So far, we have described classical field theories, using a spectral description of fermions at the one-particle level.

In order to capture the full quantum field theory this should be improved.

Recent paper w/ Chamseddine & Connes on "spectral action and entropy" is a first step.
CMP.

We will build a dictionary:

| one-particle | "second-quantized" |
|---|---|
| $A$ $*$-algebra | action of $\mathrm{Pert}(A)$ on $\{\sigma_t\}$ |
| $\mathcal{H}$ Hilbert space | $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ (Clifford algebra) |
| $D$ unbdd self-adj. op. | $\{\sigma_t\}$ 1-p. group of autom. on $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H})$ corresp. to $e^{itD} \in \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$ |
| Spectral action | entropy of $KMS_\beta$-state for $\sigma_t$ |

We start by replacing $\mathcal{H}$ by the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$.

$\mathcal{H}_{\mathbb{R}}$ : real vector space underlying $\mathcal{H}$

so $\mathcal{H}_{\mathbb{R}}$ Eucl. vector space.

$$g : \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}} \to \mathbb{R}$$

$$(v, w) \mapsto \langle v, w \rangle.$$

$\mathrm{Cliff}(\mathcal{H}_{\mathbb{R}})$ : $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2g(v,w)$

Complexification $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}) = \mathrm{Cliff}(\mathcal{H}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$.

Next, consider operator $D$.
It gives rise to one-parameter family of orthogonal transfo. $\{e^{itD}\}$ on $\mathcal{H}_{\mathbb{R}}$.

This induces a one-param. family of automorphisms of $Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ via

$$\sigma_t(\gamma(v)) = \gamma(\sigma_t(v))$$

This is how $D$ manifests itself at 2nd quantized level.

Inner perturbations map $D \mapsto D_A$ and accordingly $\sigma_t^D \to \sigma_t^{D_A}$.

| | |
|---|---|
| $\mathcal{A}$ | $\sigma_t^D \mapsto \sigma_t^{D_A}$. |
| $\mathcal{H}$ | $Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ |
| $D$ | $\{\sigma_t^D\}$ arising from $e^{itD}$ |

$C^*$-dynamical system $(\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}), \sigma_t^0)$

$\exists!$ $KMS_\beta$-state : $\varphi : \mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}) \to \mathbb{C}$

$$\varphi(a\,\sigma_t(b))\Big|_{t=i\beta} = \varphi(ba)$$

$a, b$ norm-dense subalgebra (of $\sigma$-analytic vectors)

Example: $M_2(\mathbb{C}) = \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2)$

$\hookrightarrow$ $\sigma^1, \sigma^2$ Pauli.

$\varphi : M_2(\mathbb{C}) \to \mathbb{C}$ $\quad \sigma_t = e^{i\beta H}(\cdot)e^{-i\beta H}$.

$\varphi(a) = \mathrm{tr}(\rho a)$ $\quad KMS_\beta \Rightarrow \rho = e^{-\beta H}$

More generally, for $M_N(\mathbb{C})$. $\quad H \geq 0$.

## Representations of $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$. [GVF'01]

$V = \mathcal{H}_{\mathbb{R}}$. + complex structure $I$

$\rightsquigarrow$ complexification $V_I$ $\qquad I^2 = -1$

$$\gamma_I : \mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}) \to \mathcal{L}(\Lambda(V_I))$$

$$v \longmapsto a_I^*(v) + a_I(v)$$

$a_I^*(v) = v \wedge \ldots \qquad a_I$ adj..

$\Omega_I \in \Lambda^0(V_I)$

**Prop.** $\gamma_I$ is irreducible

Physical Hilbert space: instead of given complex structure on $\mathcal{H}$ we consider

$$I := i(E_+ - E_-) \qquad E_\pm \text{ spectral proj. of } D.$$

This amounts to $e^{itD}$ acting as $e^{it|D|}$

**Prop.** (i) $\gamma_I(\sigma_t(a)) = \Lambda(e^{it|D|})\,\gamma_I(a)\,\Lambda(e^{-it|D|})$

(ii) $KMS_\beta$-state $\varphi_\beta(a) = \frac{1}{Z} tr\left(\Lambda(e^{-\beta|D|})\,\gamma_I(a)\right)$

$a \in Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$

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Density matrix $\rho = \Lambda(e^{-\beta|D|})$

We consider von Neumann entropy

$$S(\phi) = -tr\, \rho \log \rho$$

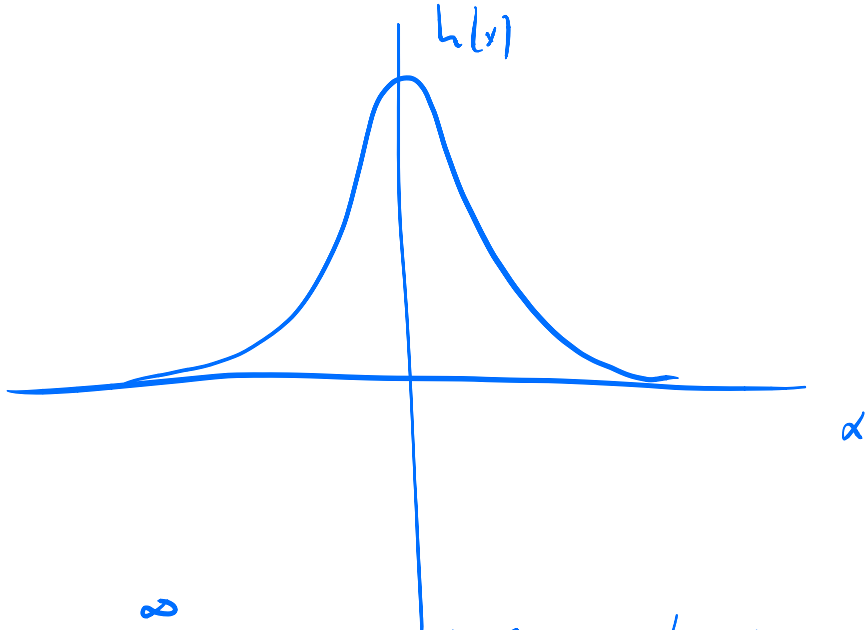
**Thm** Entropy of $KMS_\beta$-state $\varphi_\beta$ is given by spectral action for the function $h(x) = \mathcal{E}(e^{-x})$ where $\mathcal{E}(x)$ is entropy of partition of unit interval in two intervals with size of ratio $x$.

$$\mathcal{E}(x) = \log x+1 - \frac{x \log x}{x+1}. \qquad \rho = \frac{1}{1+x}\binom{1}{x}$$

$$= -\rho \log \rho = -\frac{1}{1+x}\log\frac{1}{1+x} - \frac{x}{1+x}\log\frac{x}{1+x} = \frac{1+x}{1+x}\log 1+x - \frac{x}{1+x}\log x$$

Consider $h(x)$:

$$h(x) = \mathcal{E}(e^{-x}) = \frac{x}{1+e^{x}} + \log(1+e^{-x})$$




$$\begin{cases} h(x) = \int_0^\infty g(t)\, e^{-tx^2} \quad \text{Laplace transf.} \\ g(t) = \frac{-1}{8\sqrt{\pi}\, t^{5/2}} \sum_{n\in\mathbb{Z}} \underbrace{(-1)^n n^2 q^{n^2}}_{q\frac{\partial}{\partial q}\theta_4(0;q)} ; \quad q = e^{-1/4t} \end{cases}$$

**Thm**

Heat expansion: $\operatorname{tr} e^{-tD^2} \sim \sum_k t^k b_k \Rightarrow$

$$\operatorname{tr} h(\beta D) \sim \sum \beta^{2k} \gamma(k)\, b_k$$

$$\gamma(k) = \frac{1-2^{-2k}}{k} \pi^{-k} \xi(2k)$$

**Riemann $\xi$-function:**

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$\vdots$

$$\gamma(-1) = \frac{9\zeta(3)}{2}$$

$$\gamma\left(-\tfrac{1}{2}\right) = \frac{\pi^{3/2}}{3}$$

$$\gamma(0) = \log 2$$

$$\gamma\left(\tfrac{1}{2}\right) = \frac{1}{2\sqrt{\pi}}$$

$$\gamma(1) = \frac{1}{8}$$

$$\gamma\left(\tfrac{3}{2}\right) = \frac{7\zeta(3)}{8\pi^{5/2}}$$

$\vdots$