Dirac operators & spectral triples.

Motivation: Dirac's relativistic adaptation of Schrödinger's equation:

$$i\frac{\partial}{\partial t}\psi(x,t) = \left(\Delta + V(x,t)\right)\psi(x,t)$$

$$\Delta = -\left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$$

"Square root" of rhs (assume periodic b.c.)

S^1: $\Delta_{S^1} = -\left(\frac{\partial}{\partial x}\right)^2$ $\qquad D_{S^1} = i\frac{\partial}{\partial x}$.

$D_{S^1}^2 = \Delta_{S^1}$ $\qquad$ eigenvalues: $n \in \mathbb{Z}$

$$D_{S^1} e^{inx} = -n e^{inx}$$

$\mathbb{T}^2 = S^1 \times S^1$: $\Delta_{\mathbb{T}^2} = -\left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2$

$$D_{\mathbb{T}^2} = a\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2} \rightsquigarrow a^2 = b^2 = -1, \quad ab = 0$$

$\lightning$

But: matrix-valued solutions exist:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rightsquigarrow \mathbb{H}$$

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} \\ -\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & 0 \end{pmatrix} \to D_{\mathbb{T}^2}^2 = -\Delta_{\mathbb{T}^2} \cdot \mathbb{1}_2 .$$

eigenfunctions:

$$\psi_{\vec{n}}^{(\pm)}(\vec{x}) = \begin{pmatrix} e^{i\vec{n}\cdot\vec{x}} \\ \mp\frac{in_1 + n_2}{\sqrt{n_1^2+n_2^2}} e^{i\vec{n}\cdot\vec{x}} \end{pmatrix}; \quad D_{\mathbb{T}^2}\psi_{\vec{n}}^{(\pm)} = \pm\sqrt{n_1^2+n_2^2}\ \psi_{\vec{n}}^{(\pm)}$$

$\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1$

$$\Delta_{\mathbb{T}^4} = -\left(\frac{\partial}{\partial x^1}\right)^2 - \dots - \left(\frac{\partial}{\partial x^4}\right)^2$$

$$D_{\mathbb{T}^4} = a\frac{\partial}{\partial x^1} + \dots + d\frac{\partial}{\partial x^4} \qquad a^2 = \dots = d^2 = -1$$

$$\rightsquigarrow a = 1,\ b = i,\ c = j,\ d = k \quad \underset{M_2(\mathbb{H})}{} \quad \left\{ \begin{array}{l} ab + ba = 0 \\ ac + ca = 0 \dots \end{array} \right.$$

Quaternions

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} + j\frac{\partial}{\partial x^3} + k\frac{\partial}{\partial x^4} \\ -\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} + j\frac{\partial}{\partial x^3} + k\frac{\partial}{\partial x^4} & 0 \end{pmatrix}$$

$$\Rightarrow \quad D_{\mathbb{T}^4}^2 = \Delta_{\mathbb{T}^4} \mathbb{1}_4 \qquad \text{spectrum } \pm\sqrt{n_1^2 + \dots + n_4^2}$$

In general, a Dirac operator exists whenever (M,g) is a Riemannian $\underline{\text{spin}}^c$ manifold.

Then we have:

- $\Gamma^\infty(S_M)$: spinors, carrying a representation of Clifford algebra generated by $\gamma^\mu = \gamma(dx^\mu)$ satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

- ∇^{S_M}, spin connection, lift of Levi-Civita:

$$\nabla^{S_M}(\gamma(\omega) s) = \gamma(\nabla^{LC}(\omega)) s + \gamma(\omega) \nabla^{S_M}(s)$$

- Unique if M is spin; $J_M : \Gamma(\mathcal{S}_M) \to \Gamma(\mathcal{S}_M)$

- Dirac operator

$$D_M : \Gamma^{\infty}(\mathcal{S}_M) \longrightarrow \Gamma^{\infty}(\mathcal{S}_M)$$

$$s \longmapsto \sum_{\mu} \gamma^{\mu} (\nabla^{\mathcal{S}}_{\mu} s)$$

- Lichnerowicz:

$$D_M^2 = \Delta_M + \frac{1}{4} \sigma_M .$$

Spectrum: if M compact then D_M is an essentially self-adjoint operator with cpt resolvent:

$$D_M = \sum_{n \in \mathbb{Z}} \lambda_n |\psi_n\rangle\langle\psi_n|$$

If M even-dim, then $\gamma_M : \Gamma^{\infty}(\mathcal{S}) \to \Gamma^{\infty}(\mathcal{S})$ $\quad \gamma_M D_M = -D_M \gamma_M$

Math question: does D_M capture all geometric information about (M, g)?

Marc Kac (1966) "Can one hear the shape of a drum?"

No... isospectral but non-isometric examples.

Some geometric info: $\mathrm{tr}\, e^{-tD_M^2} \sim t^{-d/2} \mathrm{Vol}(M) + t^{-d/2+1} \int R \ldots$

Noncommutative geometry adds to D_M more local spectral information about M

"locally hearing the shape of a drum"

$$\left(C^\infty(M), L^2(S_M), D_M ; \gamma_M, J_M \right)$$

$\hookrightarrow$ smooth functions $f: M \to \mathbb{C}$ to localize, act on $L^2(S_M)$

Reconstruction of (M, g) is possible (C96, C08)

Let us illustrate this for the metric distance.

Def. $$d(x,y) := \sup_{f \in C^\infty(M)} \left\{ |f(x) - f(y)| : \|[D,f]\| \le 1 \right\}$$

This is a <u>metric</u> on M. $\|\gamma^\mu \partial f\|$

Moreover:

Prop. $$d(x,y) \equiv d_g(x,y)$$

$$= \inf \left\{ \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}\, dt : \gamma(0) = x,\ \gamma(1) = y \right\}$$

Proof: $$\|[D,f]\| = \|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_g(x,y)}$$

$$\Longrightarrow d(x,y) \le d_g(x,y).$$

For fixed y, this is attained by

$$f(x) = d_g(x,y).$$ $\square$

We now abstract the following notion of a noncommutative manifold.

Defn A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by
$\mathcal{A}$: $*$-algebra of bdd operators on
$\mathcal{H}$: Hilbert space
D: ess. self-adj. operator in $\mathcal{H}$

s.t. - $[D, a]$ extends to bdd commutator $\forall a \in \mathcal{A}$
- $(1+D^2)^{-1}$ is cpt. operator.

grading: γ. $\quad \gamma D = -D\gamma$

real structure: J $\quad J^2 = \varepsilon,\ JD = \varepsilon' DJ,\ J\gamma = \varepsilon'' \gamma J$

$\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$.