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Drinfel'd double Poisson structures for the Poincaré Lie group

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Overview

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1. Introduction

Objectives and main results

- Fully construction of **all non-isomorphic Drinfel'd double** Lie algebra structures for $\mathfrak{p}(2+1)$
- Construction of **Poisson-Minkowski spacetimes** associated to each Drinfel'd double which is also a **coisotropic deformation**
- Relation of the Poisson-Lie structures coming from Drinfel'd doubles with the **whole set of Poisson-Lie structures**

Why Drinfel'd doubles?

Three motivations (among others) for considering Drinfel'd doubles:

- The **Drinfel'd double structure** generate canonical classical r-matrices that fulfill the Fock-Rosly condition and are thus compatible with the CS approach to gravity coupled to point particles[1]
- They provide a **canonical coboundary (quasi-triangular) Lie bialgebra**, by means of the associated canonical r-matrix
- This canonical r-matrix allows us to study its associated **Poisson-Minkowski spacetime**

[1] A. Ballesteros, F.J. Herranz, C. Meusburger, Class. Quant. Grav., 2013.

2. Drinfel'd double Lie algebras

Drinfel'd double Lie algebras

Drinfel'd double Lie algebras

A $2d$-dimensional Lie algebra $\mathfrak{a}$ is said to be a (classical) Drinfel'd double if there exists a basis $\{Y_1, \ldots, Y_d, y^1, \ldots, y^d\}$ of $\mathfrak{a}$ in which the Lie bracket reads

$$[Y_i, Y_j] = c_{ij}^k Y_k, \qquad [y^i, y^j] = f_k^{ij} y^k, \qquad [y^i, Y_j] = c_{jk}^i y^k - f_j^{ik} Y_k,$$

with an associative quadratic form:

$$\langle Y_i, Y_j \rangle = 0, \qquad \langle y^i, y^j \rangle = 0, \qquad \langle y^i, Y_j \rangle = \delta_j^i, \qquad \forall i, j$$

A quadratic Casimir operator for $\mathfrak{a}$ is:

$$C = \tfrac{1}{2} \sum_i (y^i\, Y_i + Y_i\, y^i)$$

Drinfel'd double - Lie bialgebra correspondence

$$[Y_i, Y_j] = c_{ij}^k Y_k, \qquad \delta(Y_n) = f_n^{lm} Y_l \otimes Y_m$$

Drinfel'd double Lie algebras

The Lie algebra $\mathfrak{a}$ is endowed with a **canonical coboundary (quasi-triangular) Lie bialgebra** structure by means of the canonical classical r-matrix:

$$r = \sum_{i=1}^{d} y^i \otimes Y_i$$

which defines the cocommutator

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r], \quad \forall X \in \mathfrak{a}.$$

Explicitly, in the Drinfel'd double basis,

$$\delta(y^i) = c^i_{jk}\, y^j \otimes y^k, \qquad \delta(Y_i) = f^{jk}_i\, Y_j \otimes Y_k\ .$$

In fact this cocommutator only depends on the **skew-symmetric component** r'

$$r' = \tfrac{1}{2} \sum_i y^i \wedge Y_i$$

of the r-matrix r.

3. Poincaré r-matrices and Poisson Minkowski spacetimes

Minkowski spacetime in (2+1) dimensions

Introducing a kinematical basis of these Lie algebras, we can write the **(2+1) dimensional Poincaré Lie algebra** $\mathfrak{p}(2+1)$ as:

$$
\begin{array}{lll}
[J, K_1] = K_2, & [J, K_2] = -K_1, & [K_1, K_2] = -J, \\
[J, P_0] = 0, & [J, P_1] = P_2, & [J, P_2] = -P_1, \\
[K_1, P_0] = P_1, & [K_1, P_1] = P_0, & [K_1, P_2] = 0, \\
[K_2, P_0] = P_2, & [K_2, P_1] = 0, & [K_2, P_2] = P_0, \\
[P_0, P_1] = 0, & [P_0, P_2] = 0, & [P_1, P_2] = 0.
\end{array}
$$

where $\{J, K_1, K_2, P_0, P_1, P_2\}$ are, respectively, the generators of rotations, boosts, time translation and space translations. Here $\mathfrak{h} = Lie(H) = \langle J, K_1, K_2 \rangle$.

With the previous description, we can describe **Minkowski spacetime** as the quotient of the Poincaré group G by the Lorentz group H: $M = G/H$.

Poisson Homogeneous Spaces: Coisotropy and subgroup conditions

When constructing non-commutative spacetimes as quantum homogeneous spaces, restrictions appear at the Hopf algebra level[2]. The Poisson counterpart of quantum homogeneous spaces was studied in detail recently[3] and the **coisotropy condition** $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$ is obtained at the level of the Lie bialgebra, where $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup H.

If moreover $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$ we say that the deformation is a **Poisson subgroup**.

Given that every r-matrix r for $\mathfrak{g} = Lie(G)$ defines a **Poisson structure compatible with multiplication in G, by means of the Sklyanin bracket**, and using coisotropy conditions we construct Poisson-Minkowski spacetimes compatible with the canonical projection $\pi : G \to G/H$.

[2] V.G. Drinfel'd, *Theoret. Math. Phys.*, 1993

[3] A. Ballesteros, C. Meusburger, P. Naranjo, J. Phys. A, 2017 (See references therein for quantum homogeneous space constructions.)

Quantum deformations of the Poincaré Lie algebra and their Poisson-Lie counterparts have been thoroughly constructed and classified in the literature:

- In (3+1) dimensions the classification of r-matrices (coboundary Lie bialgebras) was done by S. Zakrzewski[4]
- In Poincaré (2+1) all bialgebras are coboundaries, and they were classified by P. Stachura[5]
- Six dimensional Drinfel'd double structures (relevant for the (2+1) dimensional case) have been classified[6] [7]

[4] S. Zakrzewski, Comm. Math. Phys., 1997.
[5] P. Stachura, J. Phys. A, 1998.
[6] X. Gomez, J. Math. Phys., 2000.
[7] L. Snobl and H. Hlavaty, Inter. J. Mod. Phys. A, 2002.

4. Drinfel'd doubles for $\mathfrak{p}(2+1)$

A known Drinfel'd double structure on $\mathfrak{p}(2+1)$

- The Drinfel'd double corresponding to the Lie bialgebra $D(\mathfrak{sl}(2,\mathbb{R}))$, i.e. $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and $\delta(X) = 0 \; \forall X \in \mathfrak{g}$ is known to be isomorphic to $\mathfrak{p}(2+1)$. This corresponds to **Case 0** in the following classification
- Its **euclidean analogue** has also been considered[8] in relation with (2+1) euclidean gravity
- In the remaining of the talk we present **the rest of Drinfel'd double structures for $\mathfrak{p}(2+1)$**

[8] E. Batista, S. Majid, J. Math. Phys., 2003.

Classification of Drinfel'd double structures on $\mathfrak{p}(2+1)$

Eight nonisomorphic Drinfel'd double structures on $\mathfrak{p}(2+1)$[9] [10]:

Poisson subgroup, 'True' coisotropic case:

	Case 0	Case 1	Case 2	Case 3
$[Y_0, Y_1]$	$2Y_1$	Y_1	Y_1	Y_1
$[Y_0, Y_2]$	$-2Y_2$	Y_2	Y_2	$Y_1 + Y_2$
$[Y_1, Y_2]$	Y_0	0	0	0
$[y^0, y^1]$	0	y^0	0	λy^2
$[y^0, y^2]$	0	y^1	y^1	0
$[y^1, y^2]$	0	y^2	0	0
$[y^0, Y_0]$	0	$-Y_1$	0	0
$[y^0, Y_1]$	y^2	$-Y_2$	$-Y_2$	0
$[y^0, Y_2]$	$-y^1$	0	0	$-\lambda y^1$
$[y^1, Y_0]$	$2y^1$	$Y_0 + y^1$	y^1	$y^1 + y^2$
$[y^1, Y_1]$	$-2y^0$	$-y^0$	$-y^0$	$-y^0$
$[y^1, Y_2]$	0	$-Y_2$	0	$\lambda Y_0 - y^0$
$[y^2, Y_0]$	$-2y^2$	y^2	y^2	y^2
$[y^2, Y_1]$	0	Y_0	Y_0	0
$[y^2, Y_2]$	$2y^0$	$Y_1 - y^0$	$-y^0$	$-y^0$

$$[Y_i, Y_j] = c_{ij}^k Y_k, \qquad [y^i, y^j] = f_k^{ij} y^k, \qquad [y^i, Y_j] = c_{jk}^i y^k - f_j^{ik} Y_k \ .$$

[9] Gomez X J, *Math. Phys.* (2000).

[10] Snobl L, Hlavaty L, *Int. J. Mod. Phys. A* (2002).

Classification of Drinfel'd double structures on $\mathfrak{p}(2+1)$

Eight nonisomorphic Drinfel'd double structures on $\mathfrak{p}(2+1)$[11 12]:

	Case 4	Case 5	Case 6	Case 7
$[Y_0, Y_1]$	$-Y_2$	Y_1	Y_1	$-Y_2$
$[Y_0, Y_2]$	Y_1	$Y_1 + Y_2$	Y_2	Y_1
$[Y_1, Y_2]$	0	0	0	0
$[y^0, y^1]$	0	λy^2	0	0
$[y^0, y^2]$	$-y^0$	0	y^1	$-y^0$
$[y^1, y^2]$	$\lambda y^0 - y^1$	$2\omega y^0$	$2\omega y^0$	$-y^1$
$[y^0, Y_0]$	Y_2	0	0	Y_2
$[y^0, Y_1]$	$-y^2 + \lambda Y_2$	0	$-Y_2$	0
$[y^0, Y_2]$	$y^1 + Y_0 - \lambda Y_1$	$-\lambda Y_1$	0	0
$[y^1, Y_0]$	0	$y^1 + y^2 - 2\omega Y_2$	$-2\omega Y_2 + y^1$	y^2
$[y^1, Y_1]$	$-Y_2$	$-y^0$	$-y^0$	Y_2
$[y^1, Y_2]$	$-y^0 + Y_1$	$-y^0 + \lambda Y_0$	0	$-y^0$
$[y^2, Y_0]$	0	$y^2 + 2\omega Y_1$	$2\omega Y_1 + y^2$	$-Y_0 - y^1$
$[y^2, Y_1]$	y^0	0	Y_0	$-Y_1 + y^0$
$[y^2, Y_2]$	0	$-y^0$	$-y^0$	0

$$[Y_i, Y_j] = c_{ij}^k Y_k, \qquad [y^i, y^j] = f_k^{ij} y^k, \qquad [y^i, Y_j] = c_{jk}^i y^k - f_j^{ik} Y_k \, .$$

[11] Gomez X J, *Math. Phys.* (2000).

[12] Snobl L, Hlavaty L, *Int. J. Mod. Phys. A* (2002).

An example: Case 1

Isomorphism between DD and kinematical basis:

$$J = y^0 + y^1 + y^2, \quad K_1 = y^0 + y^1, \quad K_2 = -y^1 - y^2,$$
$$P_0 = y^0 + y^1 + Y_0 - Y_1 + Y_2, \quad P_1 = y^0 + y^1 + Y_0 - Y_1, \quad P_2 = y^0 - Y_1 + Y_2.$$

Classical r-matrix:

$$r_1 = \sum_{i=0}^{2} y^i \otimes Y_i = K_1 \wedge J + K_1 \wedge K_2 + J \otimes P_0 + K_2 \otimes P_1 - K_1 \otimes P_2.$$

Anti-symmetric r-matrix:

$$r_1' = K_1 \wedge J + K_1 \wedge K_2 + (J \wedge P_0 + K_2 \wedge P_1 + P_2 \wedge K_1) = r_t + r_0$$

where r_t is the non-standard deformation of $\mathfrak{so}(2,1)$ while r_0 is the r-matrix associated to Case 0.

An example: Case 1

Cocommutator

$$
\begin{aligned}
\delta(J) &= K_2 \wedge J, \\
\delta(K_1) &= J \wedge K_1 + K_2 \wedge K_1, \\
\delta(K_2) &= J \wedge K_2, \\
\delta(P_0) &= J \wedge P_1 + P_2 \wedge K_1 + K_2 \wedge P_1 + P_1 \wedge P_2, \\
\delta(P_1) &= J \wedge P_0 + K_2 \wedge P_0 + P_2 \wedge K_1 + P_0 \wedge P_2, \\
\delta(P_2) &= P_0 \wedge K_1 + K_1 \wedge P_1 + P_1 \wedge P_0.
\end{aligned}
$$

Clearly is a **Poisson subgroup space** because $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$, where $\mathfrak{h} = \text{span}\{J, K_1, K_2\}$ is the Lorentz subalgebra.

An example: Case 1

r-matrix with generic parameters:

$$r'_{1,(\alpha_1,\beta_1)} = \alpha_1 \left(J \wedge K_1 + K_2 \wedge K_1\right) + \beta_1 \left(P_0 \wedge J + K_1 \wedge P_2 + P_1 \wedge K_2\right)$$

Poisson spacetime

$$\{x^0, x^1\} = -\alpha_1 x^2(x^0 + x^1) + 2\beta_1 x^2$$
$$\{x^0, x^2\} = \alpha_1 x^1(x^0 + x^1) - 2\beta_1 x^1$$
$$\{x^1, x^2\} = \alpha_1 x^0(x^1 + x^0) - 2\beta_1 x^0$$

Clearly there are **two contributions**: the one with β_1, which comes from the twist part and defines a 'Snyder type' (which is exactly the one corresponding to Case 0) and the one with α_1 which defines the quadratic part. This Case 1 defines **the only not Lie algebraic spacetime** and it is thus the most interesting one. A comparison with[13] should be performed.

[13] J. Lukierski, M. Woronowicz, Physics Letters B, 2006

Relation of Case 1 with (2+1) gravity

The relation with (2+1) gravity comes from the **canonical pairing**. For this case we have

$$\langle J, P_0 \rangle = 1, \;\; \langle K_1, P_2 \rangle = -1, \;\; \langle K_2, P_1 \rangle = 1$$

This corresponds to the first of the two **symmetric invariant bilinear forms** that appear in (2+1)-gravity[14], which is the one that permits to reformulate (2+1) gravity as a Chern-Simons gauge theory with the relevant isometry group as the gauge group, where $\langle \cdot, \cdot \rangle$ enters the CS action.[15] [16]

[14] E. Witten, Nucl. Phys. B, 1988
[15] C. Meusburger and B. Schroers, *J. Math. Phys.*, 2008
[16] C. Meusburger and B. Schroers, *Nucl. Phys. B*, 2009

Drinfel'd double r-matrices for $\mathfrak{p}(2+1)$

Poisson subgroup, 'True' coisotropic case:

$$r_0' = \frac{1}{2}(J \wedge P_0 + K_2 \wedge P_1 + P_2 \wedge K_1)$$

$$r_1' = K_1 \wedge J + K_1 \wedge K_2 + (J \wedge P_0 + K_2 \wedge P_1 + P_2 \wedge K_1)$$

$$r_2' = P_2 \wedge J + K_2 \wedge P_0 + K_2 \wedge P_2 + \frac{1}{2}(K_2 \wedge P_1 + P_0 \wedge J + P_2 \wedge K_1)$$

$$r_3' = J \wedge P_2 + K_2 \wedge P_0 + K_2 \wedge P_2 + \frac{1}{2}(J \wedge P_0 + K_1 \wedge P_2 + P_1 \wedge K_2) + \frac{1}{\lambda}(P_0 \wedge P_1 + 2(P_0 \wedge P_2 + P_2 \wedge P_1))$$

$$r_4' = P_2 \wedge J + \frac{1}{2}(P_0 \wedge J + P_2 \wedge K_1 + K_2 \wedge P_1) + \lambda P_0 \wedge P_2$$

$$r_5' = P_1 \wedge J + \frac{1}{2}(J \wedge P_0 + P_1 \wedge K_2 + K_1 \wedge P_2) + \frac{1}{\lambda} P_1 \wedge P_0$$

$$r_6' = P_0 \wedge K_2 + \frac{1}{2}(P_2 \wedge K_1 + P_1 \wedge K_2 + J \wedge P_0)$$

$$r_7' = P_2 \wedge J + \frac{1}{2}(J \wedge P_0 + K_1 \wedge P_2 + P_1 \wedge K_2)$$

Poisson spacetimes for $\mathfrak{p}(2+1)$ from Drinfel'd doubles

Poisson subgroup

	Case 0	Case 1
$\{x^0, x^1\}$	$-x^2$	$-\alpha_1 x^2(x^0 + x^1) + 2\beta_1 x^2$
$\{x^0, x^2\}$	x^1	$\alpha_1 x^1(x^0 + x^1) - 2\beta_1 x^1$
$\{x^1, x^2\}$	x^0	$\alpha_1 x^0(x^1 + x^0) - 2\beta_1 x^0$

'True' coisotropic case

	Case 2	Case 6	Case 7
$\{x^0, x^1\}$	0	0	0
$\{x^0, x^2\}$	$\alpha_2\left(-x^0 + x^2\right)$	$\alpha_6(-x^0 + x^1)$	0
$\{x^1, x^2\}$	$\beta_2\left(-x^0 + x^2\right)$	0	$-\alpha_7(x^0 + x^2)$

Classification of PL structures and limits from (A)dS

We have obtained **eight different coboundary Poisson-Lie structures on P(2+1)**. It certainly makes sense to compare them with the classification given by Stachura[17].

Out of the eight (possibly parametric) families found by Stachura, we find that all our DD generate only four of them, some with different parameters

This allows us to see that:

- Only the **'space-like' (and not the 'time-like') κ-Poincaré** deformation comes from a DD structure
- **All the DD structures for the (Anti-) de Sitter Lie algebra**[18] with the relevant bilinear form give Poincaré DD

[17] P. Stachura, J. Phys. A, 1998.

[18] A. Ballesteros, F.J. Herranz, C. Meusburger, Class. Quant. Grav., 2013.

7. Final remarks

Final remarks

- We have presented and constructed **all eight nonisomorphic DD structures for $\mathfrak{p}(2+1)$**. Two give rise to Poisson homogeneous spaces and three to 'coisotropic' homogeneous spaces. All of them have the **pairing allowing a CS reformulation of (2+1) gravity**
- We have presented the associated **Poisson spacetimes to the DDs satisfying the coisotropy condition**. Only one out of five is quadratic, while the other four are of Lie algebraic type
- We have studied the general picture of DD structures for $\mathfrak{p}(2+1)$ by relating our results with the full classification
- In the same paper we have also constructed **all non-isomorphic DD for $\overline{\mathfrak{p}(1+1)}$**, the non-trivially centrally extended Poincaré Lie algebra in (1+1) dimensions

Thanks for your attention!