Grassmannian Geometry of Scattering Amplitudes

LECTURE 4

Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP)

University of California, Davis

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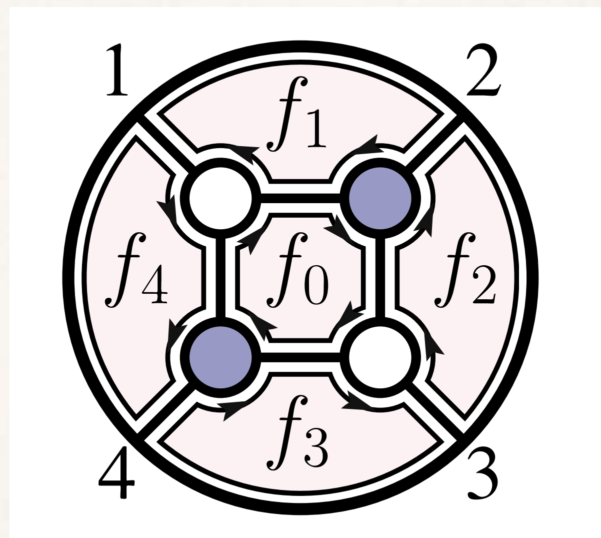
Positive Grassmannian

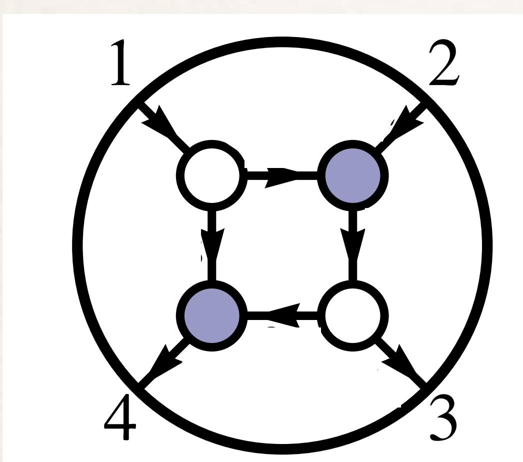
- Face variables



with the property

$$\prod_j f_j = -1$$

- Perfect orientation



Elements of $(k \times n)$matrix

$$c_{ab} = -\sum_{\Gamma} \prod_j (-f_j)$$

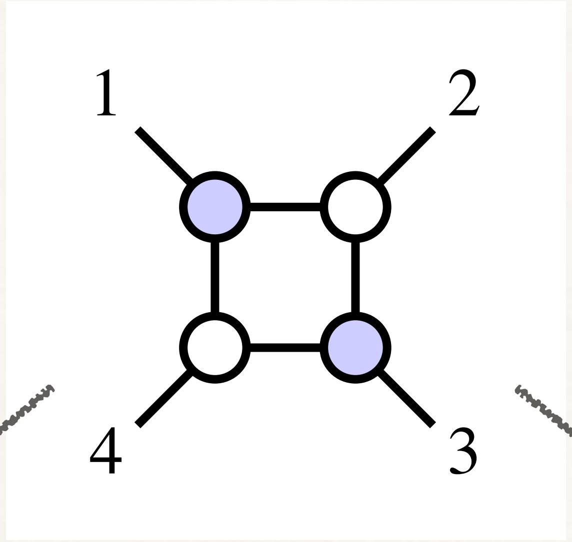
$$k \overset{n}{\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}}$$

Cell in positive Grassmannian $G_+(k,n)$

$$k \overset{k}{\begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix}} \geq 0$$

Surprising connection

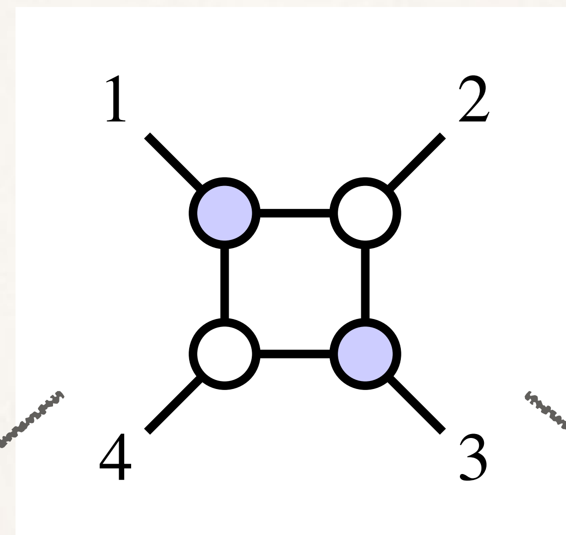
Connection



$$R = \mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\mathcal{M}_3^{tree}\mathcal{M}_4^{tree}$$

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

Connection



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$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

$$R = \int \frac{df_0}{f_0}\frac{df_1}{f_1}\frac{df_2}{f_2}\frac{df_3}{f_3}\delta(C\cdot Z) \qquad Z = (\lambda, \widetilde{\lambda}, \widetilde{\eta})$$

Momentum conservation

$$\delta(C \cdot Z) = \delta(C \cdot \widetilde{\lambda})\delta(C^{\perp} \cdot \lambda)$$

- Simple motivation: linearize momentum conservation

$$\delta(P) = \delta\left(\sum_a \lambda_a \widetilde{\lambda}_a\right)$$

- We want to write it as two linear factors

$$\delta\left(C_{ab}\widetilde{\lambda}_b\right)\,\delta\left(D_{ab}\lambda_b\right)$$

and get the condition: $D_{ab} = C^{\perp}_{ab}$

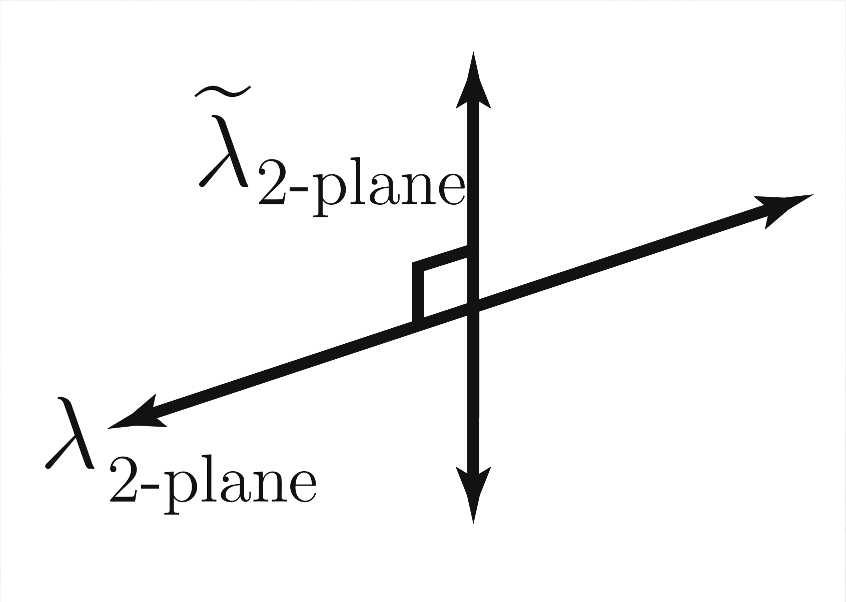
Geometry of delta function

- 2-planes for λ and $\tilde{\lambda}$ in n-dimensions

$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix}$$

same for $\tilde{\lambda}$

- Momentum conservation



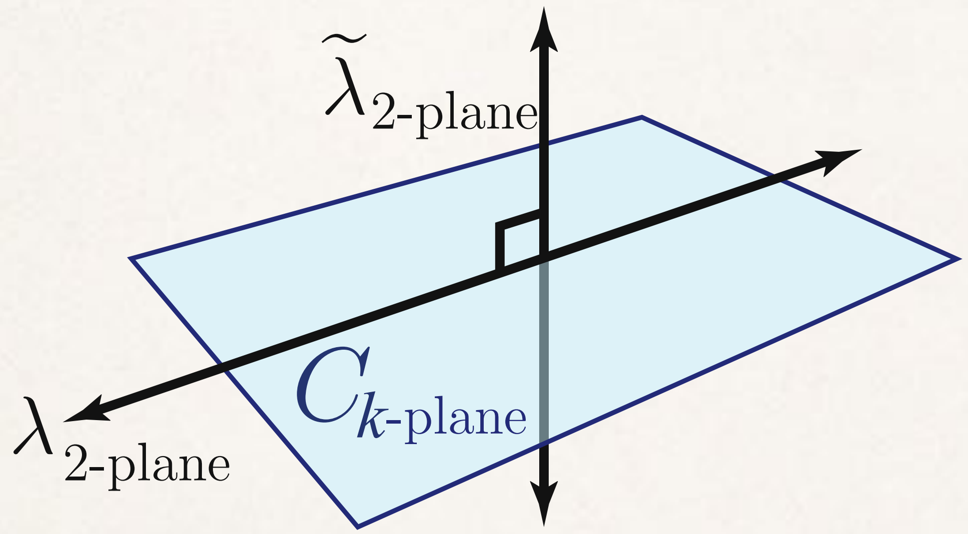
Two planes are orthogonal

$$\delta(\sum_k p_k) = \delta(\lambda \cdot \tilde{\lambda})$$

in the n-dimensional space

Geometry of delta function

- Introduce an auxiliary k-plane C in n-dimensions



C is orthogonal to $\tilde{\lambda}$

$$\delta(C \cdot \tilde{\lambda})$$

C is contains λ

$=$ λ orthogonal to $C^{\perp}$

$$\delta(C^{\perp} \cdot \lambda)$$

- This forces λ and $\tilde{\lambda}$ to be orthogonal

$$\delta(\lambda \cdot \tilde{\lambda})$$

Logarithmic form

$$R = \int \frac{df_1}{f_1}\frac{df_2}{f_2}\dots\frac{df_m}{f_m}\,\delta(C\cdot Z)$$

- Delta functions

$$\delta(C\cdot Z) \equiv \delta(C\cdot\widetilde{\lambda})\times\delta(C^{\perp}\cdot\lambda)\times\delta(C\cdot\widetilde{\eta})$$

$$\delta(\lambda\cdot\widetilde{\lambda})$$

Solves for some/all parameters f_j

For example

$$\delta\left(f_1 - \frac{\langle 12\rangle}{\langle 13\rangle}\right)$$

Then use $\displaystyle\int\frac{df}{f}\delta(f-f_0) = \frac{1}{f_0}$

Logarithmic form

$$R = \int \frac{df_1}{f_1}\frac{df_2}{f_2}\dots\frac{df_m}{f_m}\,\delta(C\cdot Z)$$

- Delta functions

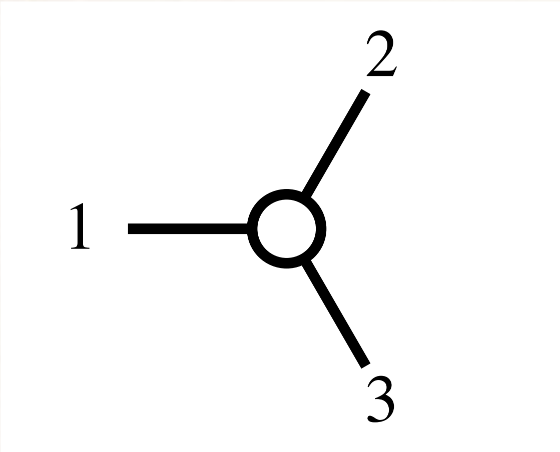
$$\delta(C\cdot Z) \equiv \delta(C\cdot\widetilde{\lambda}) \times \delta(C^\perp\cdot\lambda) \times \delta(C\cdot\widetilde{\eta})$$

Depending on the dimensionality

Polynomial in fermonic $\widetilde{\eta}$

- We solve for all f_j
- We solve for all f_j and in addition delta functions for $\lambda, \widetilde{\lambda}$
- Not enough delta functions to solve for all f_j - some left unspecified

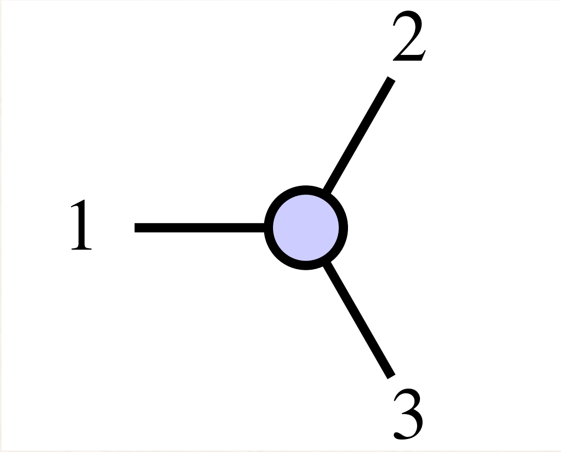
Derivation: starting with 3pt



$$C = \begin{pmatrix} 1 & z_1 & z_2 \end{pmatrix}$$

$$A_3^{(1)} = \int \frac{dz_1\, dz_2}{z_1 z_2} \delta^{(1)}(C \cdot Z)$$

$$\delta^{(1)}(C \cdot Z) = \delta^{1\times 2}(C \cdot \widetilde{\lambda})\delta^{2\times 2}(\lambda \cdot C^{\perp})\delta^{1\times 4}(C \cdot \widetilde{\eta})$$



$$C = \begin{pmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \end{pmatrix}$$

$$A_3^{(2)} = \int \frac{dz_1\, dz_2}{z_1 z_2} \delta^{(2)}(C \cdot Z)$$

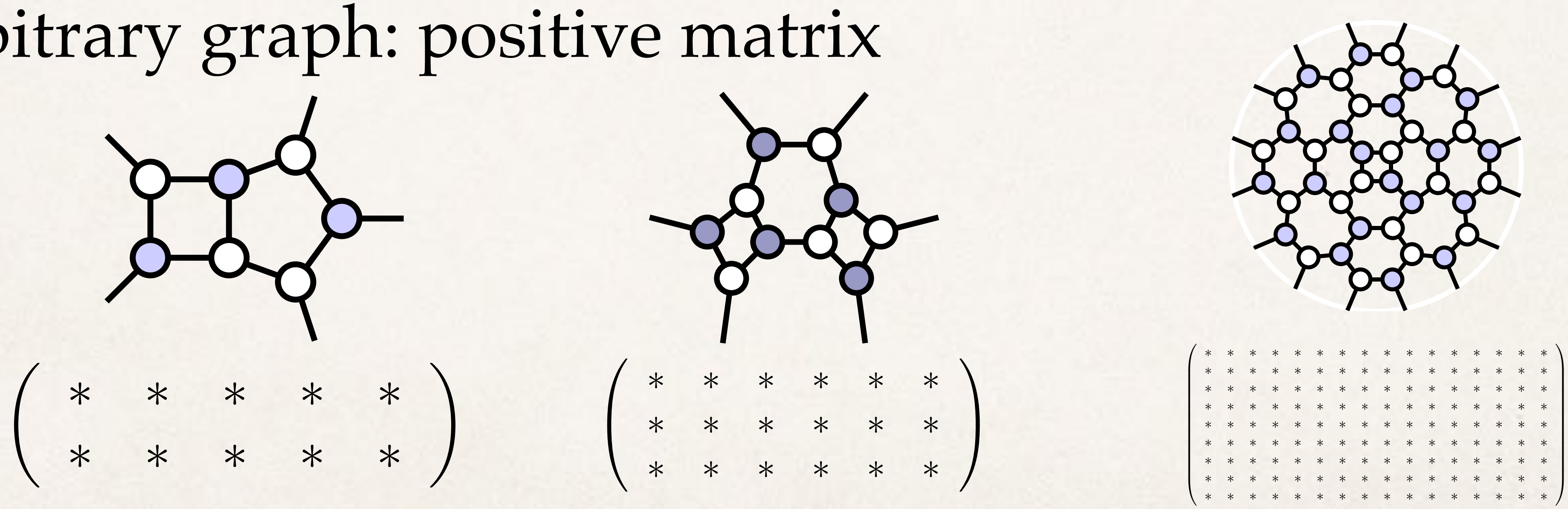
$$\delta^{(2)}(C \cdot Z) = \delta^{2\times 2}(C \cdot \widetilde{\lambda})\delta^{2\times 1}(\lambda \cdot C^{\perp})\delta^{2\times 4}(C \cdot \widetilde{\eta})$$

Amalgamation procedure

- Construct big positive matrix from small ones

Gluing preserves positivity of minors

- Arbitrary graph: positive matrix



New procedure

- Write the amplitude as a sum of on-shell diagrams using recursion relations
- For each on-shell diagram construct the C-matrix using the boundary measurement
- Write logarithmic form which calculates the diagram

Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

N=4 SYM:

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\dots\frac{d\alpha_n}{\alpha_n}\delta(C\cdot Z)$$

Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

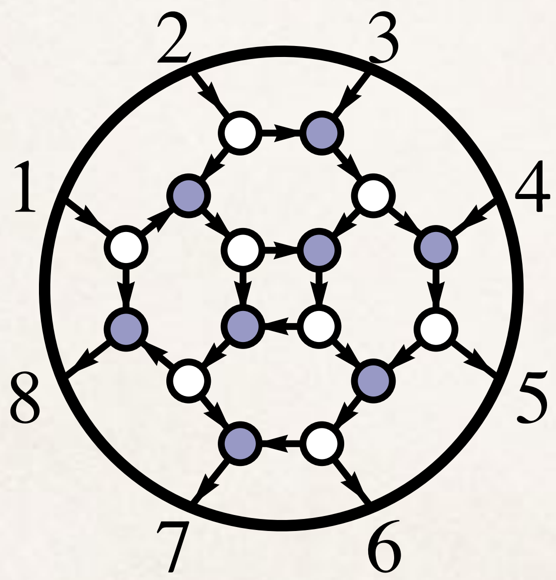
General QFT:

$$\Omega = F(\alpha)\,\delta(C\cdot Z)$$

- In a sense $F(\alpha)$ defines a theory (as Lagrangian does)

Exploring space of theories

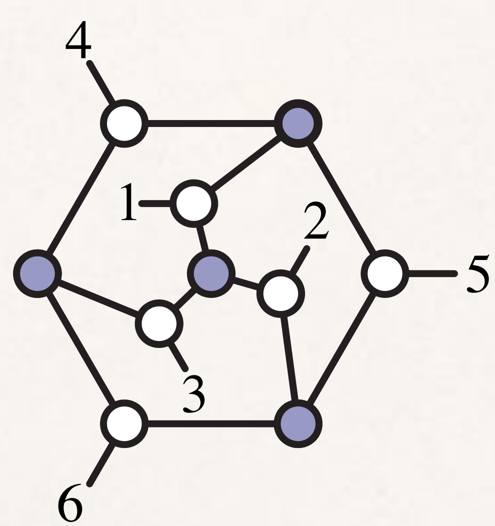
- One step at a time away from N=4 SYM
- **Case 1**: planar N<4 SYM (including QCD)



$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\dots\frac{d\alpha_n}{\alpha_n}\cdot\mathcal{J}(\alpha)\delta(C\cdot Z)$$

Exploring space of theories

- **Case 2:** Non-planar N=4 SYM



same form as planar

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\dots\frac{d\alpha_n}{\alpha_n}\delta(C\cdot Z)$$

less is known on mathematical side

- **Case 3**: N=8 Supergravity

Both have natural extensions to lower SUSY

$$\Omega = \frac{d\alpha_1}{\alpha_1^3}\frac{d\alpha_2}{\alpha_2^3}\dots\frac{d\alpha_m}{\alpha_m^3}\prod_v \Delta_v\cdot\delta(C\cdot Z)$$

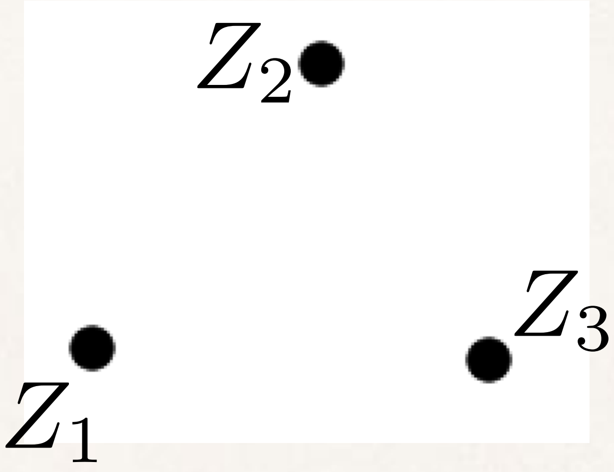
Back to planar N=4 SYM theory

- Each on-shell diagram has a nice geometric interpretation
- To get the amplitude we use the recursion relations = consequence of the factorization of tree-level amplitudes
- From geometric point of view: why this particular sum?
- Goal: to find the geometric formulation for the full amplitude

Inside of the triangle

Inside of the triangle

- Let us consider three points in a projective plane

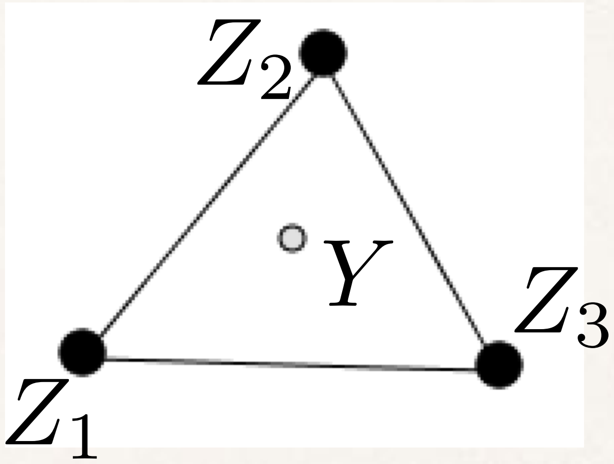


$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim tZ_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

Inside of the triangle

- Point inside the triangle



$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim t Z_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

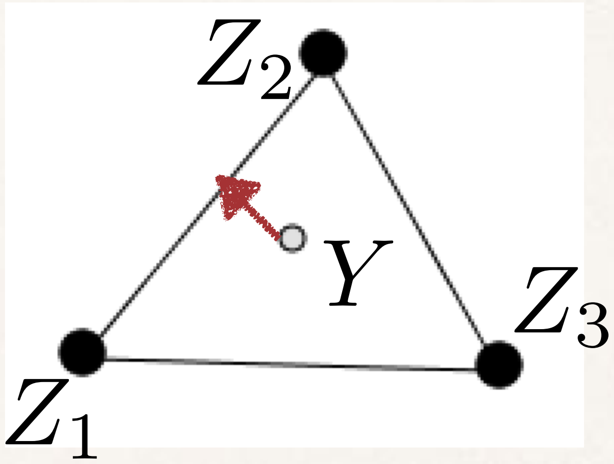
- Point inside the triangle

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \qquad c_1, c_2, c_3 > 0$$

Projective: one of c_j can be fixed to 1

Inside of the triangle

- Point inside the triangle



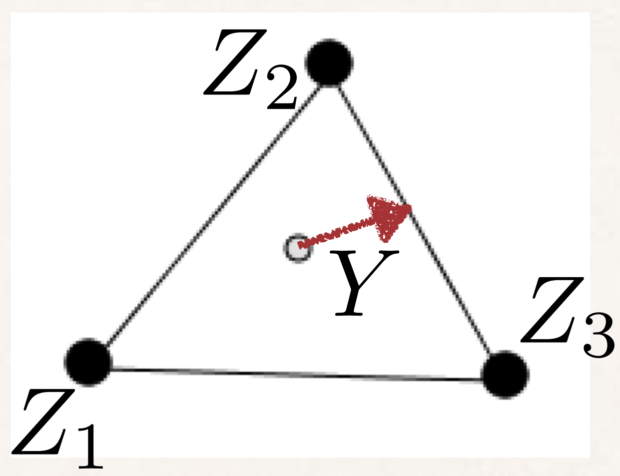
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_3 = 0$$

Inside of the triangle

- Point inside the triangle



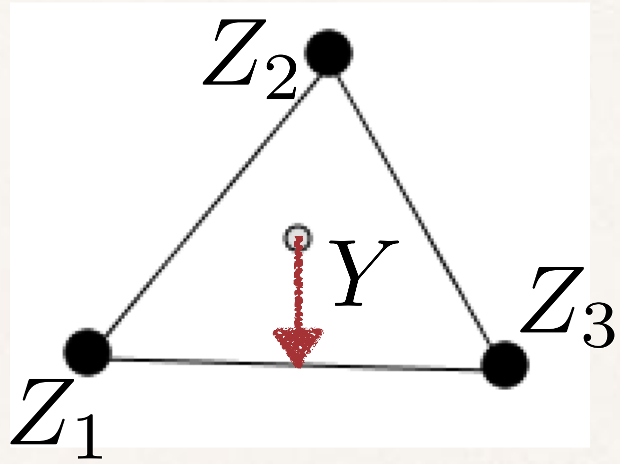
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_1 = 0$$

Inside of the triangle

- Point inside the triangle



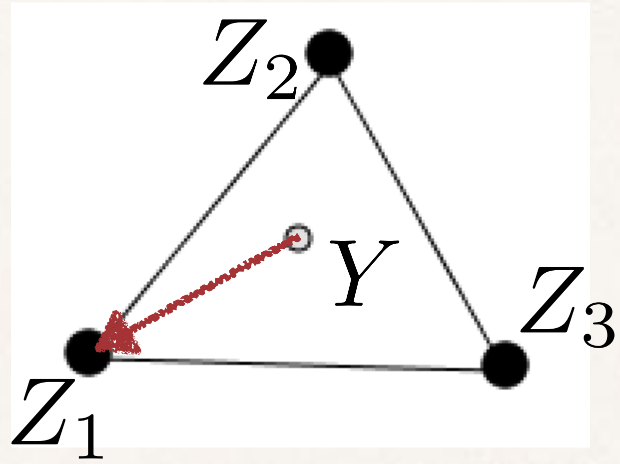
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = 0$$

Inside of the triangle

- Point inside the triangle



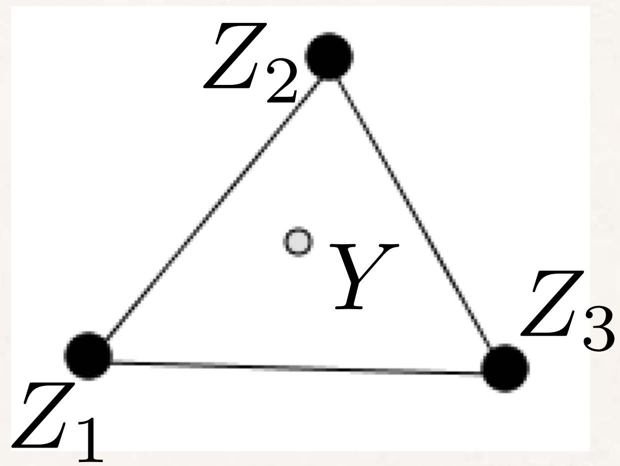
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = c_3 = 0$$

Logarithmic form

- Point inside the triangle



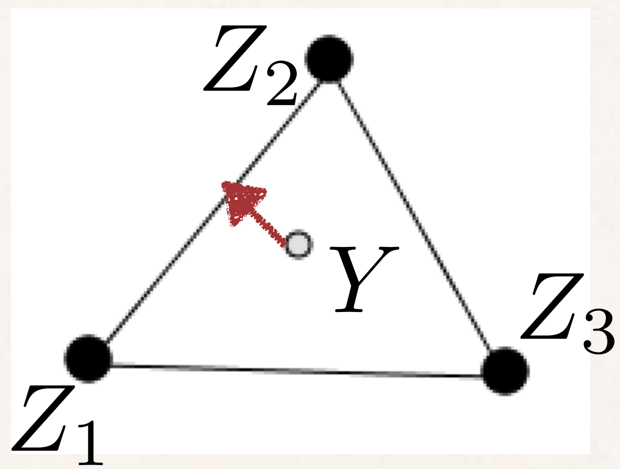
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_3 = 0$$

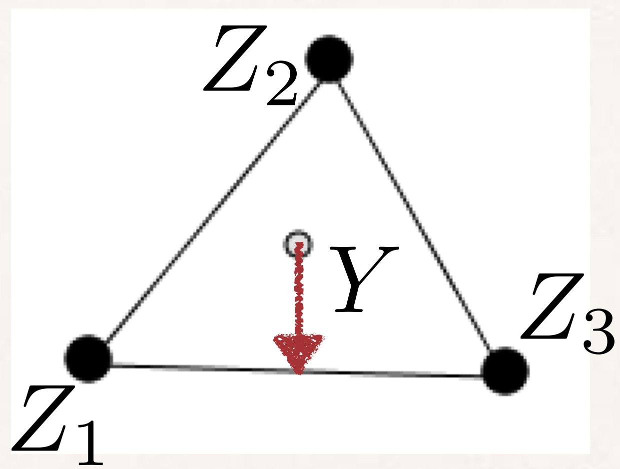
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_2}{c_2}$$

Logarithmic form

- Point inside the triangle



$$c_2 = 0$$

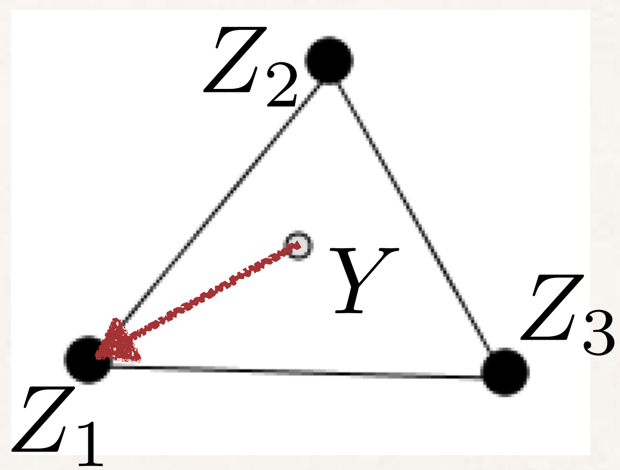
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

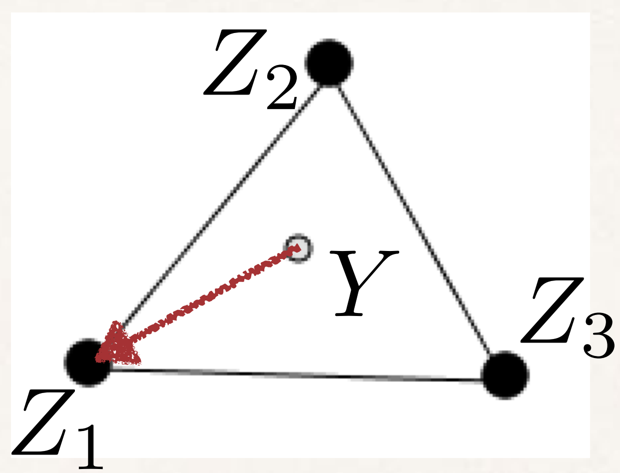
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3} \to 1$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3} \to 1$$

- Other boundaries can correspond to $c_2, c_3 \to \infty$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

$$\langle X_1 X_2 X_3 \rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

↓

$$c_2 = \frac{\langle Y13 \rangle}{\langle Y23 \rangle} \quad c_3 = \frac{\langle Y12 \rangle}{\langle Y23 \rangle} \quad dc_2\, dc_3 = \frac{\langle Y\, d^2Y \rangle \langle 123 \rangle^2}{\langle Y23 \rangle^3}$$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

$$\langle X_1 X_2 X_3 \rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2 Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

$$\downarrow$$

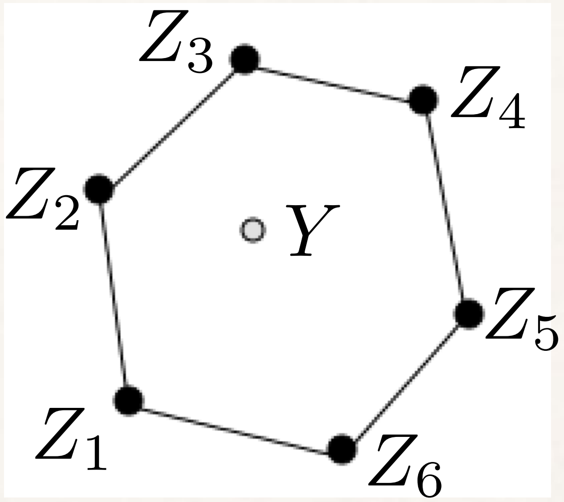
$$\Omega = \frac{\langle Y\, d^2 Y \rangle \langle 123 \rangle^2}{\langle Y 12 \rangle \langle Y 23 \rangle \langle Y 31 \rangle}$$

Projective in all variables

Polygon

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

$$\Downarrow$$

$$c_j > 0$$

interior of the polygon

- Convex polygon: condition on points Z_i

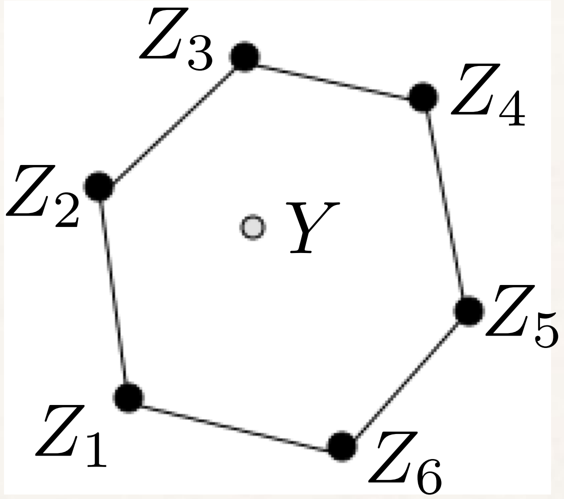
$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \dots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

All main minors positive

$$\begin{vmatrix} * & * & * \\ * & * & * \\ * & * & * \end{vmatrix} > 0$$

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

Space of all points inside convex polygon

More formally:

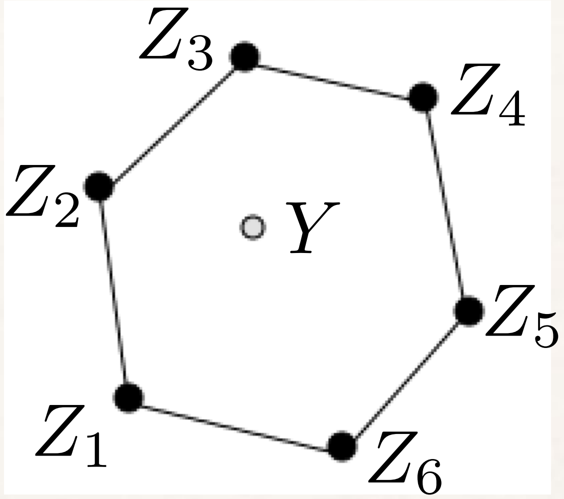
$$\boxed{Y = C \cdot Z}$$

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \end{pmatrix} \in G_+(1,n)$$

$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \ldots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \in M_+(3,n)$$

Logarithmic form

- Form with logarithmic singularities on boundaries



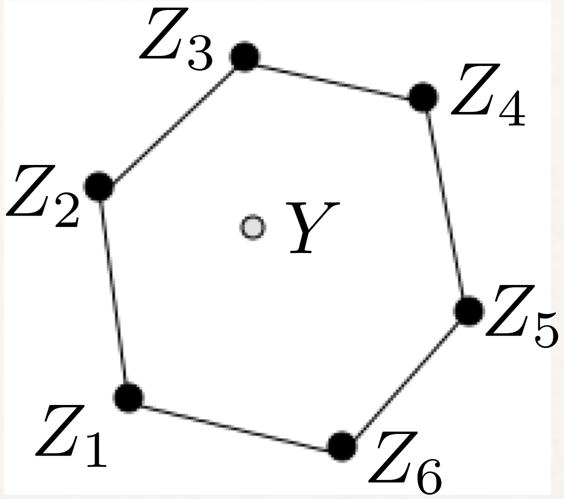
$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

First guess

$$\Omega = \frac{dc_1}{c_1}\frac{dc_2}{c_2}\dots\frac{dc_n}{c_n}$$

Logarithmic form

- Form with logarithmic singularities on boundaries



$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

First guess

$$\Omega = \frac{dc_1}{c_1}\frac{dc_2}{c_2}\dots\frac{dc_n}{c_n}$$

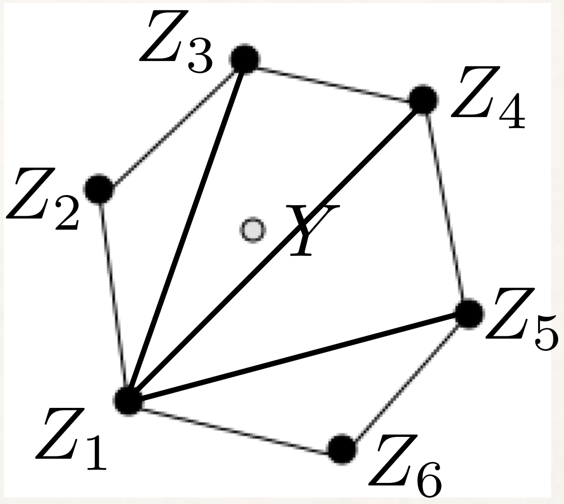
Space of all Y is only two-dimensional

- Two-form with n poles

$$\Omega \sim \frac{dc_1\, dc_2\, N(c_1, c_2)}{D(c_1, c_2)}$$

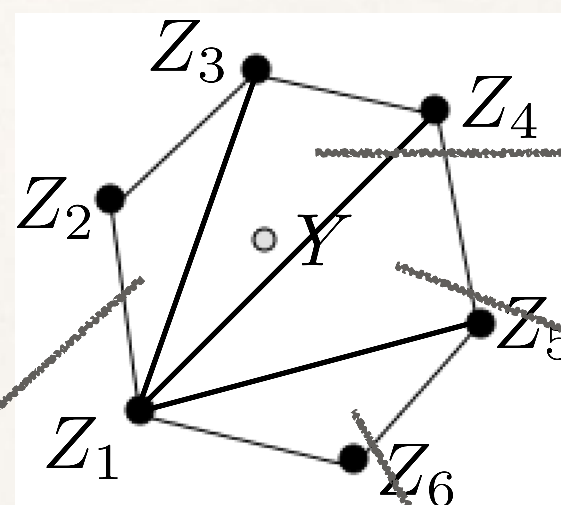
Logarithmic form

- Easiest way how to write the form is to triangulate



Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{dc_3}{c_3}\frac{dc_4}{c_4} \quad c_3, c_4 \geq 0$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{dc_4}{c_4}\frac{dc_5}{c_5} \quad c_4, c_5 \geq 0$$

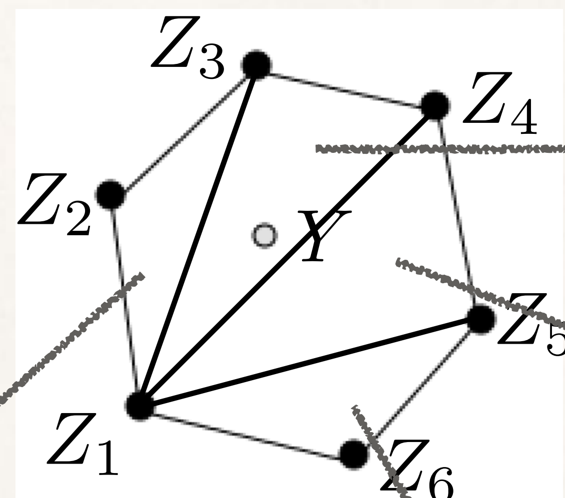
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \quad c_2, c_3 \geq 0$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{dc_5}{c_5}\frac{dc_6}{c_6} \quad c_5, c_6 \geq 0$$

How to sum them?

Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{\langle Y\, d^2Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{\langle Y\, d^2Y\rangle \langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle}$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{\langle Y\, d^2Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{\langle Y\, d^2Y\rangle \langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

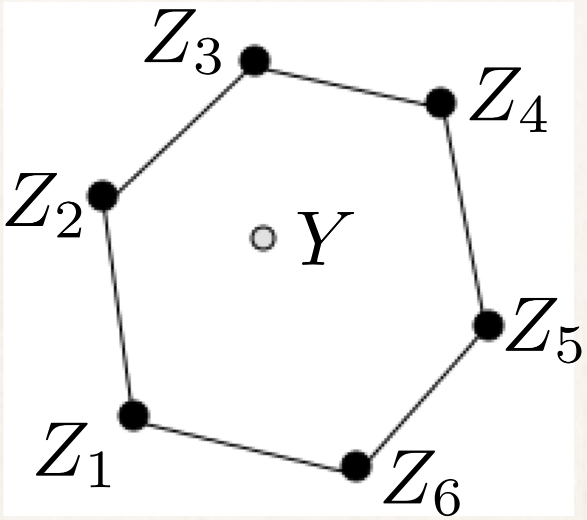
Write in projective form

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$

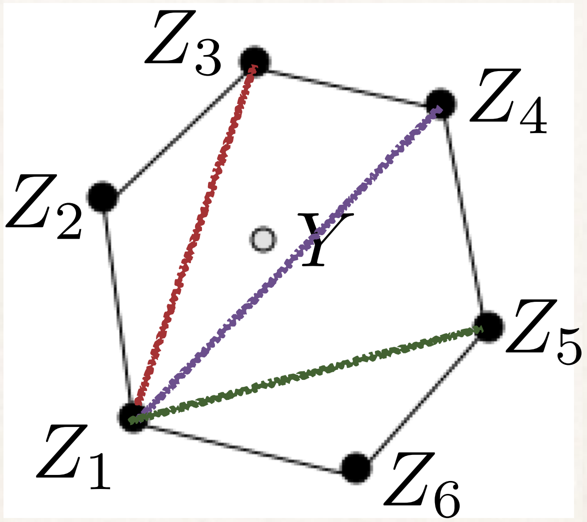


Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$



Spurious poles

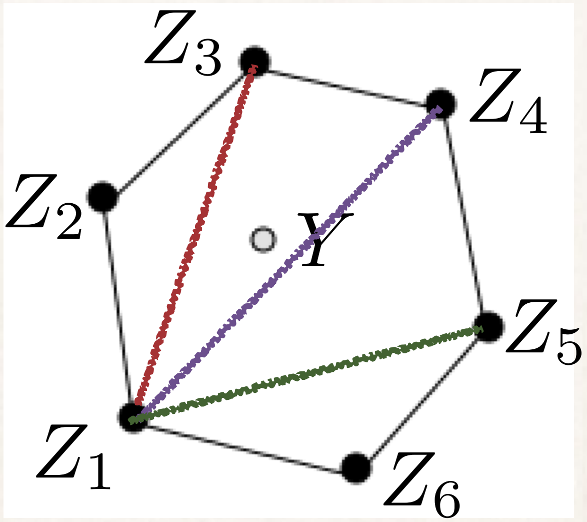
Cancel in the sum

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

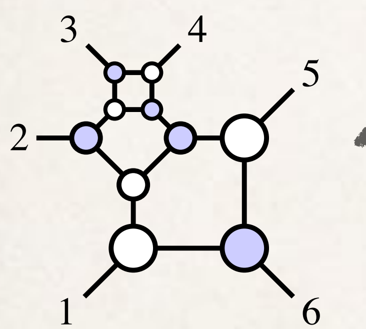
- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$

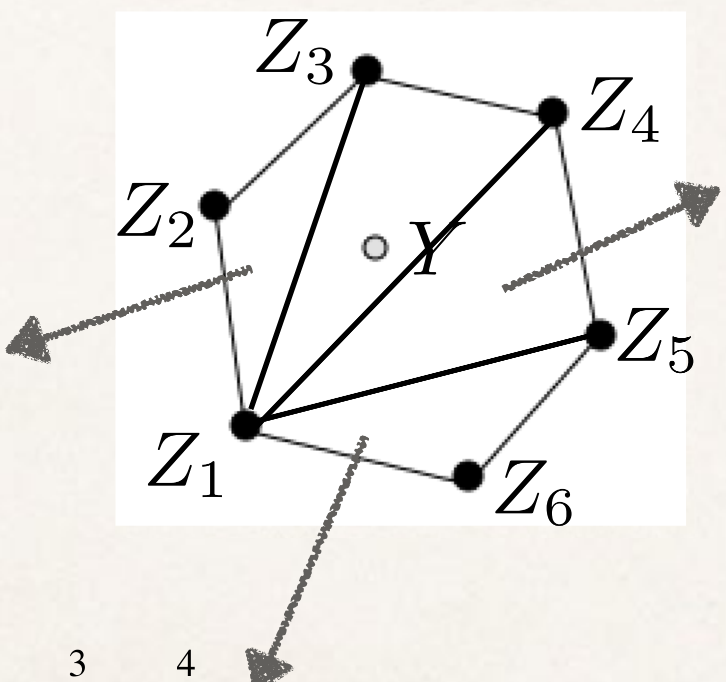


$$\Omega = \frac{\langle Y\, d^2Y\rangle\, \mathcal{N}(Y, Z_j)}{\langle Y12\rangle\langle Y23\rangle\langle Y34\rangle\langle Y45\rangle\langle Y56\rangle\langle Y61\rangle}$$

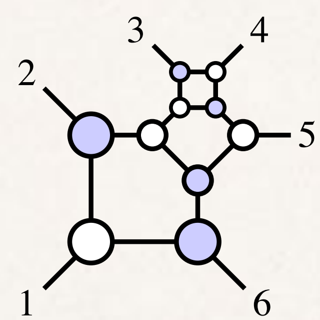
Similarities with on-shell diagrams

- Notice some similarities with recursion relations and on-shell diagrams





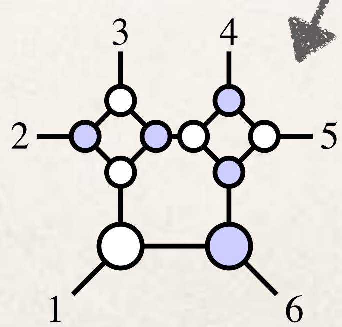
$$C = \begin{pmatrix} 1 & 0 & 0 & c_4 & c_5 & 0 \end{pmatrix} \in G_+(1,6)$$



$$\Omega = \frac{dc_4}{c_4}\frac{dc_5}{c_5}$$ looks similar

$$Y = C \cdot Z \qquad \delta(C \cdot Z)$$

$$\Omega = \frac{\langle Y\, d^2Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$



Amplituhedron

Amplituhedron

- On-shell diagrams triangulate the bigger object which represents the scattering amplitude: Amplituhedron

tree $Y_\alpha^I = C_{\alpha a} Z_a^I$

$$C_{\alpha a} \in G_+(k,n)$$
$$Z_a^I \in M_+(n, n+4)$$
$$Y \in G(k, k+4)$$

- Z - kinematical variables (momentum twistors)

- Extension to the loop integrand

Amplitude from Amplituhedron

- Amplitude is associated with th form with logarithmic singularities on the boundaries of the Amplituhedron

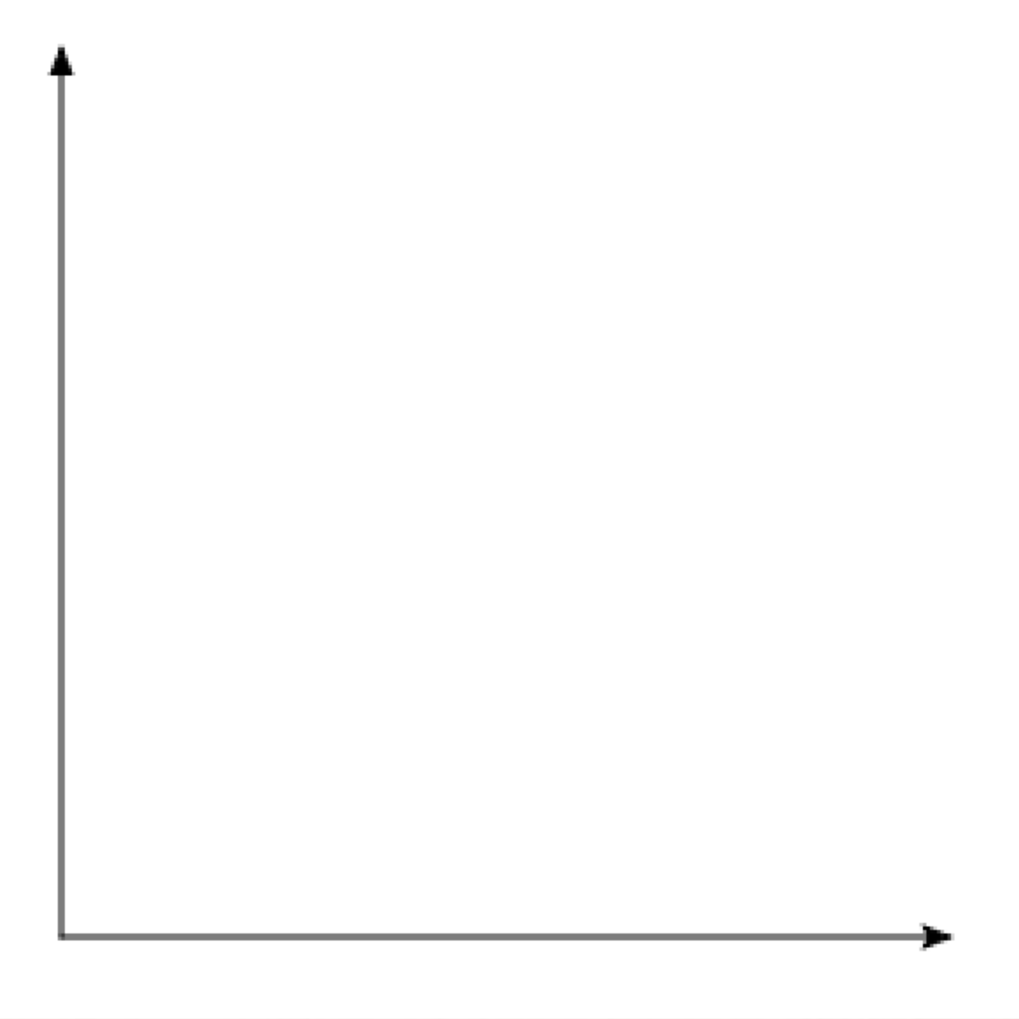
$$\Omega \sim \frac{dx}{x} \qquad \text{near} \quad x = 0$$

- More than just simple poles: $\dfrac{dx\,dy}{xy(x+y)} \xrightarrow{x=0} \dfrac{dy}{y^2}$
- The main work is to triangulate the space
- Instead of explaining the precise definition: example of the 4pt scattering to all loops

High school problem $gg \to gg$

High school problem $gg \to gg$

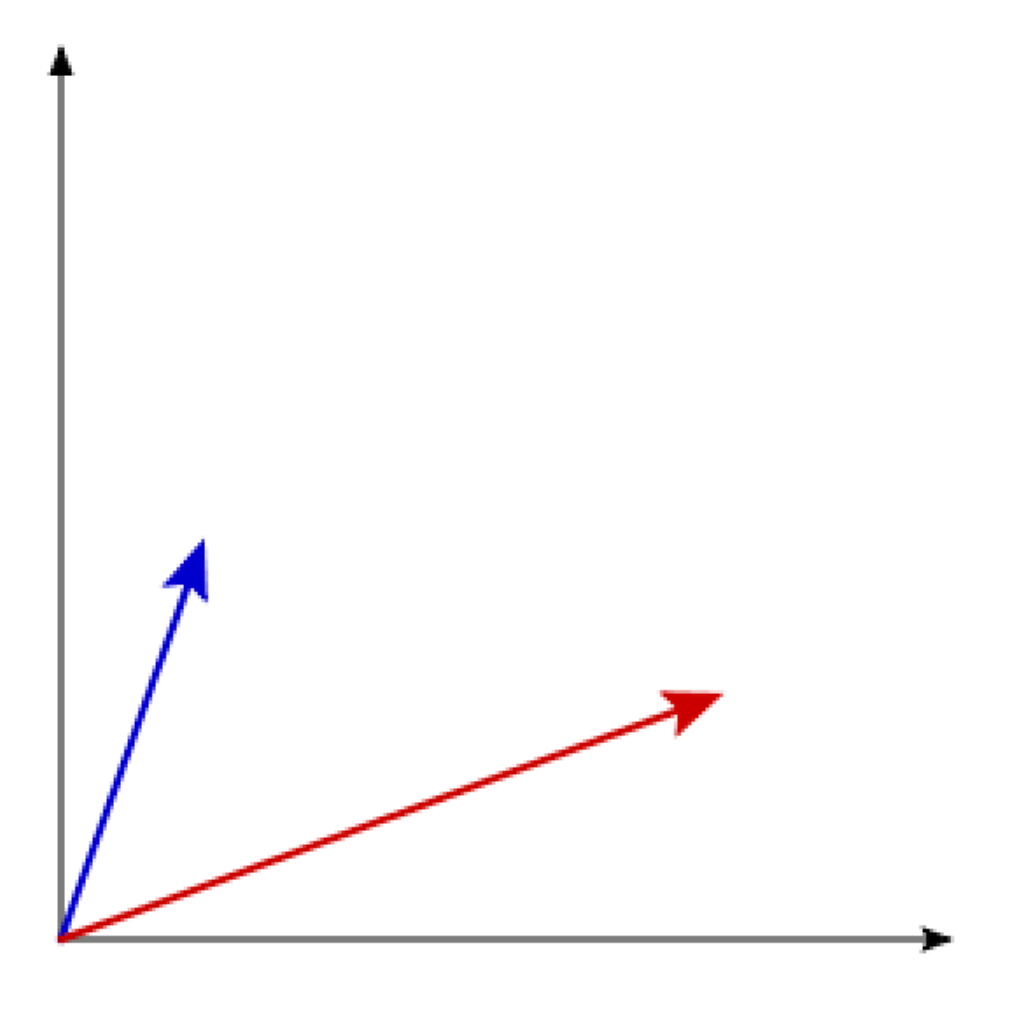
- Positive quadrant



High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

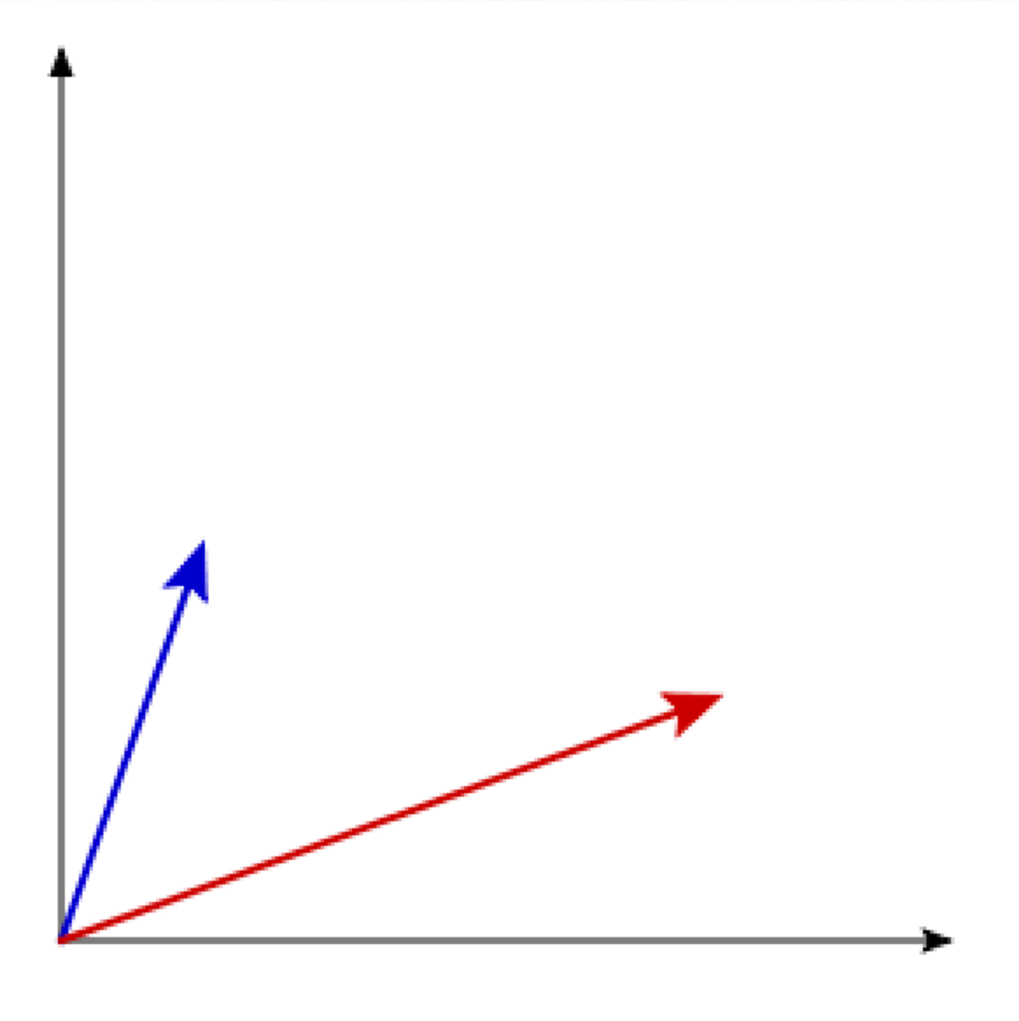


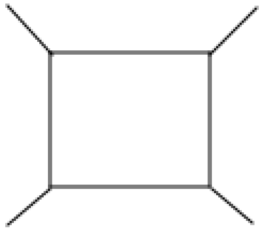
$$\mathrm{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



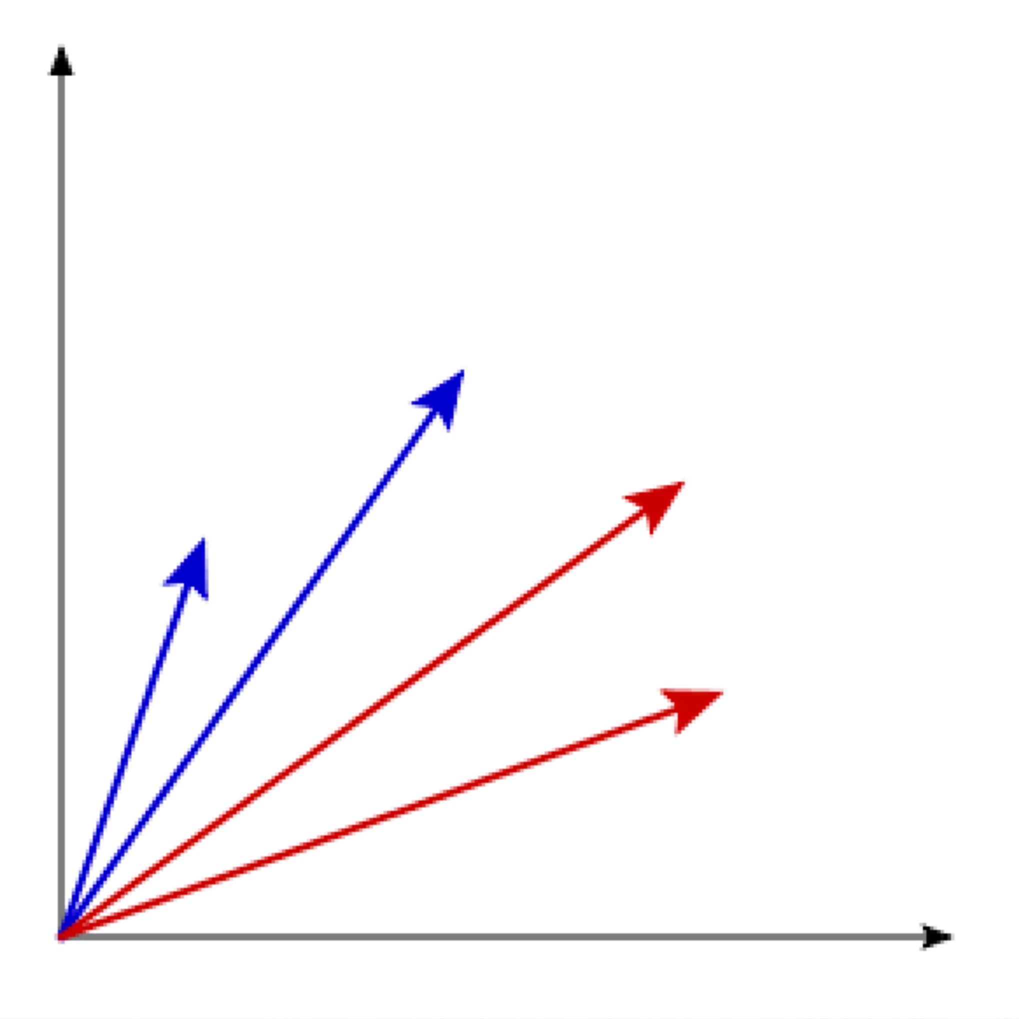
$$\mathrm{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1} =$$ 

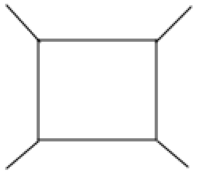
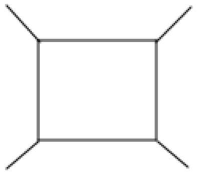
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$[\mathrm{Vol}\,(1)]^2 = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2} =$$  $\times$ 

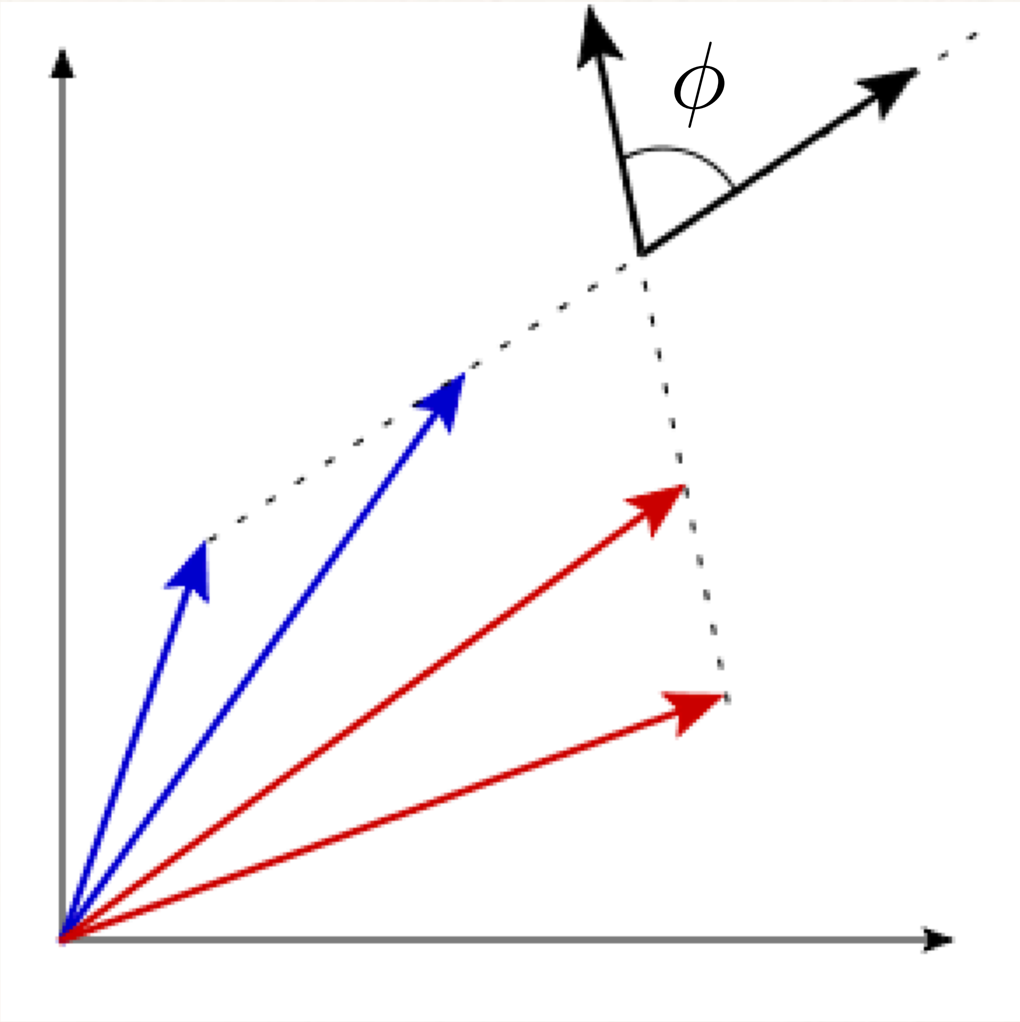
High school problem $gg \to gg$

- Positive quadrant

- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



- Impose: $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1) \leq 0$ $\qquad \phi > 90^o$

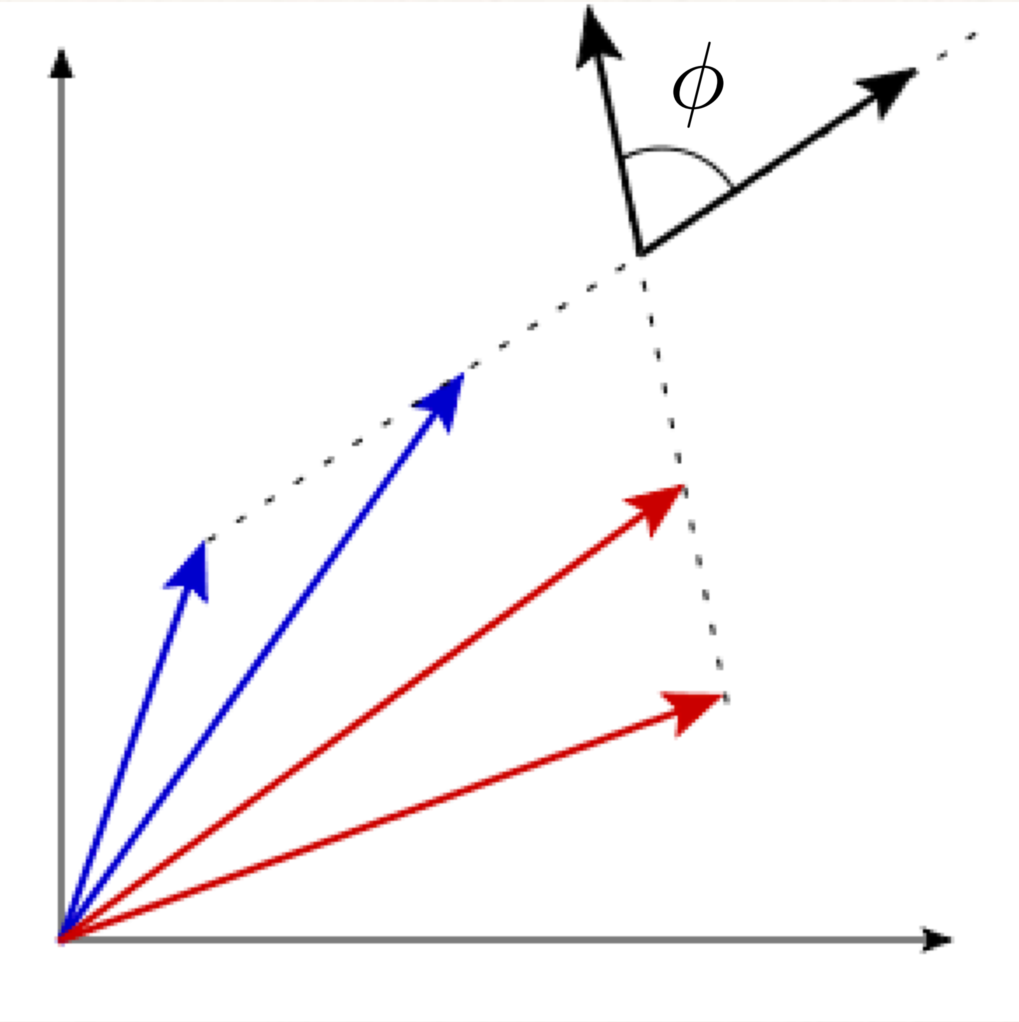
Subset of configurations allowed: triangulate

High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



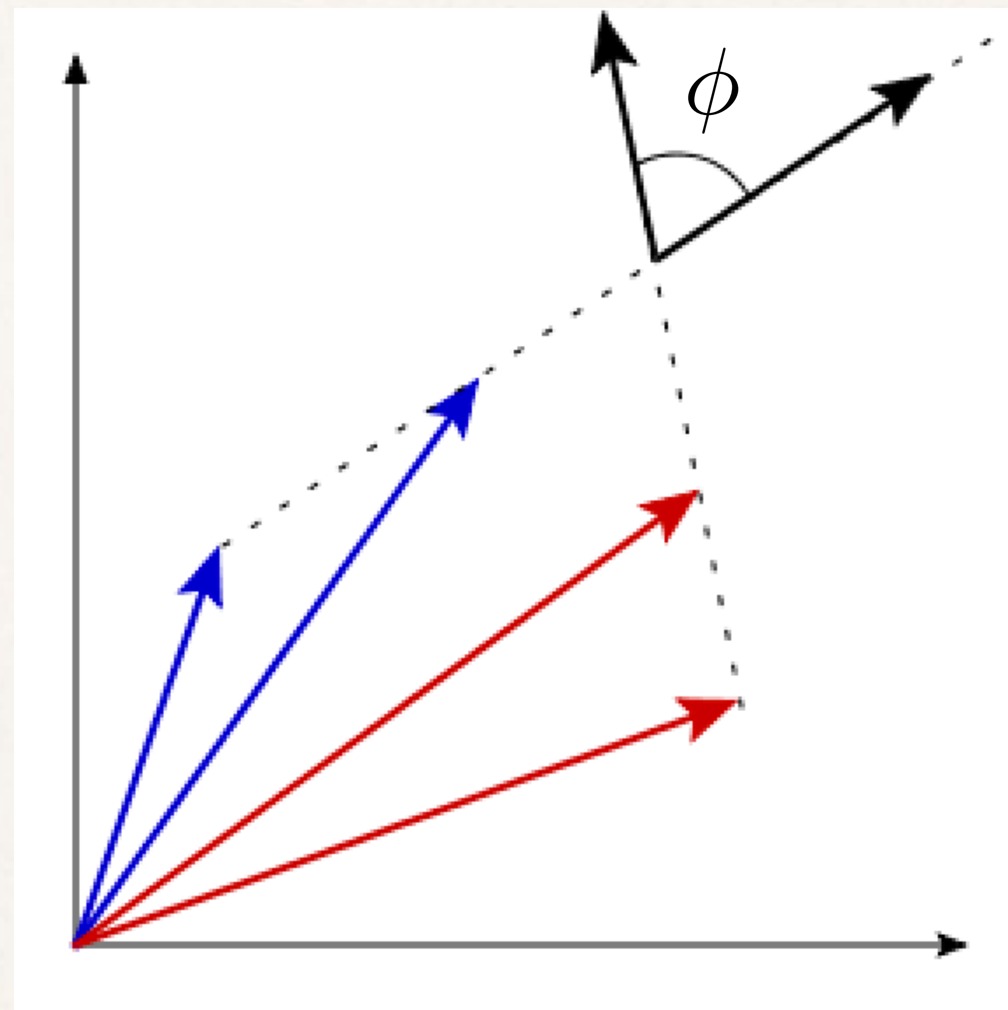
$$\mathrm{Vol}\,(2) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}\left[\frac{\vec{a}_1\cdot\vec{b}_2+\vec{a}_2\cdot\vec{b}_1}{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_2-\vec{b}_1)}\right]$$

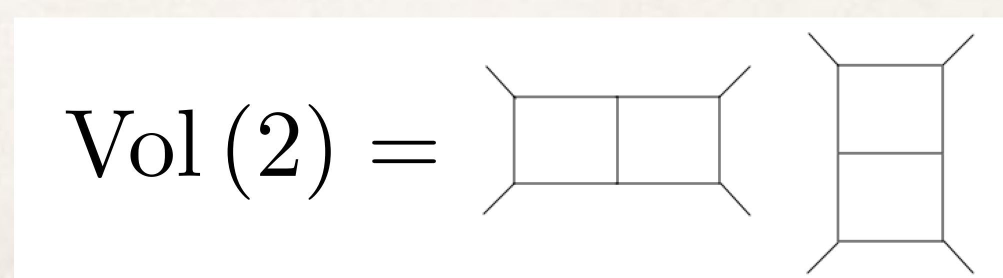
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\color{red}{\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}} \quad \color{blue}{\vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}}$$

$$\color{red}{\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}} \quad \color{blue}{\vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}}$$





High school problem $gg \to gg$

- Positive quadrant

- Vectors

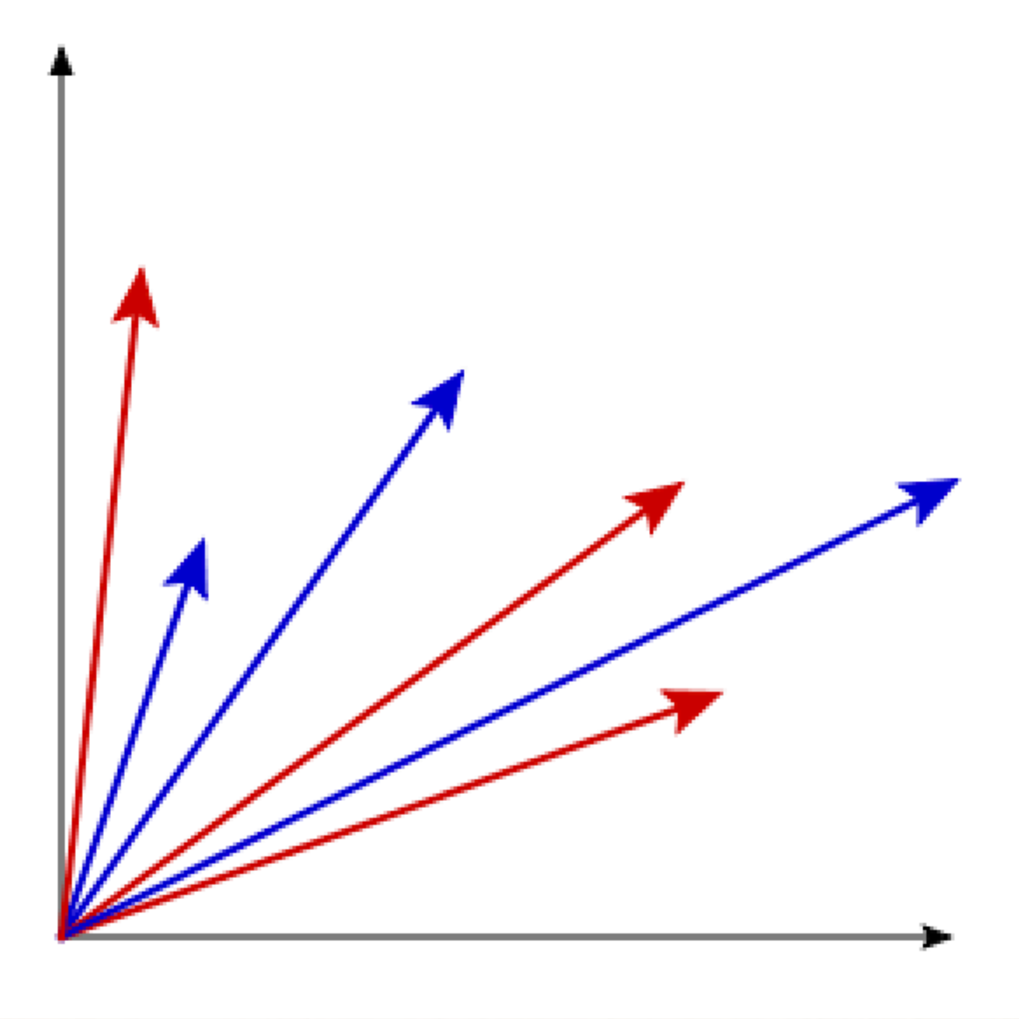
  $\vec{a}_1, \vec{a}_2, \vec{a}_3 \qquad \vec{b}_1, \vec{b}_2, \vec{b}_3$

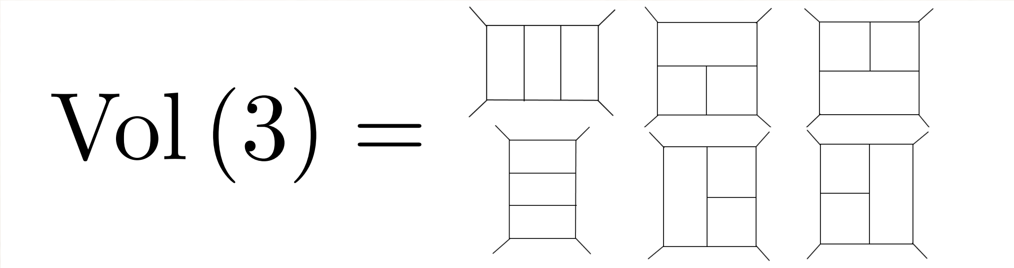
- Conditions

$$(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2) \leq 0$$

$$(\vec{a}_1 - \vec{a}_3) \cdot (\vec{b}_1 - \vec{b}_3) \leq 0$$

$$(\vec{a}_2 - \vec{a}_3) \cdot (\vec{b}_2 - \vec{b}_3) \leq 0$$





High school problem $gg \to gg$

- Positive quadrant
- Vectors

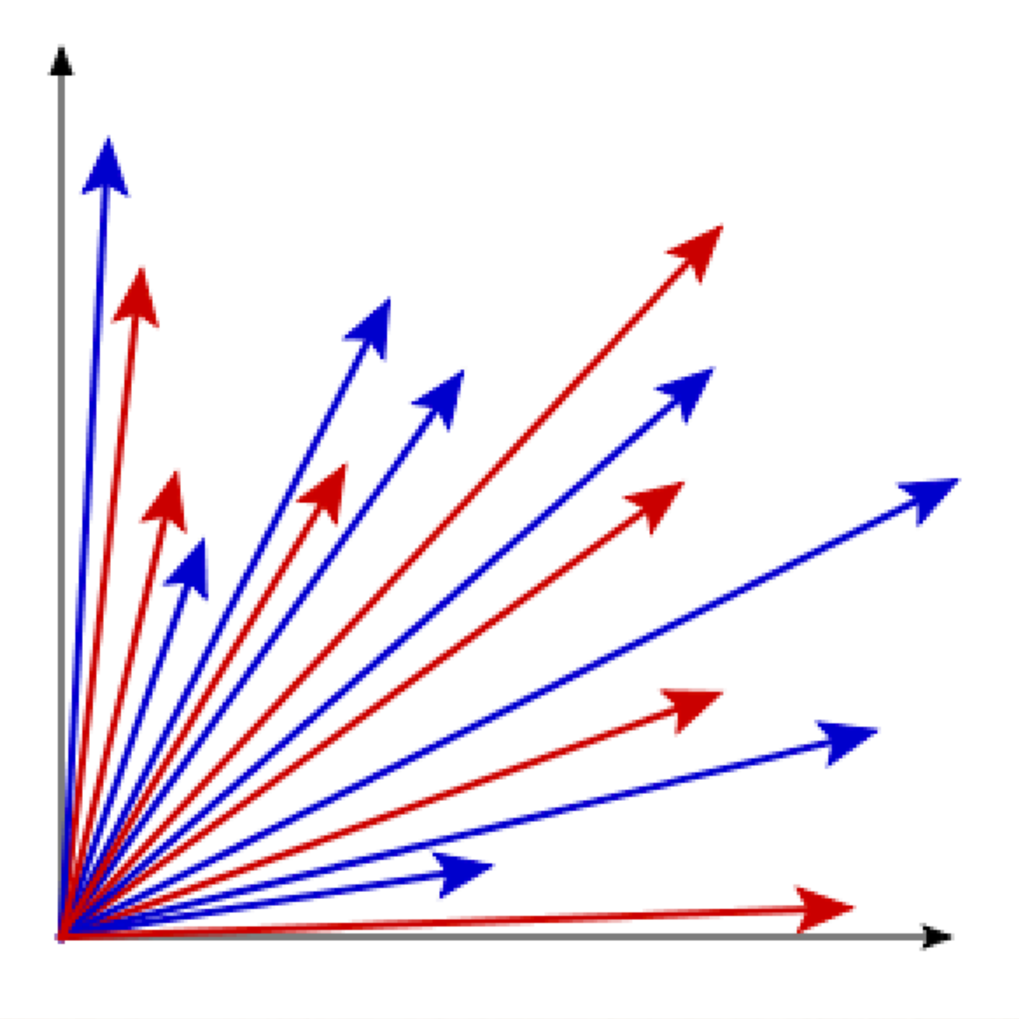
$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell \quad \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_\ell$

- Conditions

$$(\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) \leq 0$$

for all pairs i, j

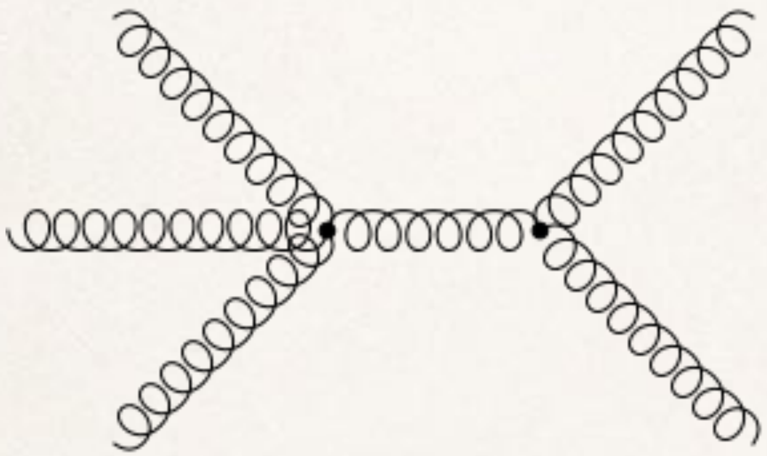
Let me know if you solve it!

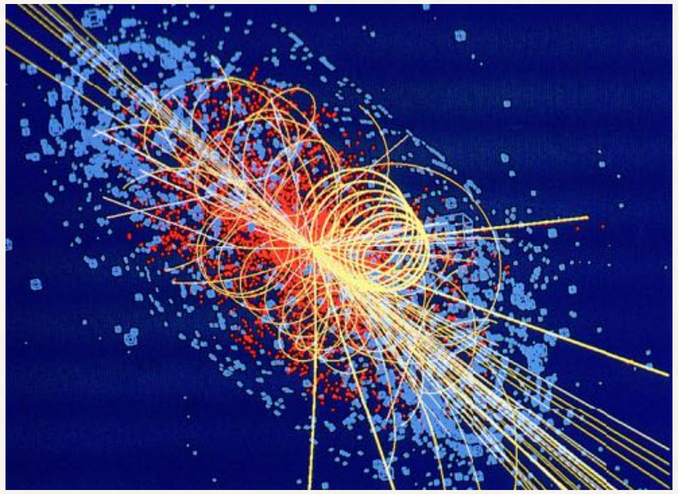


$$\mathrm{Vol}\,(\ell) = \ldots\ldots$$

Physics vs geometry

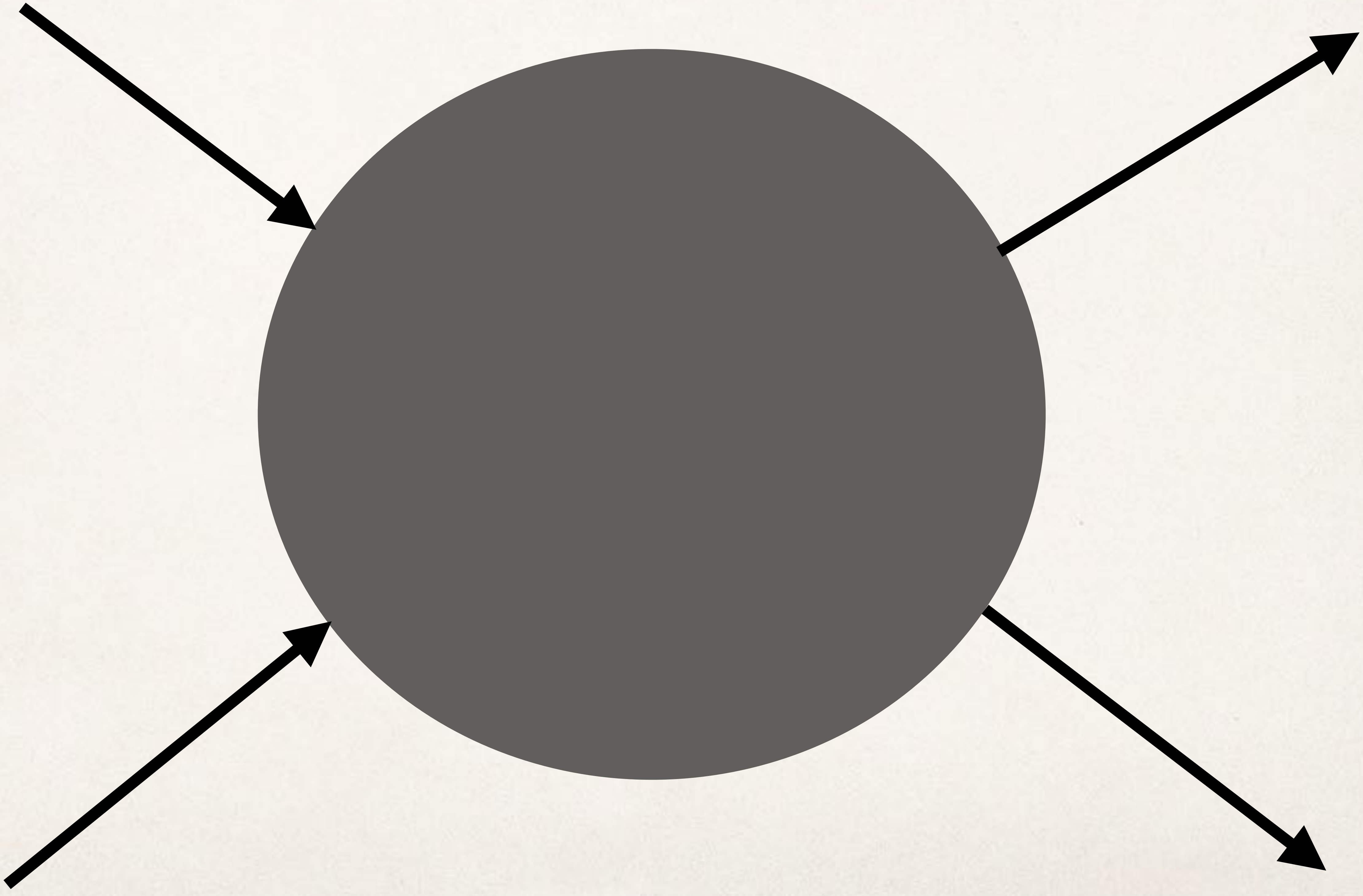
- Dynamical particle interactions in 4-dimensions



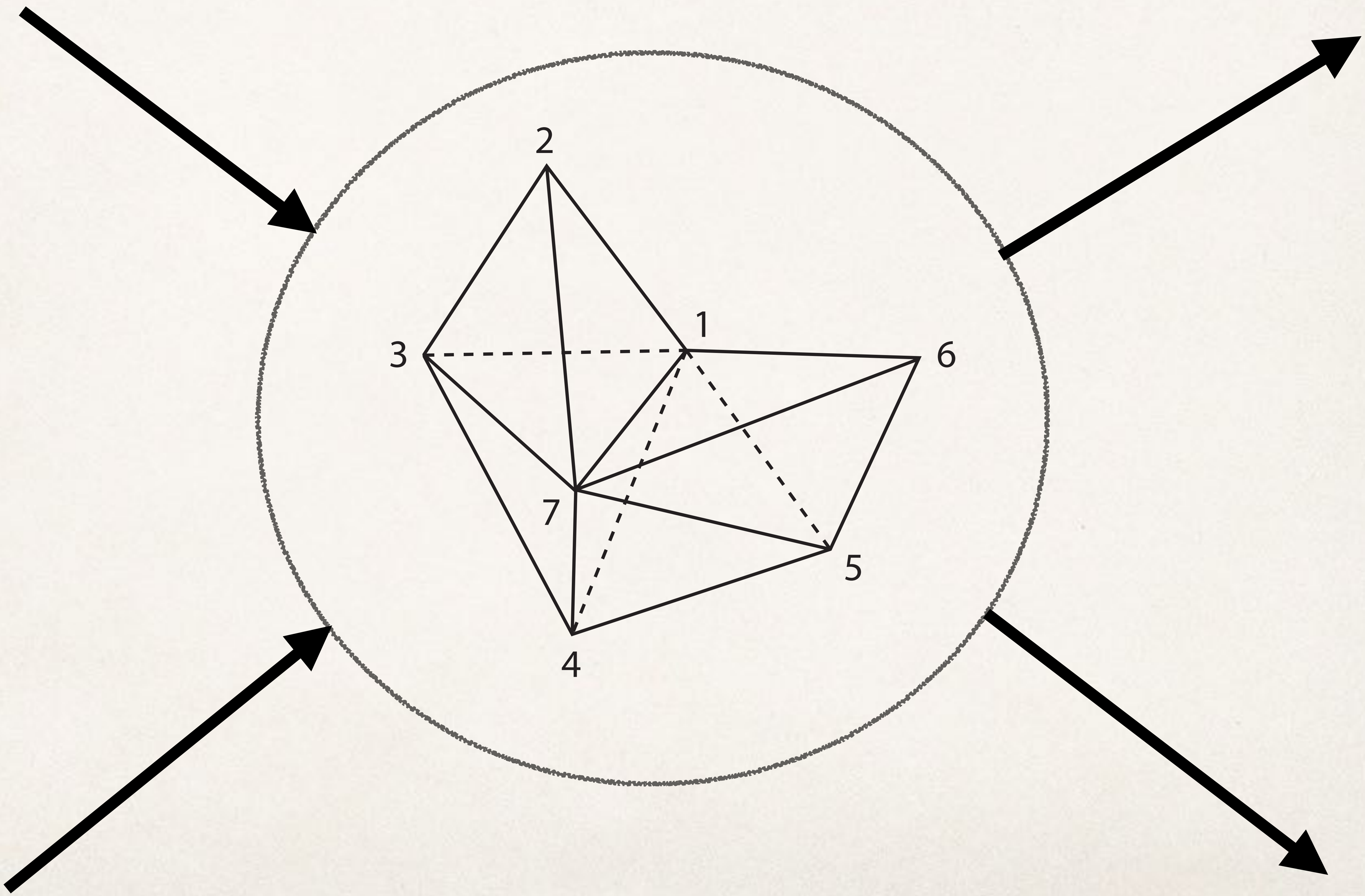


- Static geometry in high dimensional space

What is scattering amplitude?



What is scattering amplitude?

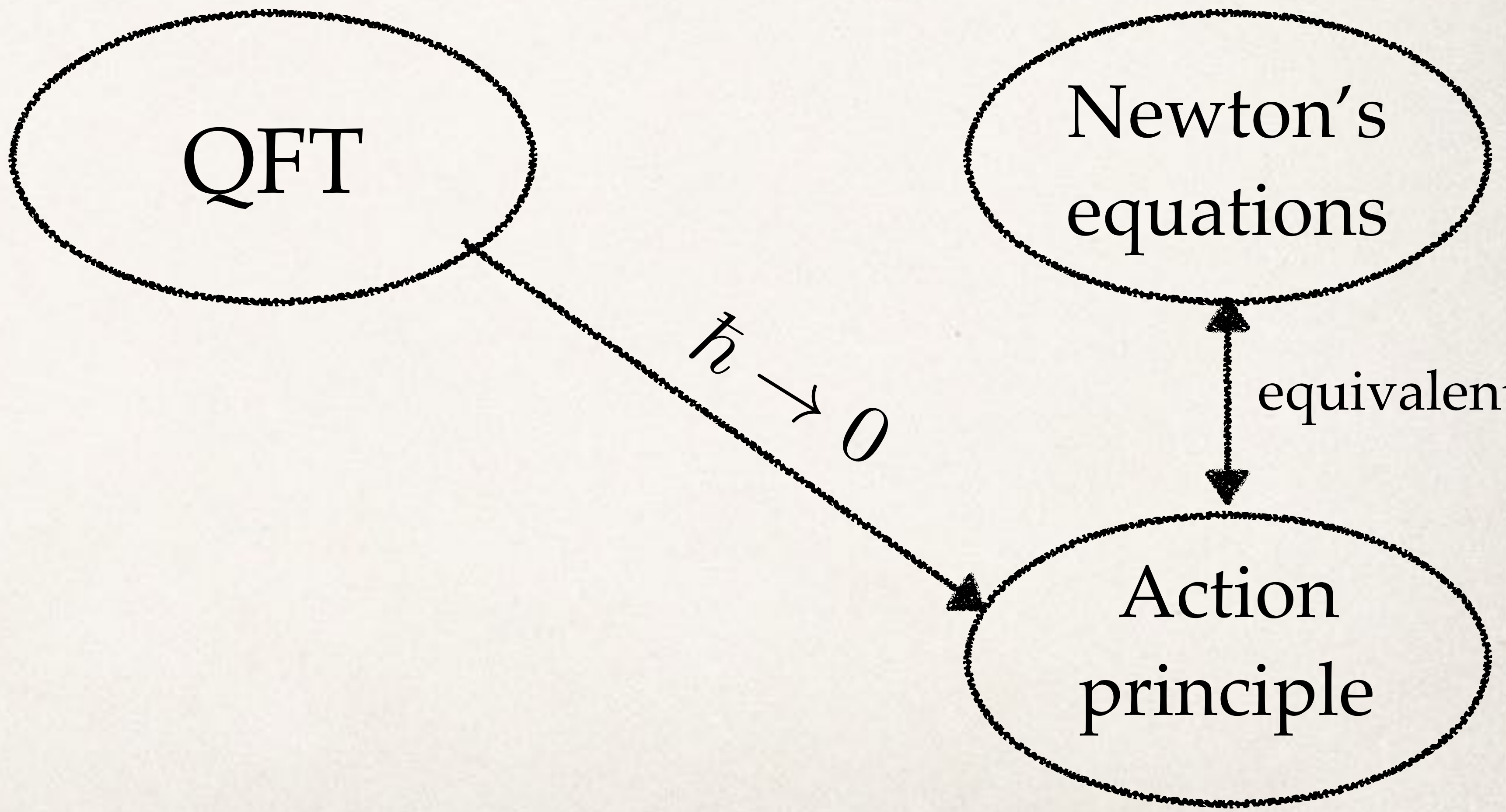


Step 1.1.1.

- It is very early to say if/how this can generalize
- Some encouraging news but more work needed
- Extend to other theories, beyond the integrand,….

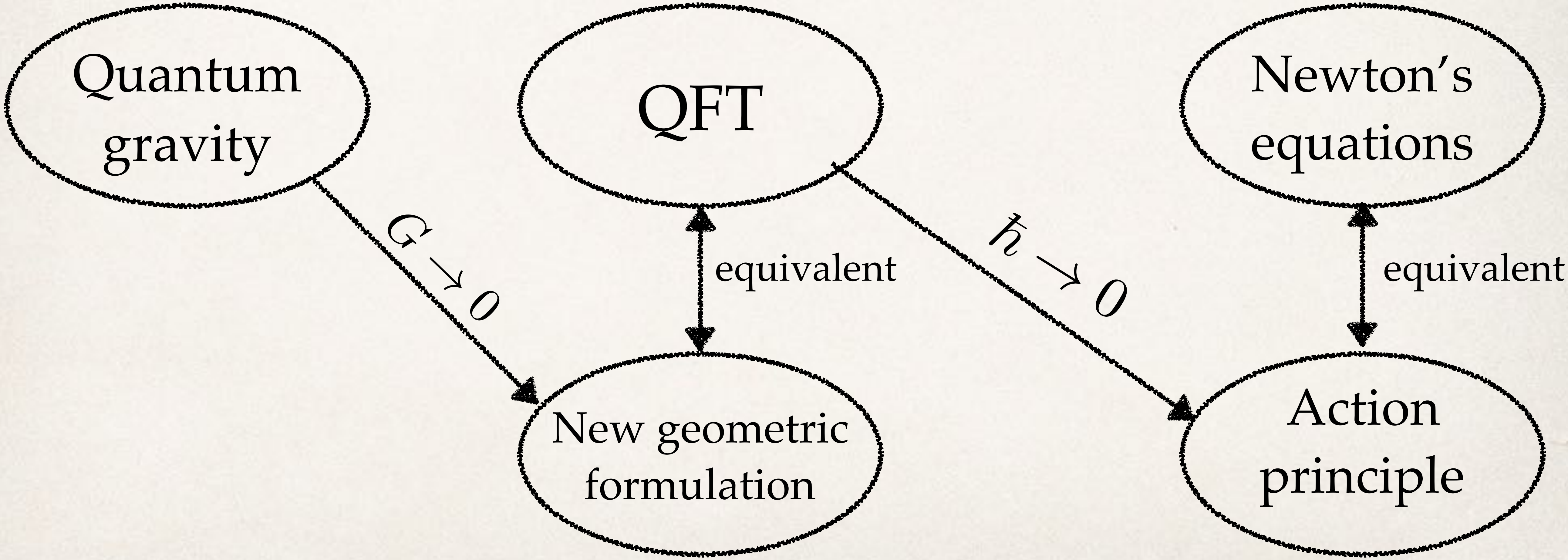
Fantasy

- Beyond understanding QFT better there is one more motivation



Fantasy

- Beyond understanding QFT better there is one more motivation

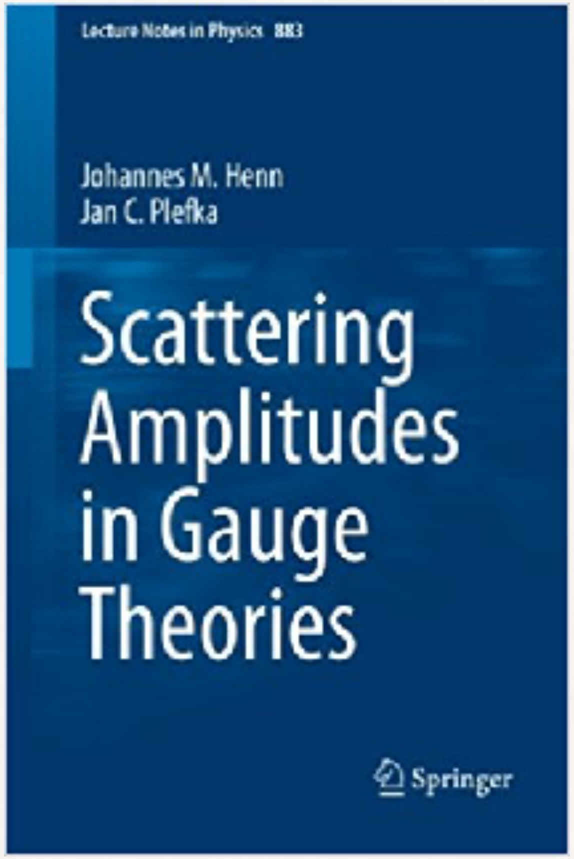


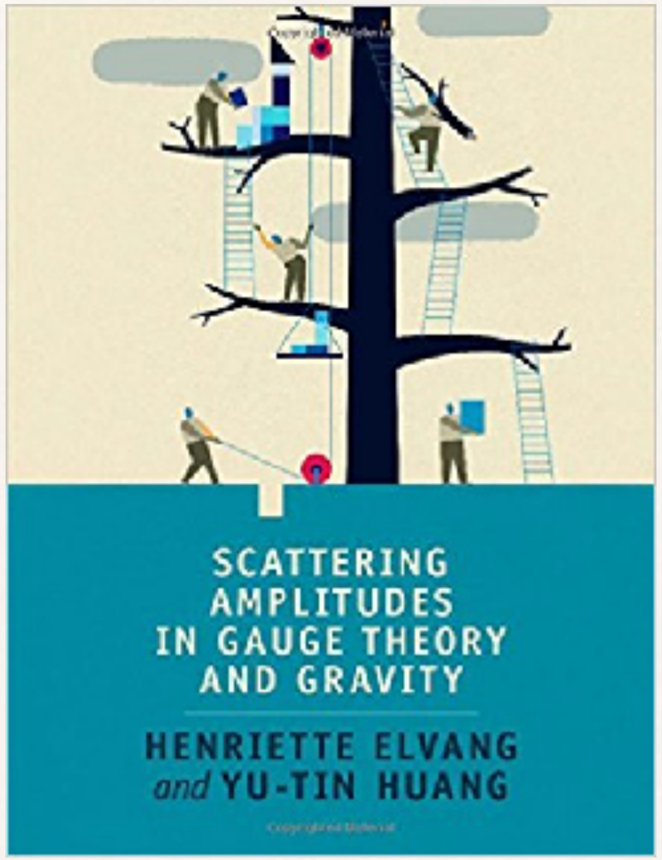
Amplitudes as a new field

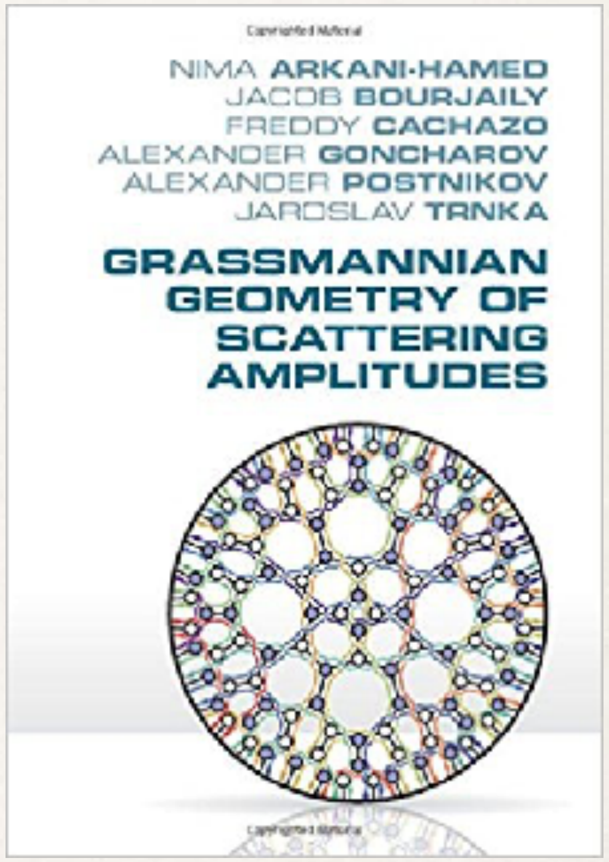
- This is one of the directions in fast developing field
- Scattering equations, BCJ duality, string amplitudes, supergravity amplitudes, hexagon bootstrap, cluster polylogarithms, worldsheet models, integration techniques, soft theorems, LHC calculations,…..
- Zeroth order problems open, many chances for young people to make big discoveries!

Resources for amplitudes

- Books



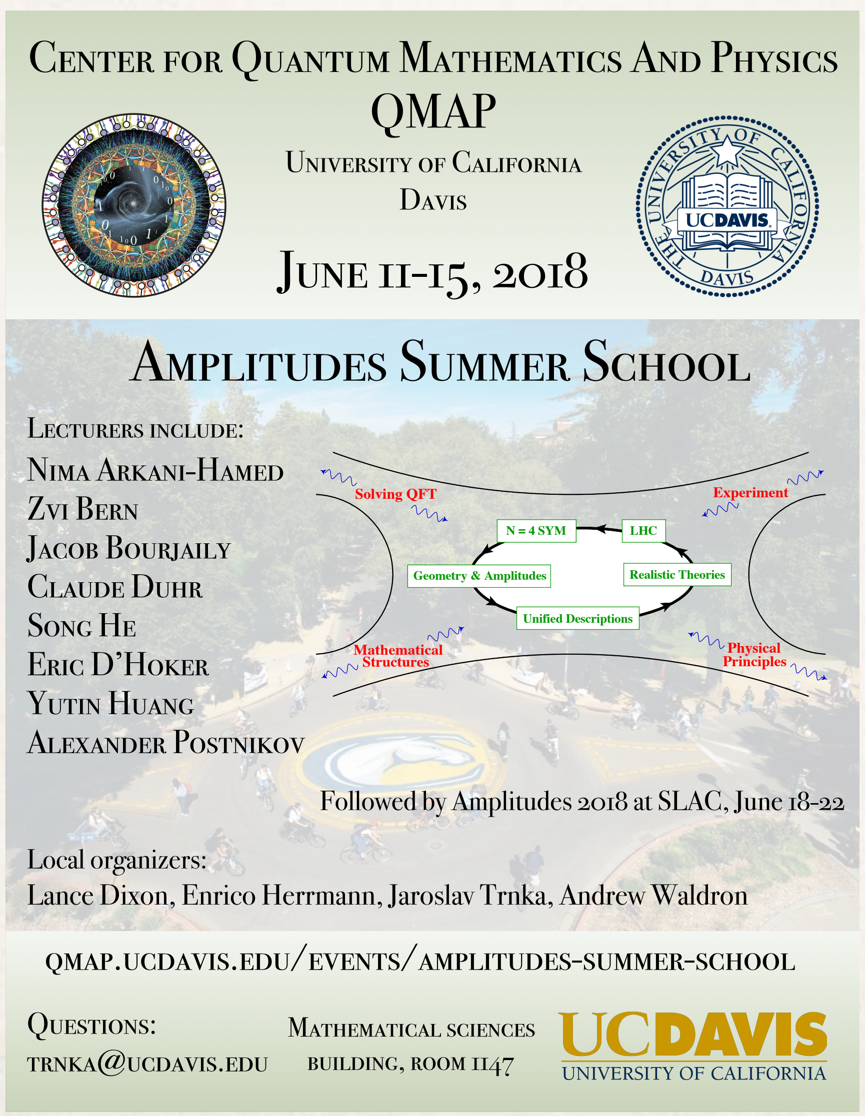




- Many conferences, workshops, summer schools,….

Videos on youtube

- In June 2018: QMAP Amplitudes Summer School at UC Davis



Lectures online:
35 hours on various topics in amplitudes including 11-hour marathon lecture by Nima Arkani-Hamed

Thank you!