Deformation theory & structures up to homotopy (1)

(1) Deformations, cohomology & graded Lie algebras

(2) (Non-commutative) formal geometry

(3) structures up to homotopy & operads

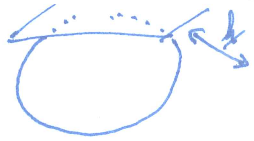
(4) ~~...~~ Ex: Simultaneous deformations

a) Motivation: Quantization and Moyal product.

$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}$

$\{a, b\cdot c\} = \{a,b\}\cdot c + b\{a,c\}$

Classical		Quantum
$\mathbb{R}^{2n}$		$\mathcal{H} = L^2(\mathbb{R}^n)$
$C^\infty(\mathbb{R}^{2n}), \cdot, \{\ \}$	$\xrightarrow[\text{Quantization}]{1-1}$	$\mathcal{O}(\mathcal{H}), \circ, [\ ]$
$\frac{df}{dt} = \{h, f\}$		$\frac{d\mathcal{O}}{dt} = [H, \mathcal{O}]$



(2)

$p \longmapsto \hat{p} : \varphi \longmapsto -i\hbar \frac{\partial}{\partial x}\varphi$

$q \longmapsto \hat{q} : \varphi \longmapsto x\varphi$

$f *_\hbar g := \left(\hat{f} \circ \hat{g}\right)^{1-1}$

-Groenewold

Thm (Moyal):

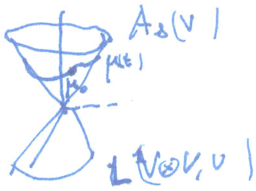
$$\boxed{*_\hbar = \cdot + \hbar B_1(\ ,\ ) + \hbar^2 B_2(\ ,\ ) + \dots}$$

i.e. $*_\hbar$ is a deformation of $\cdot$

b) (formal) Deformations of associative algebras

V k-vector space

$As(V) = \{\mu \in L(V\otimes V, V) \ / \ \mu(\mu(a,b),c) = \mu(a,\mu(b,c))\}$



Def: a formal deformation of $\mu_0 \in As(V)$ is a formal series such that

(3)

$\mu(t) = \mu_0 + t\mu_1 + \dots$ $\quad \mu_i \in L(V\otimes V, V)$

$\mu(t)$ defines an associative structure on $V \otimes k[[t]]$.

Reformulation:

$\mu(a_0 + a_1 t + a_2 t^2 + \dots, b_1 + b_2 t + \dots)$

$= \sum_{n\in\mathbb{N}} t^n \sum_{i+j=n} \mu(a_i, b_j)$

$\Rightarrow$ suffices to know on elements of V.

$\mu(t)$ associative:

order 0:

$\cancel{\mu_t(\mu_t(a,b),c) =}$

$\mu_t(\mu_t(a,b),c) = \mu_t(\mu_0(a,b) + t\mu_1(a,b) + \dots, c)$

$= \mu_0(\mu_0(a,b) + t\mu_1(a,b) + \dots, c)$

[illegible] $(a,b)+\dots, c) + \dots$

(4)

i.e at order n:

$$\mu_0(\mu_n(a,b),c) + \sum_{\substack{i+j=n\\ i,j\neq 0}} \mu_i(\mu_j(a,b),c)$$

$$= \mu_0(a,\mu_n(b,c)) + \sum_{\substack{i+j=n\\ i,j\neq 0}} \mu_i(a,\mu_j(b,c))$$

Recovering $\{\ \}$ from $[\ ]$

Notations: $[a,b]_i := \frac{1}{2}(\mu_i(a,b) - \mu_i(b,a))$

$\boxed{\text{Prop: } [\ ]_1 \text{ is Poisson}}$

Proof: $*_t$ associative $\Rightarrow [\ ]_t$ Lie, i.e

$[a,[b,c]_t]_t = [[a,b]_t,c]_t + [b,[a,c]_t]_t \quad (J)$

- (J) at order 2: ($[\ ]_0 = 0$ as $\cdot$ commutative) $\Rightarrow [\ ]_1$ alg de Lie.
- $[A, B\circ C] = [A,B]\circ C + B\circ[A,C]$

$ABC - BCA = ABC - BAC + BAC - BCA$

at order 1: $[\ ]_1$ Poisson. $\square$

Deformation quantization problem: (5)

given a $(\{\ \},\cdot)$, $\exists$? μ_t ?

"Any Poisson manifold is quantizable"

(Corollary of formality thm of Kontsevich)

$\uparrow$ homotopy .

Plato

Rq: Mathematics $=$ Physics

Arnold

quantum quantum

q: how to quantize a mathematical concept?

a: Express it in terms of algebras and then deform it.

Ex quantum groups:

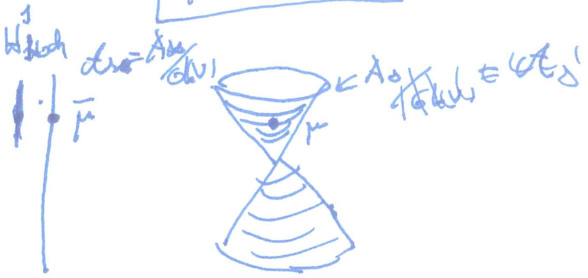
$G \longleftrightarrow \mathcal{C}^\infty(G)$

$\cdot \qquad \Delta$

$e \qquad \varepsilon$

Tools: cohomology and gLa (6)

Thm 1 $\boxed{T_{\bar\mu}\, As_d = H^1_{Hoch}}$



$GL(V)$ action:

a) $GL(V)$ acts on $L(V\otimes V, V)$

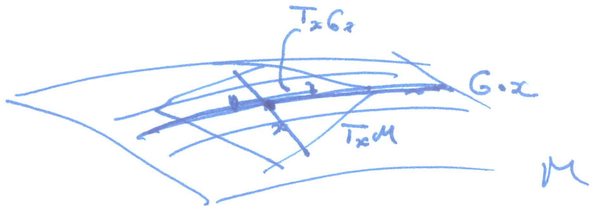
by $\boxed{(g\cdot\mu)(a,b) := g(\mu(g^{-1}a, g^{-1}b))}$

b) As_d is preserved by this action

$g(\mu(g^{-1}g\mu(g^{-1}a, g^{-1}b), g^{-1}c)) = g\mu(g^{-1}a, g^{-1}g\mu(g^{-1}b, g^{-1}c))$ ✓

Remark: if G acts on M (7)

then: ① $T_x G\cdot x \subset T_x M$



② $T_{\bar x}\, M/G = T_x M / T_x G\cdot x$

Hence, to prove Thm 1, it suffices to identify

i) $T_\mu As_d = Z^1_{Hoch}$

ii) $T_{\bar\mu}\, GL(V)\cdot\mu = B^1_{Hoch}$