A Chern–Simons view of noncommutative scalar theory

Leonardo Santilli
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Grupo de **F**isica **M**atematica
Universidade de Lisboa

QSpace Training School
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Exact solution and relation to Chern–Simons theory

1. Introduction

Noncommutative theories: generalities

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Product of functions is modified to obtain

$$\left[x^i, x^j\right] = \mathrm{i}\theta^{ij}(x).$$

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- **Fuzzy sphere**: $\mathbb{S}^2$ with truncation of algebra of functions. $\theta^{ij}(x)$ related to generators of rotations in $\mathbb{R}^3$.
- $\mathbb{R}^3_\lambda$: 3d Euclidean space with coordinates replaced by $\mathfrak{su}(2)$ generators multiplied by a length scale λ. It foliates into fuzzy spheres of discrete radii.

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Throughout this talk the focus will be on the latter.

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2. LSZ model

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Each 2d subspace allows a harmonic oscillator (h.o.) description.
We will take a mixture of all ingredients to obtain **two commuting copies** of h.o.

The LSZ scalar theory

We define an action for a scalar field with quadratic and quartic interaction:

$$S_{\mathrm{LSZ}} = \int_{R^2_\theta} \left\{ \frac{1}{2}\Phi^\dagger \left(-\sigma D^2 - \tilde{\sigma}\tilde{D}^2 \right) \Phi + \frac{1}{2}\Phi \left(-\sigma D^2 - \tilde{\sigma}\tilde{D}^2 \right) \Phi^\dagger \right.$$
$$\left. m_0^2 \Phi^\dagger \Phi + \frac{g_0^2}{2} \left(\Phi^\dagger \Phi \right)^2 \right\}, \tag{2.1}$$

where D_i is the covariant derivative and $\tilde{D}_i$ involves a reflection $B \mapsto -B$.

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$$S_{\mathrm{LSZ}} = N \mathrm{Tr} \left\{ M^\dagger E M + M \tilde{E} M^\dagger + \hat{m}^2 M^\dagger M + \frac{\hat{g}^2}{2} \left(M^\dagger M \right)^2 \right\}, \tag{2.2}$$

where the $\hat{\cdot}$ means adimensional coupling and the external matrices $E, \tilde{E}$ have h.o. spectra.

Our approach to LSZ

Nevertheless, we will solve the LSZ matrix model in a **generalized setting**, for external matrices with arbitrary spectra, and eventually recover the original model by setting eigenvalues of both to be the Landau levels.

3. Exact solution and relation to Chern–Simons theory

Generalized solution, step I

The partition function of the matrix model obtained above can be exactly solved.
We do it in a generalized sense, allowing the external matrices $E, \tilde{E}$ to have eigenvalues of the form of arbitrary integers $\eta_\ell, \tilde{\eta}_\ell$ times a factor $4\pi/N$.

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We use the decomposition

$$M = U_1^\dagger \mathrm{diag}\left(\lambda_1, \dots, \lambda_N\right) U_2,$$

with unitary U_α. The Jacobian of this transformation introduces a squared Vandermonde determinant.

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After simplifying with the Vandermonde squared, we arrive to:

$$Z\left(E,\tilde{E}\right) = \frac{\mathcal{C}_N}{\Delta_N\left[\eta\right]\Delta_N\left[\tilde{\eta}\right]} \int \prod_{\ell=1}^{N} \mathrm{d}y_\ell e^{-\sum\left(m^2 y_\ell + \frac{g^2}{2} y_\ell^2\right)} \times \det\left(e^{-\eta_\ell y_m}\right) \det\left(e^{-\tilde{\eta}_\ell y_m}\right), \tag{3.1}$$

with coefficient $\mathcal{C}_N = 2^{-N(N+1)/2}/\mathrm{Vol}(U(N))$.

Generalized solution, step III

We rewrite the integers eigenvalues $\eta_\ell, \tilde{\eta}_\ell$ in the form

$$\begin{aligned} \eta_\ell &= -\mu_\ell - N + \ell, \\ \tilde{\eta}_\ell &= -\nu_\ell - N + \ell, \end{aligned}$$

for arbitrary partitions μ, ν.

Generalized solution, step IIIb

In this way, the determinants in the partition function can be rewritten in terms of Schur polynomials. We arrive to:

$$Z\left(E, \tilde{E}\right) = \frac{\mathcal{C}_N}{\Delta_N[\eta]\,\Delta_N[\tilde{\eta}]} \int \prod_{\ell=1}^{N} \mathrm{d}y_\ell e^{-\sum\left(\beta y_\ell + \frac{g^2}{2} y_\ell^2\right)} \prod_{\ell<m} \left(2\sinh\left(\frac{y_\ell - y_m}{2}\right)\right)^2 \times s_\mu\left(e^{y_1}, \dots, e^{y_N}\right) s_\nu\left(e^{y_1}, \dots, e^{y_N}\right), \tag{3.2}$$

where $\beta = m^2 - N + 1$.

Generalized solution, step IIIc

We recognise an expression similar to the Hopf link invariant for $U(N)$ Chern–Simons theory on $\mathbb{S}^3$ [Dolivet–Tierz, 2006].
For the two expression to coincide, we ought to conjugate the partition $\nu \mapsto \nu^*$.

Generalized solution, step IV

Doing the conjugation, one finally arrives to

$$Z\left(E, \tilde{E}\right) = C \cdot \langle W_{\mu\nu^*} \rangle, \tag{3.3}$$

where $C \propto \frac{S_{00}}{\dim \mu \dim \nu}$ and $\langle W_{\mu\nu^*} \rangle = (TST)_{\mu\nu^*}$ is the Hopf link average.

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$$Z\left(E, \tilde{E}\right) = C \cdot \langle W_{\mu\nu^*} \rangle, \qquad (3.3)$$

where $C \propto \frac{S_{00}}{\dim \mu \dim \nu}$ and $\langle W_{\mu\nu^*} \rangle = \left(TST\right)_{\mu\nu^*}$ is the Hopf link average.

This is a topological invariant of $\mathbb{S}^3$.

LSZ solution

The original LSZ model is recovered turning off both partitions.

$$Z_{\mathrm{LSZ}} \propto Z_{\mathrm{CS}}.$$

The partition function of LSZ is then proportional to CS partition function and to the so-called Witten–Reshetikhin–Turaev topological invariant.

This is the end of the presentation.

Thank you for your attention