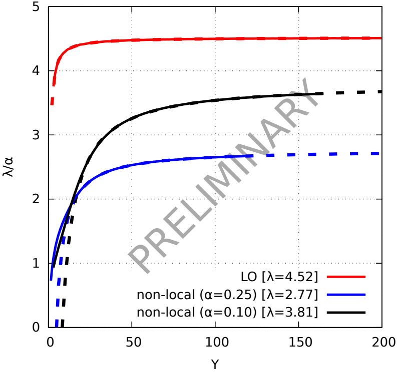


dashed: NLO; solid: res; $\bar{\alpha}_s = 0.3$

- "Resummed": all-order resummation of the double-logs in the kernel

Back to the non-local equation

- The resummation of the double-logs captures the main effects of the rapidity shift $\bar{\Delta}$ in the linear regime ...
- ... but not also in the approach to saturation, where "soft-to-hard" matters as well ("Levin-Tuchin law for the approach to the black-disk limit")
- Numerically solve the non-local equation in η (here, fixed coupling)



- saturation exponent $\lambda_s/\bar{\alpha}_s$
- leading order: $\lambda_s/\bar{\alpha}_s \simeq 4.88$
- non-local equation: $\lambda_s/\bar{\alpha}_s$ decreases with $\bar{\alpha}_s$
- the decrease in λ_s is of $\mathcal{O}(\bar{\alpha}_s)$
- the other NLO effects of $\mathcal{O}(\bar{\alpha}_s)$ must be "simply" added

Conclusions

- The rapidity of the dilute but hard projectile is a “bad” variable for studying the high energy evolution beyond leading order
  - instability requiring for all-order resummations (in both the kernel and the initial condition)
  - alternatively: non-local evolution equation, formulated as a boundary-value problem
  - saturation fronts/physics is meaningful in the target rapidity anyway
- The problem is much better behaved when the evolution time is the rapidity η of the comparatively soft target
  - still projectile evolution, but in a new variable
  - still non-local, but the non-locality in η is smaller
  - initial value problem
  - physics can be directly read off the solutions
- The main remaining problem: how to extend to full NLO accuracy ?