Recall: $(\mathcal{A}, \mathcal{H}, D)$

$$\mathcal{U}(\mathcal{A}) \longrightarrow \mathrm{Inn}(\mathcal{A})$$
$$u \longmapsto \alpha_u$$

$$\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Diff}(M) \ltimes \mathrm{Inn}(\mathcal{A})$$

$$D \longmapsto uDu^* = D + \underbrace{u[D, u^*]}_{\text{pure gauge field}}$$

Today:

→ allow for arbitrary gauge fields of the form:

$$\sum a_j [D, b_j]$$ "connection one-form"

→ ① Morita self-equiv.

→ ② Perturbation semigroup

→ action functional (spectral)

(1) Morita equivalence $\quad A \underset{M}{\sim} B$ (unital $\mathbb{C}$-algebras)

${}_B\mathcal{E}_A$, ${}_A\mathcal{F}_B$:

$$\mathcal{E} \otimes_A \mathcal{F} \cong B \qquad \mathcal{F} \otimes_B \mathcal{E} \cong A.$$

A, B commutative $\quad A \underset{M}{\sim} B \iff A \cong B.$

Transfer $(A, \mathcal{H}, D)$ to $(B, \mathcal{H}', D')$

$$\mathcal{H}' = \mathcal{E} \otimes_A \mathcal{H}$$

$$D' = 1 \otimes_\nabla D ,$$

$$(1 \otimes_\nabla D)(e \otimes \xi) = \nabla e \cdot \xi + e \otimes D\xi$$

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_D(A)$$

$$\left\{ \begin{array}{l} \Omega^1_D(A) = \left\{ \sum a_j [D, b_j] : a_j, b_j \in A \right\} \\ \nabla(ea) = \nabla(e)a + e \otimes [D, a]. \end{array} \right.$$

Thm $(A, \mathcal{H}, D)$ spectral triple $\Rightarrow$

$(B, \mathcal{E} \otimes_A \mathcal{H}, 1 \otimes_\nabla D)$ spectral triple.

(extends to real sp. tr.)

Morita self-equivalence. Take $B = A$, $\mathcal{E} = A$

Then $\nabla : A \to \Omega^1_D(A)$ determined by

$A = \nabla(1) = \sum a_i [D, b_i]$

$$1 \otimes_\nabla D = D_A = D + A$$

inner perturbations

Special case: $\mathcal{E} = {}_{\alpha_u(A)} A_A \rightsquigarrow$ pure gauge fields

② Semigroup of inner perturbations

Recall: $A^{op} = \{a^{op} : a \in A\}$ vect. space

$$a^{op} b^{op} = (ba)^{op}$$

Defn. Perturbation semigroup:

$$\mathrm{Pert}(A) = \left\{ \sum a_j \otimes b_j^{op} \in A \otimes A^{op} \;\middle|\; \sum a_j b_j = 1, \; \sum a_j \otimes b_j^{op} = \sum b_j^* \otimes (a_j^*)^{op} \right\}$$

Semigroup law inherited from $A \otimes A^{op}$.

N.B. $\left(\sum a_j \otimes b_j^{o}\right)\left(\sum a_k' \otimes (b_k')^{o}\right)$ normalized:

$$\sum a_j a_k' \cdot b_k' b_j = \sum a_j b_j = 1.$$

Observations:

- $\mathcal{U}(A) \to \mathrm{Pert}(A)$

  $u \mapsto u \otimes (u^*)^{op}$

- $\mathrm{Pert}(A)$ acts on self-adj. op. D:

$$D \mapsto \sum_j a_j D b_j = D + \underbrace{a_j [D, b_j]}_{\text{gauge field}}$$

So, semigroup structure on gauge fields

Ex: (manifold)

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) : \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\}$$

$$\frac{\partial}{\partial x^\mu} \longmapsto \left. \frac{\partial}{\partial y^\mu} f(x,y) \right|_{y=x} =: A_\mu \in C^\infty(M)$$

More generally, for $C^\infty(M, M_N(\mathbb{C}))$

$A_\mu \in C^\infty(M, \mathfrak{u}(N))$ $\mathfrak{u}(N)$-gauge field

semigroup structure on gauge fields

Ex two-point space

$$\left(\mathbb{C}^2, \mathbb{C}^2, D = \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}\right)$$

$$\mathrm{Pert}(\mathbb{C}^2) = \{ e_{11} \otimes e_{11} + z e_{11} \otimes e_{22} + \bar{z} e_{22} \otimes e_{11} + e_{22} \otimes e_{22} \} \cong \mathbb{C}$$

$\mathcal{U}(\mathbb{C}^2) \longrightarrow U(1) \subset \mathrm{Pert}(\mathbb{C}^2)$

$$D = \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 0 & zc \\ \bar{z}\bar{c} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \bar{\phi} \\ \phi & 0 \end{pmatrix}$$

moreover ϕ transforms under $U(1)$ as $\phi \mapsto \lambda \phi$

Ex: Noncommutative two-point space

$$\left(\mathbb{C}\oplus\mathbb{H},\ \mathbb{C}\oplus\mathbb{C}^2,\ D=\begin{pmatrix}0 & c & 0\\ \bar c & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)$$

$$\mathrm{Pert}(\mathbb{C}\oplus\mathbb{H})\cong\mathbb{H}^{\mathbb{C}}\times\mathrm{Pert}(\mathbb{H})$$

$$D\longmapsto\begin{pmatrix}0 & cz_1 & cz_2\\ \bar c\bar z_1 & 0 & 0\\ \bar c\bar z_2 & 0 & 0\end{pmatrix}=:\begin{pmatrix}0 & \bar\phi_1 & \bar\phi_2\\ \phi_1 & 0 & 0\\ \phi_2 & 0 & 0\end{pmatrix}$$

where $\begin{pmatrix}z_1\\ z_2\end{pmatrix}$ column in $M_2(\mathbb{C})\subset\mathrm{Pert}(\mathbb{C}\oplus\mathbb{H})$

Moreover, $\Phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ transforms under $\mathcal{U}(\mathbb{C}\oplus\mathbb{H})=U(1)\times SU(2)$

as $\Phi\longmapsto\bar\lambda u.\Phi \qquad \big((\lambda,u)\in U(1)\times SU(2)\big)$

Ex. AC mfds $\left(C^\infty(M)\otimes A_F,\ L^2(S_M)\otimes H_F,\ D\otimes 1+\gamma\otimes D_F\right)$

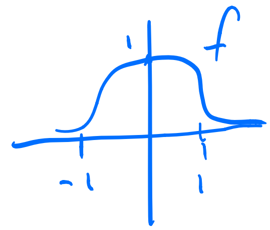
$$D\otimes 1+\gamma\otimes D_F\longmapsto D\otimes 1+\gamma\otimes D_F + \gamma^\mu A_\mu+\gamma\otimes\Phi$$

$$\begin{cases} A_\mu\in C^\infty(M,\mathfrak{u}(A_F))\\ \Phi\in C^\infty(M,\mathcal{L}(\mathcal{H}_F))\end{cases}$$

Spectral action functional

Spectral invariant associated to $(\mathcal{A}, \mathcal{H}, D)$

$$S_\Lambda[D_A] = tr\left(f\left(\frac{D_A}{\Lambda}\right)\right)$$



$$= \sum_n f\left(\frac{\lambda_n}{\Lambda}\right)$$

Smooth cutoff function $f: \mathbb{R} \to \mathbb{R}$

Laplace transform $\quad f(x) = \int_0^\infty g(t)\, e^{-tx^2}\, dt$

Allows to use heat kernel asymptotics:

$$tr\, e^{-tD^2/\Lambda^2} \sim \sum_{n \geq 0} \left(\frac{t}{\Lambda^2}\right)^{\frac{n-d}{2}} a_n(D^2)$$

This implies that $(\dim M = 4)$:

$$\operatorname{tr} f(D/\Lambda) \sim 2f_4 \Lambda^4 a_0(D^2) + 2f_2 \Lambda^2 a_2(D^2)$$

$$+ f_0\, a_4(D^2) + \mathcal{O}(\Lambda^{-2})$$

where $f_j = \int_0^\infty f(x)\, x^{j-1}\, dx$ (moments)

Prop Let D_M be Dirac operator on 4d Riem spin mfld M. Then

$$S_\Lambda[D] = \int \mathcal{L}_M \sqrt{g}\, dx$$

where

$$\mathcal{L}_M = \frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} s_M + \mathcal{O}(s_M^2)$$

$$- \frac{f(0)}{320}\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \text{topological}$$

Thm Let $M \times F$ be an AC manifold ($\dim M = 4$)

$$\left(C^\infty(M) \otimes A_F,\ L^2(S_M) \otimes \mathcal{H}_F,\ D \otimes 1 + \gamma_M \otimes D_F \right).$$

Then inner perturbations are of the form $\gamma^\mu A_\mu + \gamma^5 \Phi$ where $A_\mu(x) \in \mathcal{U}(A_F)$ and $\Phi(x) \in \mathcal{L}(\mathcal{H}_F)$. Moreover, the spectral action

$$S_\Lambda[D] = \int_M \left(N\mathcal{L}_M + \mathcal{L}_A + \mathcal{L}_\Phi \right) \sqrt{g}\, dx.$$

and $N = \dim \mathcal{H}_F$ and we defined

$$\mathcal{L}_A = \frac{f(0)}{24\pi^2} \operatorname{tr}_F F_{\mu\nu} F^{\mu\nu}$$

and

$$\mathcal{L}_\Phi = -\frac{\Lambda^2 f_2}{2\pi^2} \operatorname{tr} \Phi^2 + \frac{f(0)}{8\pi^2} \operatorname{tr} \Phi^4$$

$$+ \frac{f(0)}{48\pi^2} s_M \operatorname{tr} \Phi^2 + \frac{f(0)}{8\pi^2} \operatorname{tr} D_\mu \Phi D^\mu \Phi$$

Standard Model

$$\left\{\begin{aligned} &\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \\ &\mathcal{H}_F = \Big(\underbrace{\mathbb{C}^2}_{f,\bar f} \otimes \underbrace{\mathbb{C}^2}_{R,L} \otimes \big(\underbrace{\mathbb{C}^2}_{l} \oplus \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^3}_{q}\big)\Big)^{\oplus 3} \\ &D_F = \begin{pmatrix} S & T \\ T^* & \bar S \end{pmatrix} \end{aligned}\right.$$

$$\text{unimodular}\ \left\{\begin{aligned} &A \in C^\infty(M, u(1) \oplus su(2) \oplus su(3)) \\ &\Phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \in C^\infty(M, \mathbb{C}^2) \end{aligned}\right.$$

$U(1) \times SU(2)$ fund. repr.

Spectral action: SM Lagrangian with less parameters: GUT, Higgs mass.

Pati-Salam model

$$\begin{cases} \mathcal{A}_F = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}) \\ \mathcal{H}_F = \left(\left\{ \begin{pmatrix} \nu_e & u_1 & u_2 & u_3 \\ e & d_1 & d_2 & d_3 \end{pmatrix} \right\} \otimes \mathbb{C}^2_{R,L} \otimes \mathbb{C}^2_{f,\bar{f}} \right)^{\oplus 3} \\ D_F = \begin{pmatrix} S & T \\ T^* & \bar{S} \end{pmatrix} \end{cases}$$

Unimodular $A_\mu \in C^\infty(M, su(2)_R \oplus su(2)_L \oplus su(4))$

$$\Phi : \begin{cases} (2_R, 2_L, 1) \quad \text{2HDM} \\ (2_R, 1, 4) \\ (1, 1, 15) \end{cases}$$

Spectral action: PS model with GUT