Morita Equivalence for singular foliations

Alfonso Garmendia

KU LEUVEN

Joint work with: Marco Zambon

Introduction to singular foliations

Definition

A **singular foliation** on a manifold M is a $C_c^\infty(M)$ submodule $\mathscr{F}$ of the vector fields $\mathscr{X}(M)$ such that it is closed under the Lie bracket and locally finitely generated. The pair $(M,\mathscr{F})$ is called a **foliated manifold**.

Example

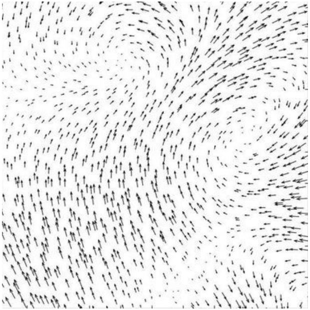
When G is a Lie group acting on a manifold M then the span of its infinitesimal action generates a foliation on M.

The foliation of the last example gives a partition of M by the orbits of G, which coincides with the group of exponentials of the infinitesimal action of G.

Example

Any finite set $X_1, \cdots, X_n \in \mathscr{X}(M)$ of involutive vector-fields span a singular foliation on M.
In particular any regular foliation is a singular foliation.

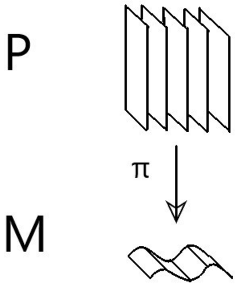
Any singular foliation on M gives a partition of M into immersed submanifolds. This partition comes following the flows of $\mathscr{F}$.



Definition

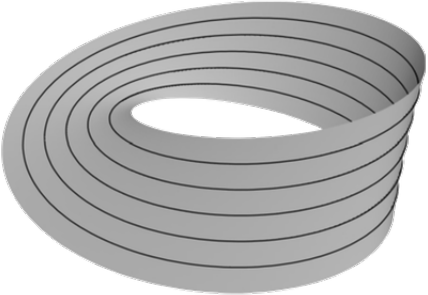
Let $(M, \mathscr{F})$ a foliated manifold and $\pi \colon P \to M$ a submersion, then the pullback foliation $\pi^{-1}\mathscr{F}$ on P is the span of the projectable vectorfields of P that projects to $\mathscr{F}$.

The pullback foliation locally looks like $\mathscr{F}$ plus the vertical direction of the fiber of π.



Transverse structure of Regular foliations

Smooth structure in the leaf space of a regular foliation.



- Holonomy groupoid of $\mathscr{F}$, denoted by $\mathscr{H}(\mathscr{F})$.
- The group of equivalent transverse structures is a Morita equivalent class of $\mathscr{H}(\mathscr{F})$.

ME for Lie groupoids

Definition

Two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are **Morita equivalent** if there exists a manifold P and two surjective submersions $\pi_M : P \to M$ and $\pi_N : P \to N$, such that $\pi_M^{-1}\mathcal{G} \cong \pi_N^{-1}\mathcal{H}$ as Lie groupoids in P.

$$
\begin{tikzcd}
\mathcal{G} \arrow[dr, shift left] \arrow[dr, shift right] & & P \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & & \mathcal{H} \arrow[dl, shift left] \arrow[dl, shift right] \\
& M & & N &
\end{tikzcd}
$$

Example

Two Lie groups $G \rightrightarrows \{*\}$, $H \rightrightarrows \{*\}$ are Morita equivalent if and only if $G \cong H$ as Lie groups.

Example

The pair groupoid $M \times M \rightrightarrows M$ is Morita equivalent to the point groupoid $\{*\} \rightrightarrows \{*\}$.

Example

Let G be a **Lie group** acting freely and proper on M. Then the **action groupoid** $G \times M$ is Morita equivalent to the **identity groupoid** on M/G.

Example

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Example

Let G be a **Lie group** acting freely and proper on M. Then the **action groupoid** $G \times M$ is Morita equivalent to the **identity groupoid** on M/G.

Proposition

If $\mathscr{G}$ is Morita equivalent to $\mathscr{H}$ then:

- $M/\mathscr{G} \cong N/\mathscr{H}$ as topological spaces.
- For $p \in P$ then $\mathscr{G}_{\pi_M(p)} \cong \mathscr{H}_{\pi_N(p)}$ are isomorphic as Lie groups.
- The categories of left modules are equivalent.

Holonomy Groupoid for a singular foliation

For a foliated manifold $(M, \mathscr{F})$:

1. I. Androulidakis and G. Skandalis defined a topological groupoid $\mathscr{H}(\mathscr{F})$ called the Holonomy groupoid. When the foliation is regular $\mathscr{H}(\mathscr{F})$ coincides with the classical notion of holonomy. (2009)
2. C. Debord proved that for every leaf L of $\mathscr{F}$ there is a canonical smooth structure on $\mathscr{H}(\mathscr{F})|_L$ making it a Lie groupoid over L. (2013)
3. I. Androulidakis and M. Zambon proved that The Holonomy groupoid of a singular foliation $\mathscr{H}(\mathscr{F})$ is a Lie groupoid if and only if $\mathscr{F}$ is a projective module. (2017)
4. M. Zambon and A.Garmendia proved that $\mathscr{H}(\mathscr{F})$ is an open source connected topological groupoid. (2018)
5. D. Pronk gave a definition of weak equivalence for topological groupoids which is similar to Morita equivalence for Lie groupoids. (1996)

Definition

Two foliated manifolds $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ are **Hausdorff Morita equivalent** if $\exists$ a manifold P and two surjective submersions with connected fibers:

$$\begin{tikzcd} & P \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M & & N \end{tikzcd}$$

such that $\pi_M^{-1}\mathscr{F}_M = \pi_N^{-1}\mathscr{F}_N$ as singular foliations on P.

Proposition

If $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ are Hausdorff Morita equivalent then $\mathscr{H}(\mathscr{F}_M)$ and $\mathscr{H}(\mathscr{F}_N)$ are weak equivalent as topological groupoids.

Definition

Two foliated manifolds $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ are **Hausdorff Morita equivalent** if $\exists$ a manifold P and two surjective submersions with connected fibers:

$$\begin{tikzcd} & P \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M & & N \end{tikzcd}$$

such that $\pi_M^{-1}\mathscr{F}_M = \pi_N^{-1}\mathscr{F}_N$ as singular foliations on P.

Proposition

If $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ are Hausdorff Morita equivalent then $\mathscr{H}(\mathscr{F}_M)$ and $\mathscr{H}(\mathscr{F}_N)$ are weak equivalent as topological groupoids.
Moreover, if $\mathscr{F}_M$ and $\mathscr{F}_N$ are projective then $\mathscr{H}(\mathscr{F}_M)$ and $\mathscr{H}(\mathscr{F}_N)$ are Morita equivalent as Lie groupoids.

Proposition

If $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ are Hausdorff Morita equivalent foliated manifolds then:

- Their spaces of leaves are homeomorphic.
- Their isotropy Lie groups and algebras are isomorphic.
- Leafwise the Lie groupoids are Morita equivalent.

Proposition

If two Poisson manifolds are Morita equivalent then their underlying singular foliations are Hausdorff Morita Equivalent.

Proposition

If two Hausdorff source connected Lie groupoids are Morita equivalent then their underlying singular foliations are Hausdorff Morita Equivalent.

1. ANDROULIDAKIS, I. AND SKANDALIS, G. "The holonomy groupoid of a singular foliation". J.Reine Angew. Math., 626:137, (2009).
2. ANDROULIDAKIS, I. AND ZAMBON, M. "Almost regular Poisson manifolds and their holonomy groupoids". Selecta Math. (N.S.), 23(3):22912330, (2017).
3. GARMENDIA, A., AND ZAMBON, M. "Hausdorff Morita equivalence of singular foliations". Published online by: Annals of Global Analysis and Geometry (Jun 2018).
4. DEBORD, C. "Longitudinal smoothness of the holonomy groupoid". C. R. Math. Acad. Sci. Paris, 351(15-16):613616, (2013).
5. PRONK, D. "Etendues and stacks as bicategories of fractions". Compositio Math. , 102(3):243303, (1996).