Grassmannian Geometry of Scattering Amplitudes

LECTURE 1

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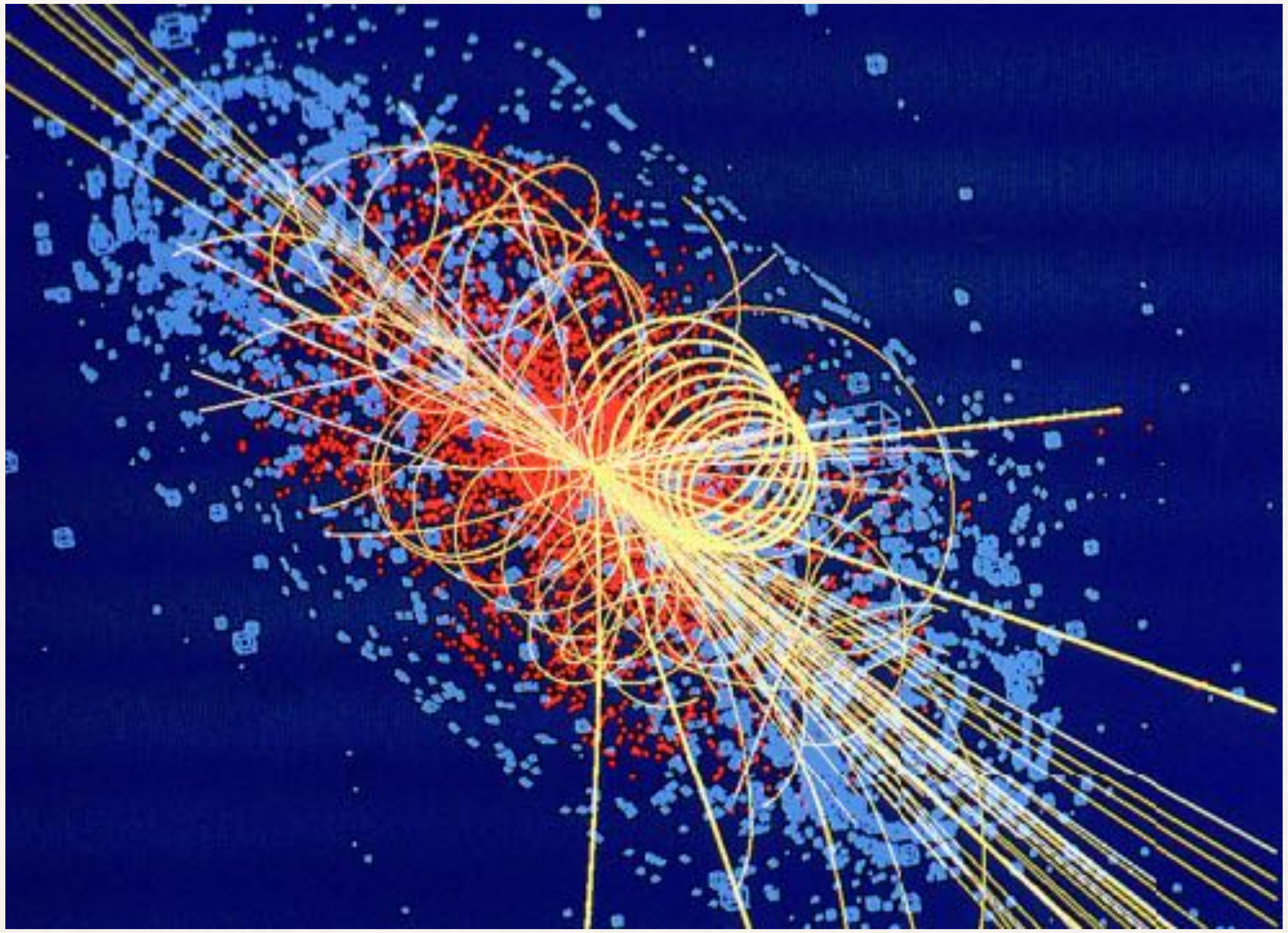
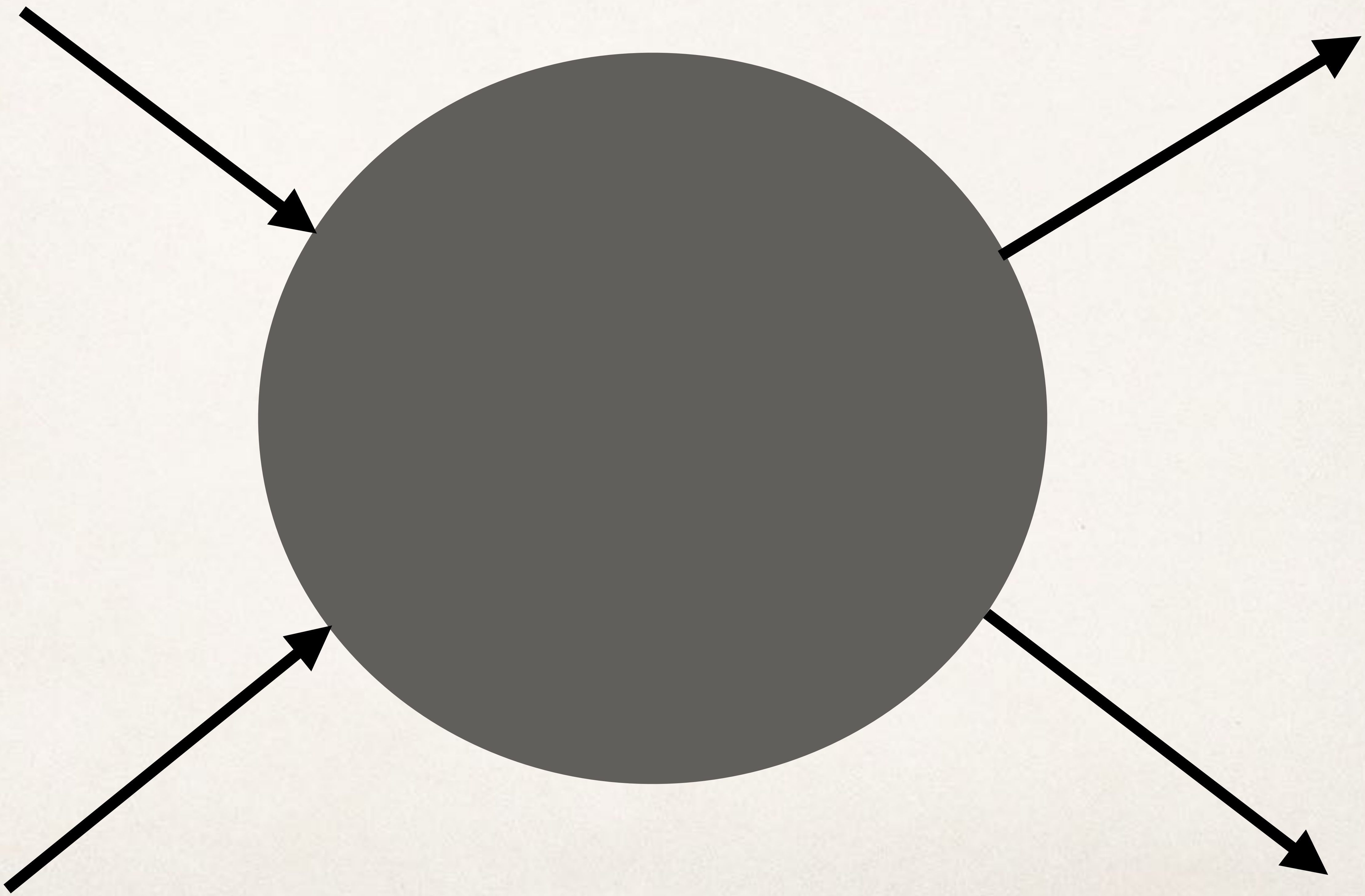
Particle experiments:
our probe to fundamental laws of Nature



What does the blob really represent?



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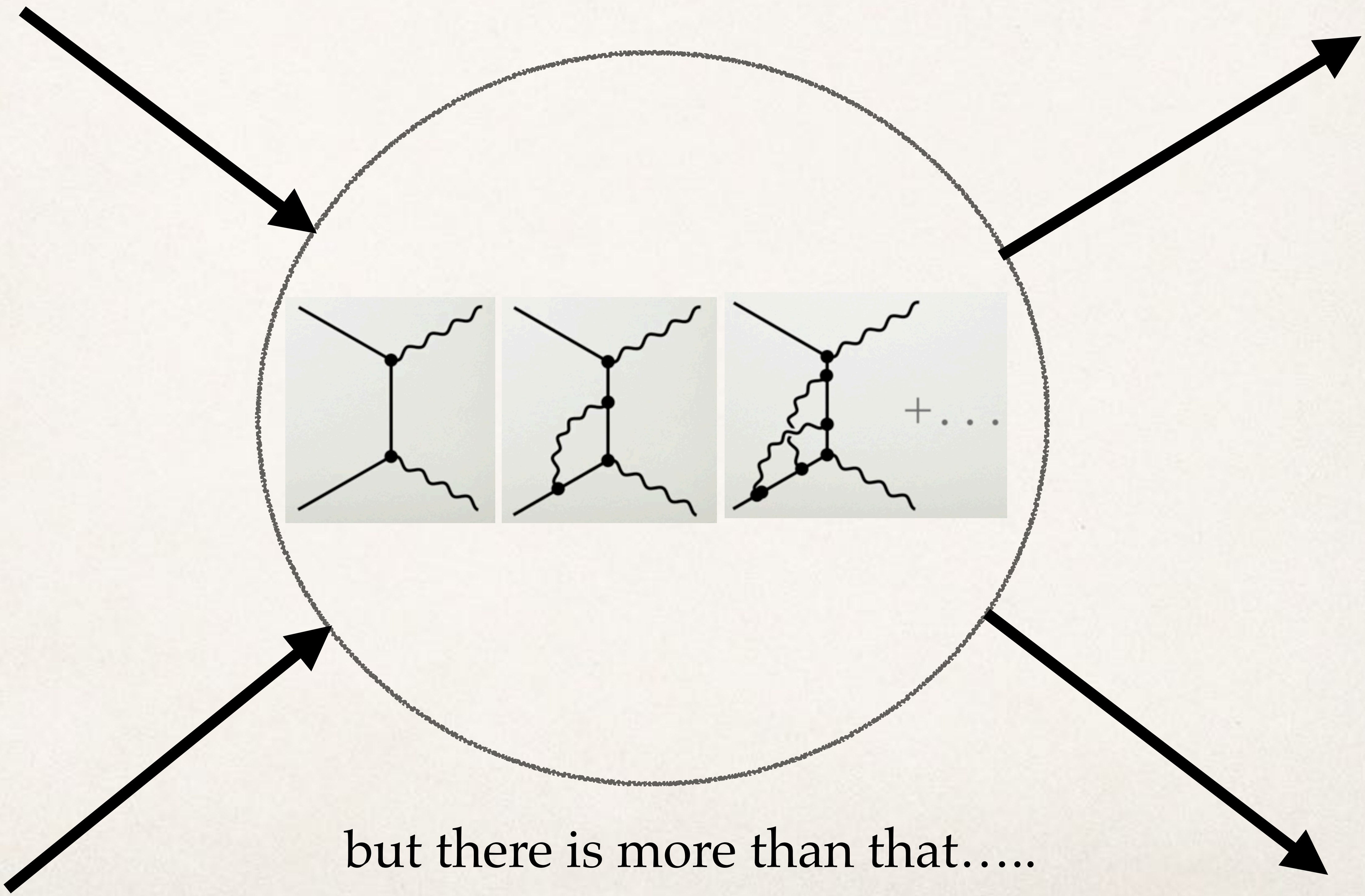
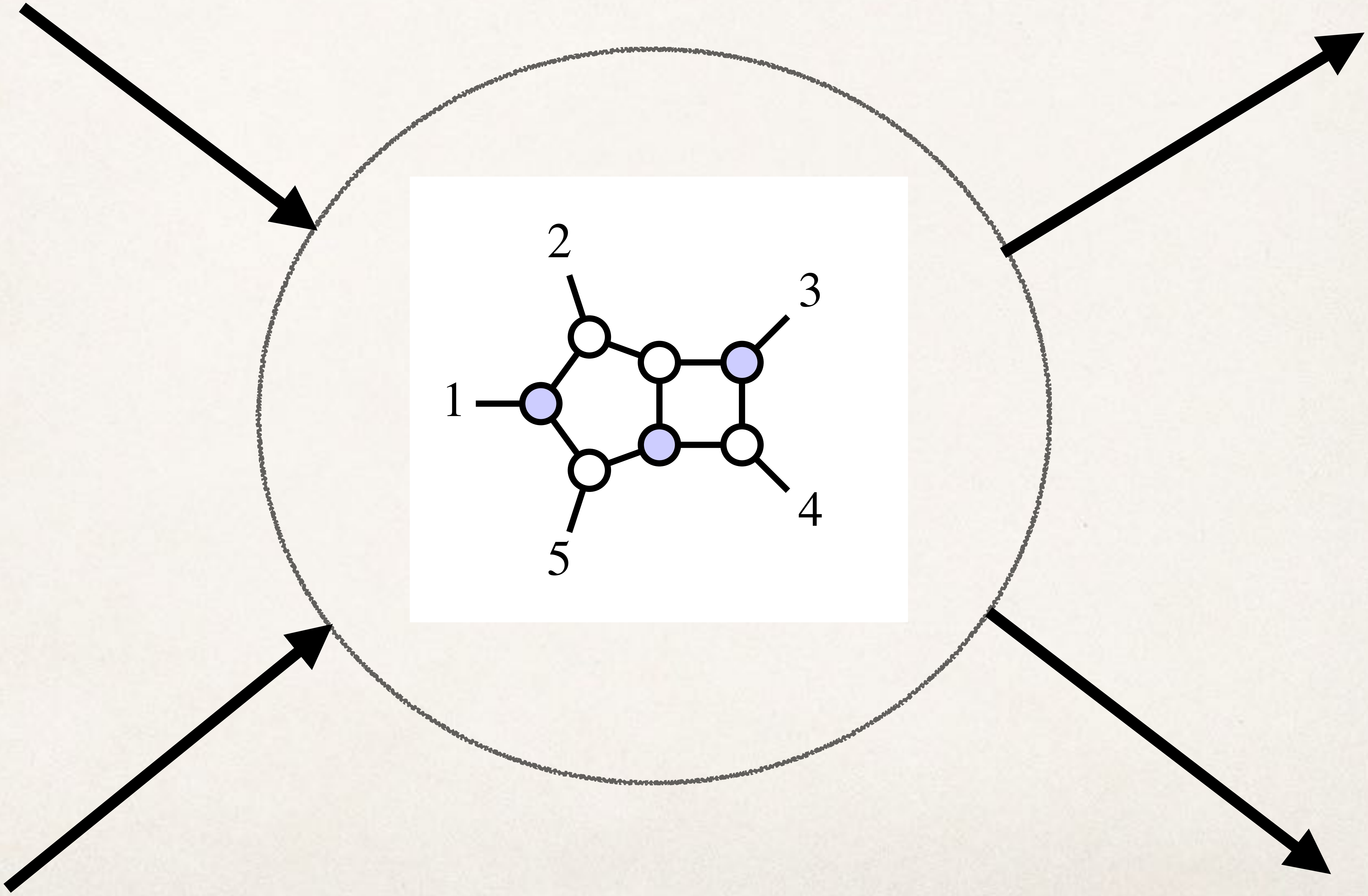
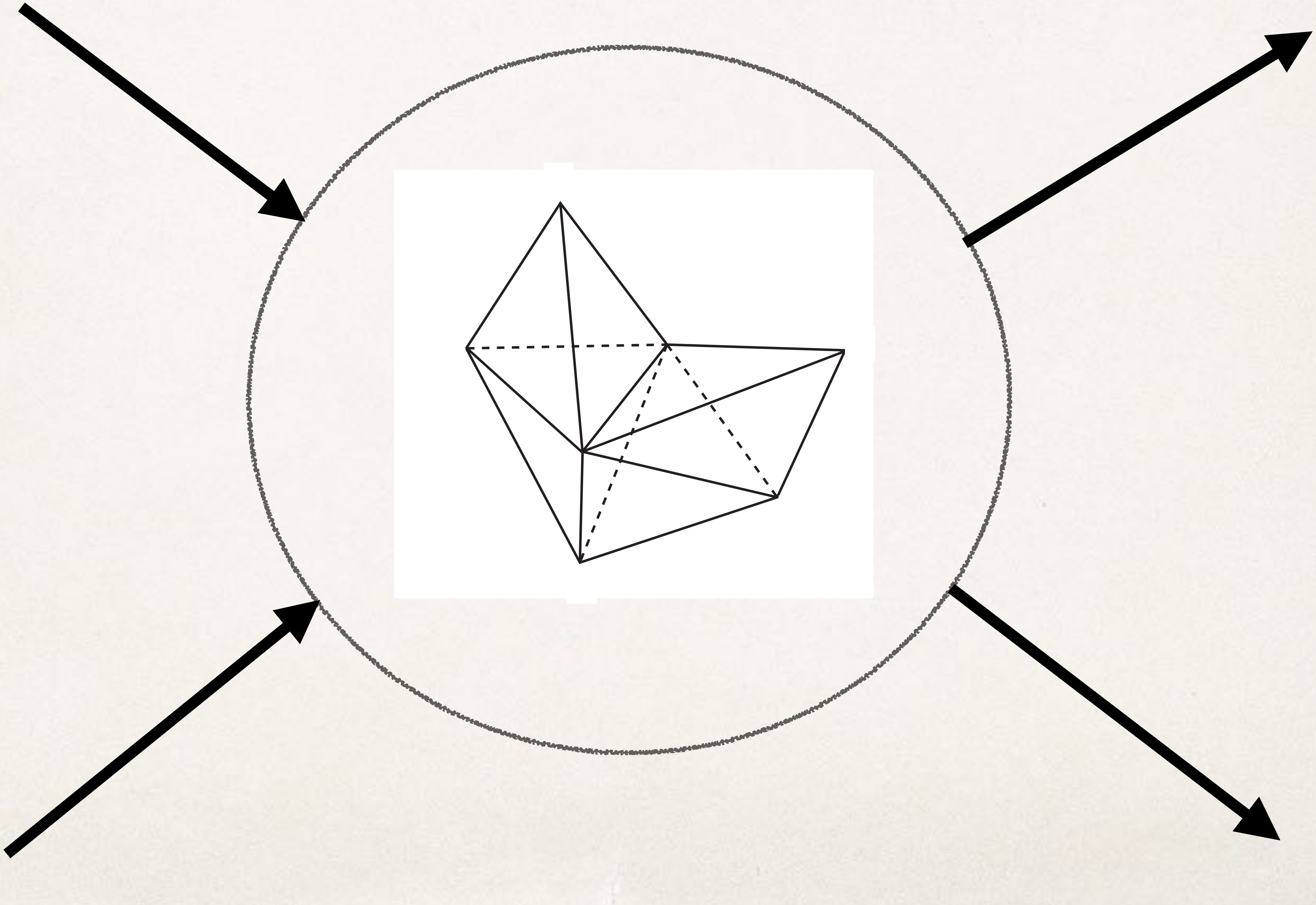


but there is more than that…..

There exists a different representation using other type of diagrams



And in a special case
even something more surprising



Overview of lectures

- Lecture 1: Review of scattering amplitudes
- Lectures 2-3: Positive Grassmannian, on-shell diagrams
- Lecture 4: Amplituhedron

Motivation

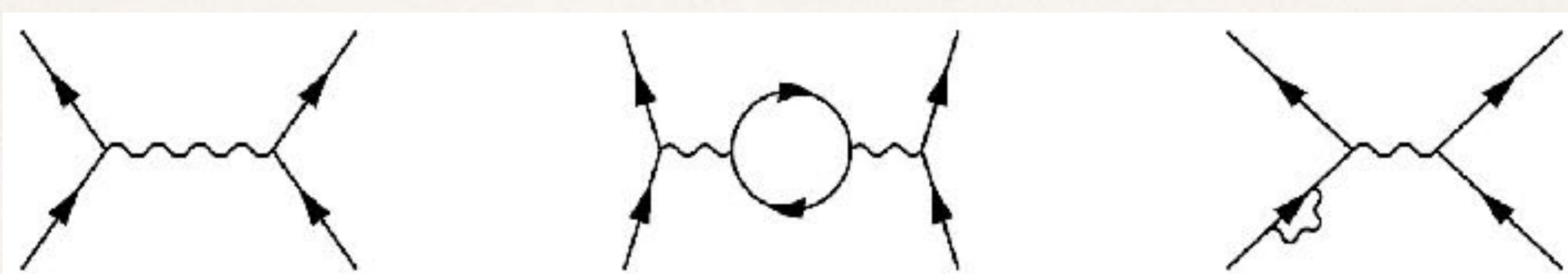
Perturbative QFT

(Dirac, Heisenberg, Pauli; Feynman, Dyson, Schwinger)

- Fields, Lagrangian, Path integral

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\not{D}\psi - m\overline{\psi}\psi \qquad \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\overline{\psi}\; e^{iS(A,\psi,\overline{\psi},J)}$$

- Feynman diagrams: pictures of particle interactions
  Perturbative expansion: trees, loops



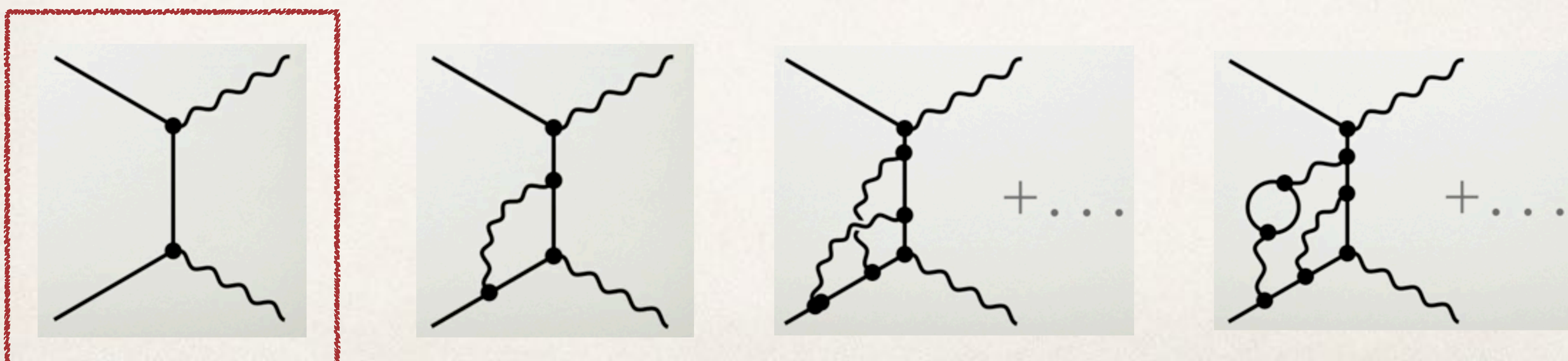
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron

1928

Theory: $g_e = 2$

Experiment: $g_e \sim 2$



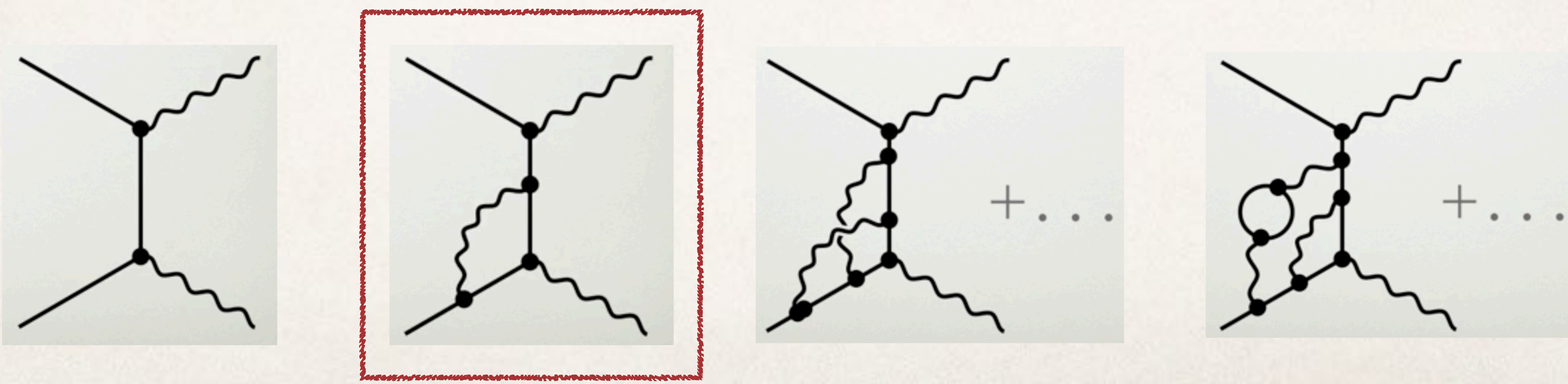
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron

1947

Theory: $g_e = 2.00232$

Experiment: $g_e = 2.0023$



Great success of QFT

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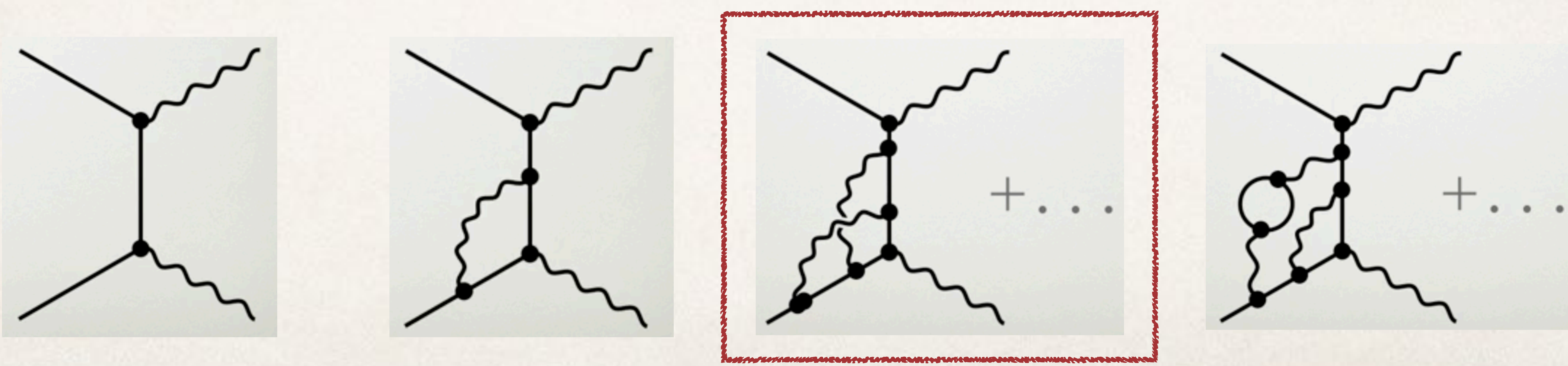
- Example: Magnetic dipole moment of electron

1957 Theory: $g_e = 2.0023193$

1972 Experiment: $g_e = 2.00231931$



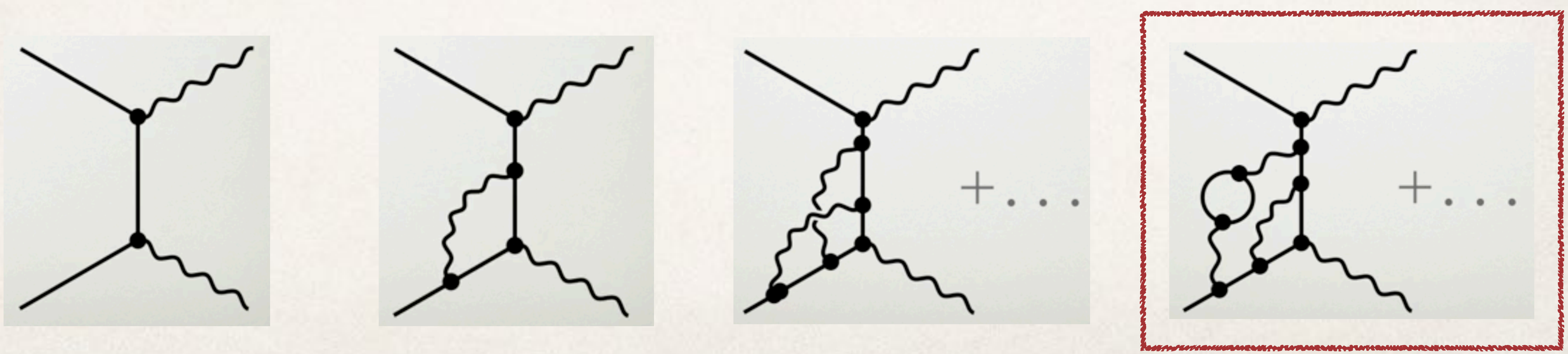
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron

1990

Theory: $g_e = 2.0023193044$

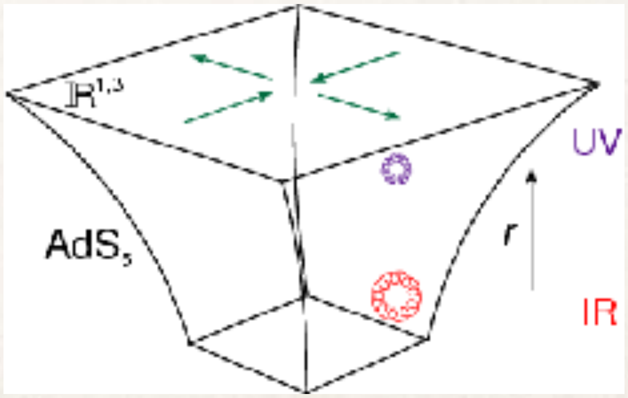
Experiment: $g_e = 2.00231930438$



Dualities

- At strong coupling: perturbative expansion breaks

- Surprises: dual to weakly coupled theory
  - Gauge-gauge dualities
    (Montonen-Olive 1977, Seiberg-Witten 1994)
  - Gauge-gravity duality
    (Maldacena 1997)



Incomplete picture

- Our picture of QFT is incomplete
- Also, tension with gravity and cosmology

> If there is a new way of thinking about QFT,
> it must be seen even at weak coupling

- Explicit evidence: scattering amplitudes

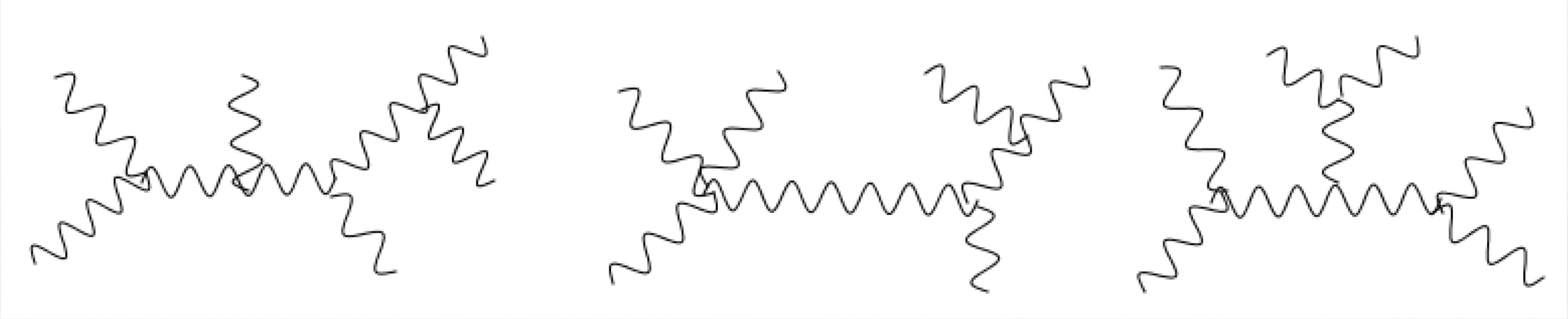
Colliders at high energies

- Proton scattering at high energies

LHC - gluonic factory

- Needed: amplitudes of gluons for higher multiplicities

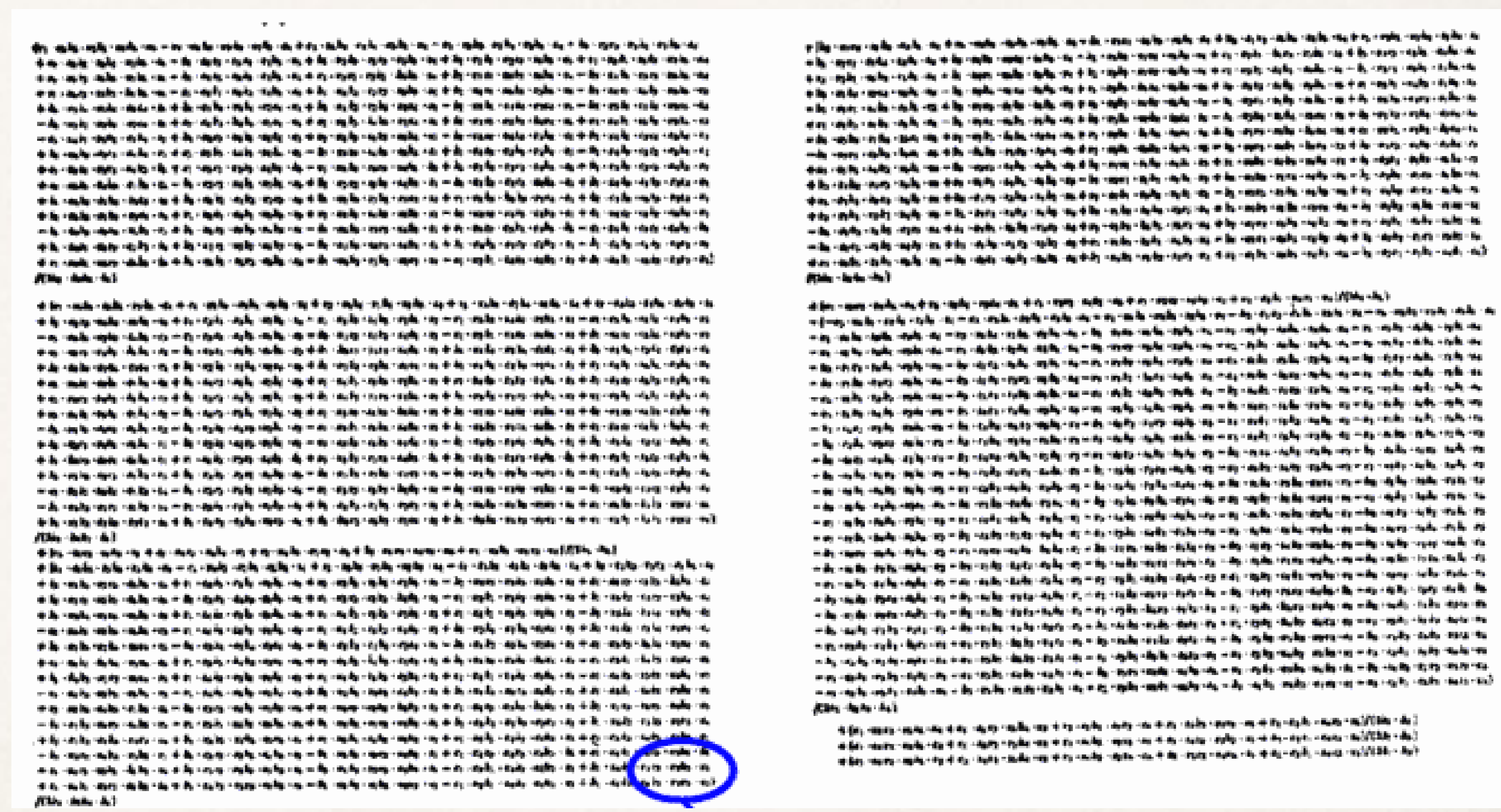
$$gg \to gg \ldots g$$



Early 80s

- Status of the art: $gg \to ggg$

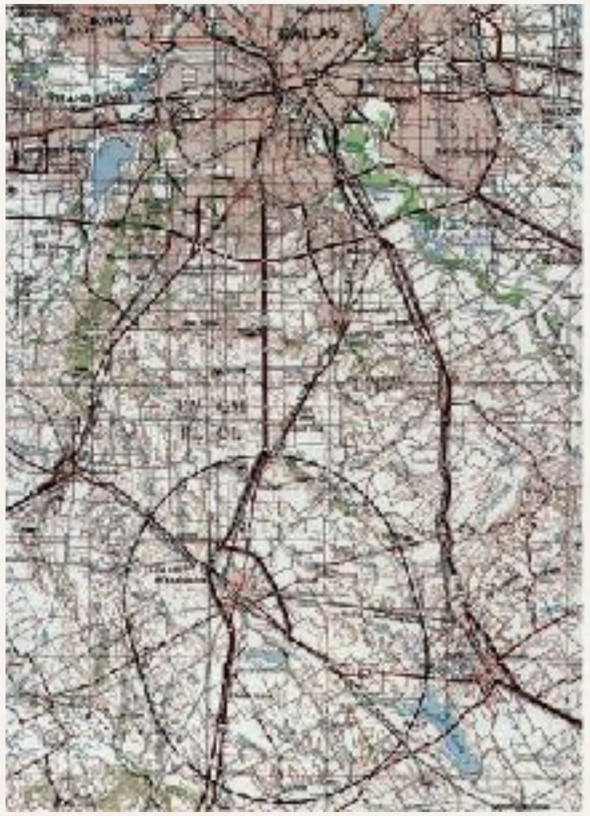
Brute force calculation
24 pages of result



$$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$$

New collider

- 1983: Superconducting Super Collider approved
- Energy 40 TeV: many gluons!

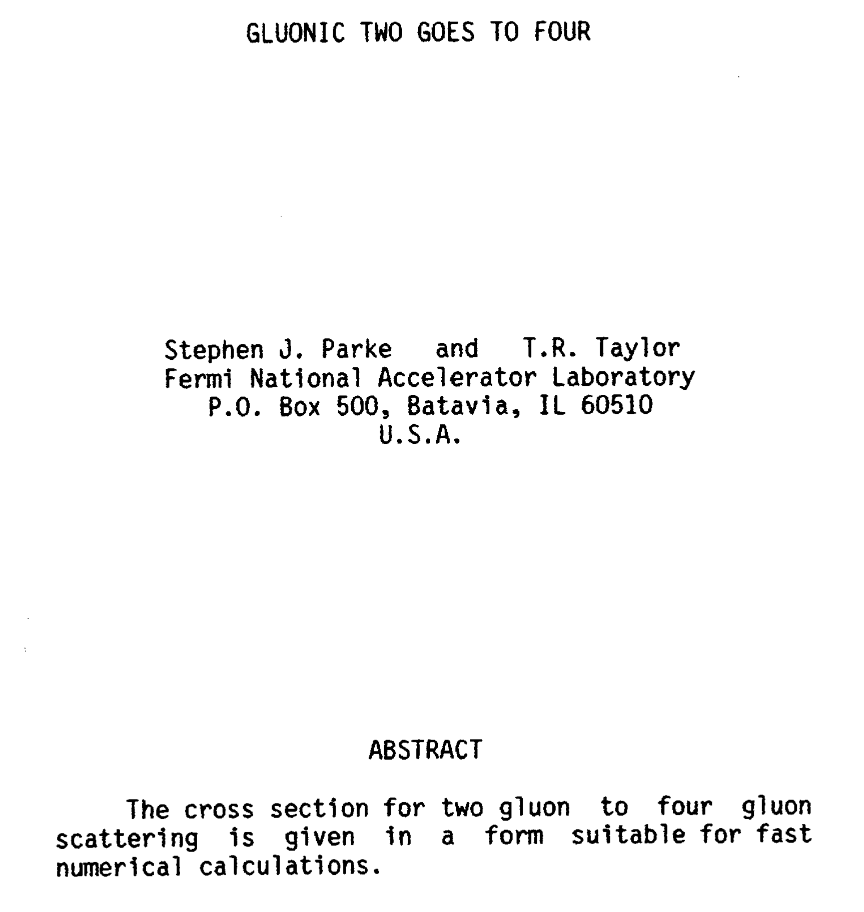


- Demand for calculations, next on the list: $gg \to gggg$

Parke-Taylor formula


(Parke, Taylor 1985)

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations
- 1985: Paper with 14 pages of result



GLUONIC TWO GOES TO FOUR

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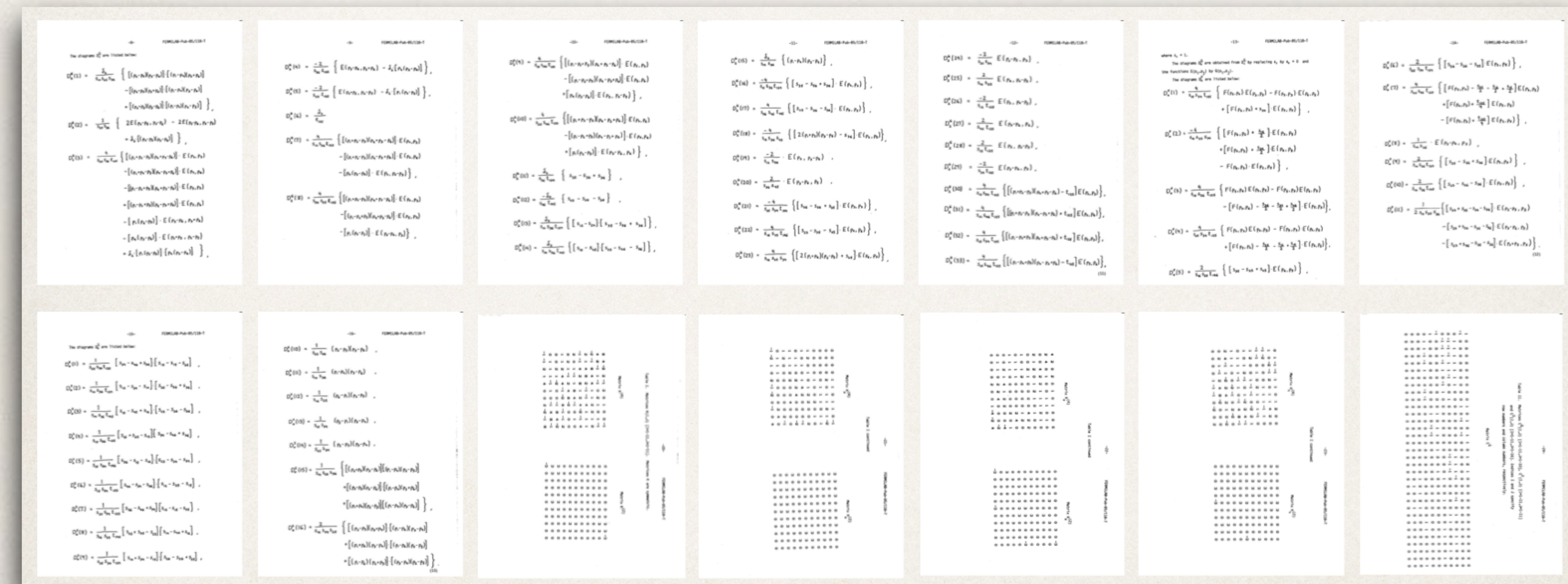
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.

Parke-Taylor formula

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations



Parke-Taylor formula

Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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- Within a year they realized

$$\mathcal{M}_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$
$$\langle 12 \rangle = \epsilon_{ab} \lambda_a^{(1)} \lambda_b^{(2)}$$
$$[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}}^{(1)} \tilde{\lambda}_{\dot{b}}^{(2)}$$

(Mangano, Parke, Xu 1987)

Parke-Taylor formula

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$$\mathcal{M}_n = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\ldots\langle n1\rangle}$$

AN AMPLITUDE FOR n GLUON SCATTERING

STEPHEN J. PARKE and T. R. TAYLOR

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P.O. Box 500, Batavia, IL 60510.

Birth of amplitudes field

- New field in theoretical particle physics

New methods and effective calculations

Uncovering new structures in QFT

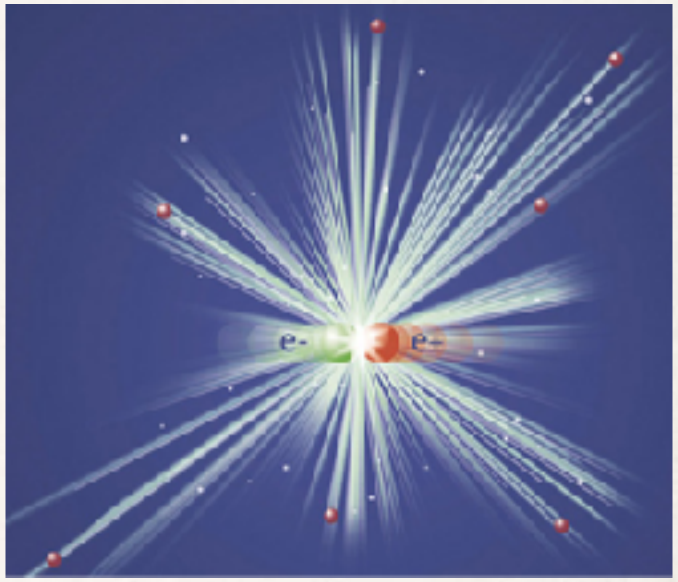
"Road map"

Explicit calculation → New structure discovered → New method which exploits it → Explicit calculation

What are scattering amplitudes

Scattering process

- Interaction of elementary particles
- Initial state $|i\rangle$ and final state $|f\rangle$
- Scattering amplitude $\quad \mathcal{M}_{if} = \langle i|f\rangle$
- Example: $e^+e^- \to e^+e^-$ or $e^+e^- \to \gamma\gamma$ etc.
- Cross section: $\sigma = \int |\mathcal{M}|^2 \, d\Omega \qquad$ probability



Quantum field theory

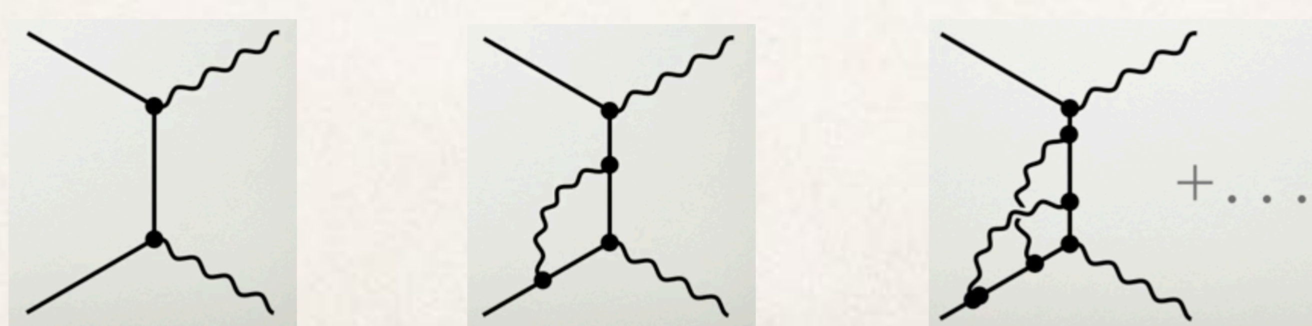
- Theory is defined by Lagrangian

$$\mathcal{L} = \mathcal{L}(\mathcal{O}_j, g_k) \qquad \mathcal{L}_{int} = e\,\overline{\psi}\gamma_\mu\psi A^\mu$$

- Weakly coupled theory

$$\mathcal{M} = \mathcal{M}_0 + g\,\mathcal{M}_1 + g^2\,\mathcal{M}_2 + g^3\,\mathcal{M}_3 + \ldots$$

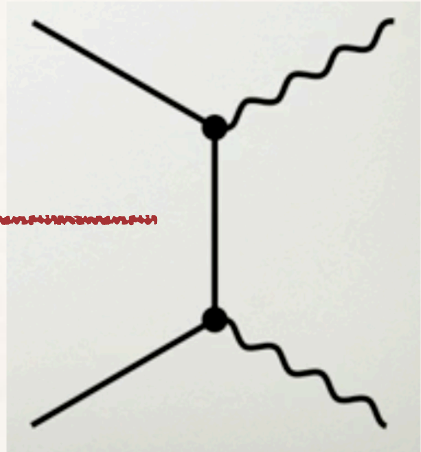
- Representation in terms of Feynman diagrams



On-shell amplitudes

- Real vs virtual particles

internal lines:
virtual particles
off-shell
$$p^2 \neq m^2$$

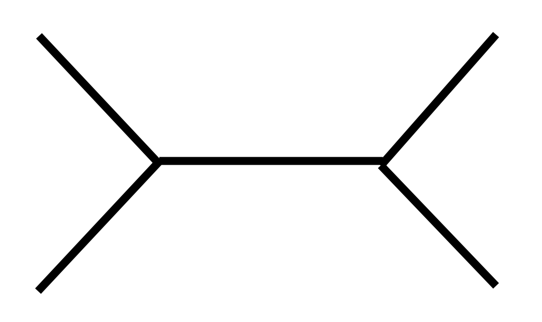


external lines:
real particles
on-shell
$$p^2 = m^2$$

convenient way how to do calculations but
virtual particles break gauge invariance

Analytic structure of amplitudes

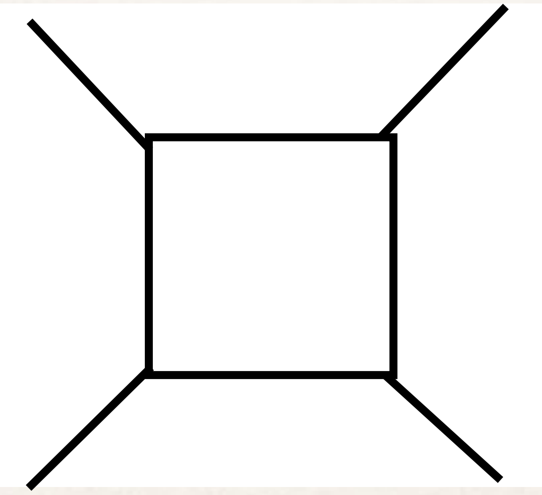
- Tree-level: rational functions



$$\sim \frac{g^2}{(p_1+p_2)^2}$$

Only poles

- Loops: polylogarithms and more complicated



$$\sim \int \frac{d^4\ell}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

Integrand

$$\sim \log^2(s/t)$$

Branch cuts

Loop amplitudes

- The problem splits into two parts: construct the integrand and integrate

$$A^{L-loop} = \int d^4\ell_1 \ldots d^4\ell_L \, \mathcal{I}$$

$$A^{1-loop} \sim \mathrm{Li}_2, \log, \zeta_2$$

$$A^{L-loop} \sim ?$$

polylogs
elliptic polylogs
beyond

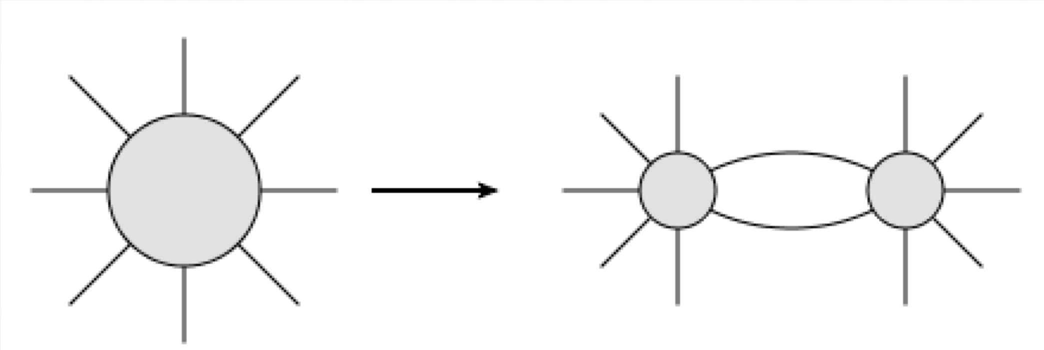
- rational function
- many variables (loop momenta)
- properties of the amplitude non-trivially encoded in the integrand

very interesting mathematics of transcendental functions

Cuts of the integrand

- Once we have the integrand we can take residues on poles: $\text{Cut} \leftrightarrow \ell^2 = 0$

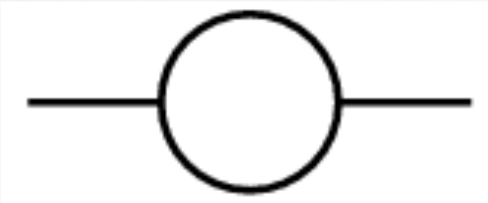
- Unitarity cut: $\ell^2 = (\ell + Q)^2 = 0$



$$\mathcal{M}^{1-loop} \xrightarrow[\ell^2=(\ell+Q)^2=0]{} \mathcal{M}_L^{tree} \frac{1}{\ell^2(\ell+Q)^2} \mathcal{M}_R^{tree}$$

Divergencies

- Loop diagrams are generally UV divergent



$$\sim \int_{-\infty}^{\infty} \frac{d^4\ell}{(\ell^2+m^2)[(\ell+p)^2+m^2]} \sim \log\Lambda$$

renormalization

- IR divergencies: physical effects, cancel in cross section

- Dimensional regularization: calculate integrals in $4+\epsilon$ dimensions

Divergencies $\sim \dfrac{1}{\epsilon^k}$

Kinematics of massless particles

Massless particles

- Parameters of elementary particles of spin S
  - Spin $s = (-S, S)$
  - Mass m
  - Momentum p^μ

On-shell (physical) particle

$$p^2 = m^2$$

- Massless particle: $m = 0$ $p^2 = 0$

spin = helicity: only two extreme values $h = \{-S, S\}$

Example: photon $h = (+, -)$

$s = 0$ missing

Spin functions

- At high energies particles are massless

  Fundamental laws reveal there

- Spin degrees of freedom: spin function

  - s=0: Scalar - no degrees of freedom
  - s=1/2: Fermion - spinor u
  - s=1: Vector - polarization vector ϵ^μ
  - s=2: Tensor - polarization tensor $h^{\mu\nu}$

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Polarization vectors

- Spin 1 particle is described by vector ϵ^μ

  2 degrees of freedom ← → 4 degrees of freedom

- Null condition: $\epsilon \cdot \epsilon^* = 0$ 3 degrees of freedom left

- We further impose: $\epsilon \cdot p = 0$

  Identification

  $$\epsilon_\mu \sim \epsilon_\mu + \alpha p_\mu$$

  Feynman diagrams depend on α

  gauge dependence

Spinor helicity variables

- Standard SO(3,1) notation for momentum

$$p^\mu = (p_0, p_1, p_2, p_3) \qquad p_j \in \mathbb{R}$$

$$p^2 = p_0^2 + p_1^2 + p_2^2 - p_3^2$$

- We use SL(2,C) representation

$$p_{ab} = \sigma^\mu_{ab} p_\mu = \begin{pmatrix} p_0 + ip_1 & p_2 + p_3 \\ -p_2 + p_3 & p_0 - ip_1 \end{pmatrix}$$

On-shell: $p^2 = \det(p_{ab}) = 0$

$$\text{Rank}\,(p_{ab}) = 1$$

Spinor helicity variables

- We can then write $p_{ab} = \lambda_a \kappa_b$
- SL(2,C): dotted notation $p_{a\dot{b}} = \lambda_a \widetilde{\lambda}_{\dot{b}}$

  $\widetilde{\lambda}$ is complex conjugate of λ

- Little group transformation

$$\lambda \to t\lambda \qquad \widetilde{\lambda} \to \frac{1}{t}\widetilde{\lambda}$$

leaves momentum unchanged $p \to p$

3 degrees of freedom

Spinor helicity variables

- Momentum invariant $\quad (p_1 + p_2)^2 = (p_1 \cdot p_2)$

$$p_1^\mu = \sigma^\mu_{a\dot{a}}\, \lambda_{1a}\widetilde{\lambda}_{1\dot{a}} \qquad p_2^\mu = \sigma^\mu_{b\dot{b}}\, \lambda_{2b}\widetilde{\lambda}_{2\dot{b}}$$

- Plugging for momenta

$$(p_1 \cdot p_2) = (\sigma^\mu_{a\dot{a}}\sigma_{\mu\, b\dot{b}})\,(\lambda_{1a}\lambda_{2b})(\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}})$$

$$\downarrow$$

$$\epsilon_{ab}\,\epsilon_{\dot{a}\dot{b}} \qquad = (\epsilon_{ab}\lambda_{1a}\lambda_{2b})(\epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}})$$

Define: $\quad \boxed{\langle 12\rangle \equiv \epsilon_{ab}\lambda_{1a}\lambda_{2b}} \qquad \boxed{[12] \equiv \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}}}$

Invariant products

- Momentum invariant

$$s_{ij} = (p_i + p_j)^2 = \langle ij \rangle [ij]$$

Square brackets

Angle brackets $\langle ij \rangle = \epsilon_{ab} \lambda_{ia} \lambda_{jb}$ $[ij] = \epsilon_{\dot{a}\dot{b}} \widetilde{\lambda}_{i\dot{a}} \widetilde{\lambda}_{j\dot{b}}$

- Antisymmetry

$$\langle 21 \rangle = -\langle 12 \rangle \qquad [21] = -[12]$$

- More momenta

$$(p_1 + p_2 + p_3)^2 = \langle 12 \rangle [12] + \langle 23 \rangle [23] + \langle 13 \rangle [13]$$

Invariant products

- Shouten identity

$$\langle 13\rangle\langle 24\rangle = \langle 12\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle$$

- Mixed brackets

$$\langle 1|2+3|4] \equiv \langle 12\rangle[24] + \langle 13\rangle[34]$$

- Momentum conservation

$$\sum_{i=1}^{n} \lambda_{ia}\widetilde{\lambda}_{i\dot{a}} = 0$$

Non-trivial conditions:
Quadratic relation
between components

Polarization vectors

- Two polarization vectors

$$\epsilon_+^\mu = \sigma^\mu_{a\dot{a}} \frac{\eta_a \widetilde{\lambda}_{\dot{a}}}{\langle \eta \lambda \rangle} \qquad \epsilon_-^\mu = \sigma^\mu_{a\dot{a}} \frac{\lambda_a \widetilde{\eta}_{\dot{a}}}{[\widetilde{\eta}\, \widetilde{\lambda}]}$$

where $\eta, \widetilde{\eta}$ are auxiliary spinors

Note that

$$(\epsilon_+ \cdot \epsilon_-) = 1$$

- Freedom in choice of $\eta, \widetilde{\eta}$ corresponds to

$$\epsilon^\mu \sim \epsilon^\mu + \alpha\, p^\mu$$

- Gauge redundancy of Feynman diagrams

Scaling of amplitudes

- Consider some amplitude $A(-+--+\cdots-)$

$$A = (\epsilon_1 \epsilon_2 \dots \epsilon_n) \cdot Q$$

depends only on momenta

- Little group scaling

$$\lambda \to t\lambda \qquad \widetilde{\lambda} \to \frac{1}{t}\widetilde{\lambda}$$

$$p \to p \qquad \epsilon_+ \to \frac{1}{t^2} \cdot \epsilon_+ \qquad \epsilon_- \to t^2 \cdot \epsilon_-$$

$$A(i^-) \to t^2 \cdot A(i^-) \qquad A(i^+) \to \frac{1}{t^2} \cdot A(i^+)$$

Back to Parke-Taylor formula

- Let us consider $A(1^-2^-3^+4^+5^+6^+)$
- Scaling $A\left(t\lambda_i, \frac{1}{t}\widetilde{\lambda}_i\right) = t^2 \cdot A(\lambda_i, \widetilde{\lambda}_i)$ for particles 1,2

$$A\left(t\lambda_i, \frac{1}{t}\widetilde{\lambda}_i\right) = \frac{1}{t^2} \cdot A(\lambda_i, \widetilde{\lambda}_i) \quad \text{for particles 3,4,5,6}$$

- Check for explicit expression

$$A_6 = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

If only $\langle ij\rangle$ allowed the form is unique

Helicity amplitudes

- In Yang-Mills theory we have + or - "gluons"

$$A_6(1^- 2^- 3^+ 4^+ 5^+ 6^+)$$

- We denote k: number of - helicity gluons

- Some amplitudes are zero

$$A_n(+ + + \cdots +) = 0$$
$$A_n(- + + \cdots +) = 0$$
$$A_n(- - - \cdots -) = 0$$
$$A_n(+ - - \cdots -) = 0$$

First non-trivial: k=2

$$A_n(- - + \cdots +)$$

Parke-Taylor formula for tree level

Three point amplitudes

Three point kinematics

- Gauge invariant building blocks: on-shell amplitudes

$$p_1^2 = p_2^2 = p_3^2 = 0 \qquad\qquad p_1 + p_2 + p_3 = 0$$

- Plugging second equation into the first

$$(p_1 + p_2)^2 = (p_1 \cdot p_2) = 0$$

- Similarly we get for other pairs

$$(p_1 \cdot p_2) = (p_1 \cdot p_3) = (p_2 \cdot p_3) = 0$$

These momenta are very constrained!

Three point kinematics

- Use spinor helicity variables trivializes on-shell condition

$$p_1 = \lambda_1 \widetilde{\lambda}_1, \quad p_2 = \lambda_2 \widetilde{\lambda}_2, \quad p_3 = \lambda_3 \widetilde{\lambda}_3$$

- The mutual conditions then translate to

$$(p_1 \cdot p_2) = \langle 12 \rangle [12] = 0$$

- And similarly for other two pairs

$$(p_1 \cdot p_3) = \langle 13 \rangle [13] = 0 \qquad (p_2 \cdot p_3) = \langle 23 \rangle [23] = 0$$

Two solutions

- We want to solve conditions

$$\langle 12\rangle[12] = \langle 13\rangle[13] = \langle 23\rangle[23] = 0$$

- Solution 1: $\langle 12\rangle = 0$ which implies $\lambda_2 = \alpha\lambda_1$

  Then we also have $\langle 23\rangle = \alpha\langle 13\rangle$

  And we set $\langle 13\rangle = 0$ by demanding $\lambda_3 = \beta\lambda_1$

$$\boxed{\lambda_1 \sim \lambda_2 \sim \lambda_3}$$

Two solutions

- Solution 2: $[12] = [23] = [13] = 0$

$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

- Let us take this solution

$$p_1 = \lambda_1 \widetilde{\lambda}_1, \quad p_2 = \alpha \lambda_2 \widetilde{\lambda}_1, \quad p_3 = (-\lambda_1 - \alpha \lambda_2)\widetilde{\lambda}_1$$

complex momenta

<u>No solution for real momenta</u>

Three point amplitudes

- Gauge theory: scattering of three gluons (not real)
- Building blocks: $\langle 12\rangle, \langle 23\rangle, \langle 13\rangle, [12], [23], [13]$
- Mass dimension: each term $\sim m$
- Three point amplitude $A_3 \sim \epsilon^3 p \sim p \sim m$

Three point amplitudes

- Two options

$$A_3^{(1)} = \langle 12\rangle^{a_1}\langle 13\rangle^{a_2}\langle 23\rangle^{a_3} \qquad A_3^{(2)} = [12]^{b_1}[13]^{b_2}[23]^{b_3}$$

- Apply to $A_3(1^-,2^-,3^+)$

$$A_3(t\lambda_1, t^{-1}\widetilde{\lambda}_1) = t^{a_1+a_2}\cdot A_3 \qquad a_1+a_2=2 \qquad a_1=3$$

$$A_3(t\lambda_2, t^{-1}\widetilde{\lambda}_2) = t^{a_1+a_3}\cdot A_3 \longrightarrow a_1+a_3=2 \longrightarrow a_2=-1$$

$$A_3(t\lambda_3, t^{-1}\widetilde{\lambda}_3) = t^{a_2+a_3}\cdot A_3 \qquad a_2+a_3=-2 \qquad a_3=-1$$

Three point amplitudes

- Similarly for $A_3(1^+, 2^+, 3^-)$
- Two fundamental amplitudes

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

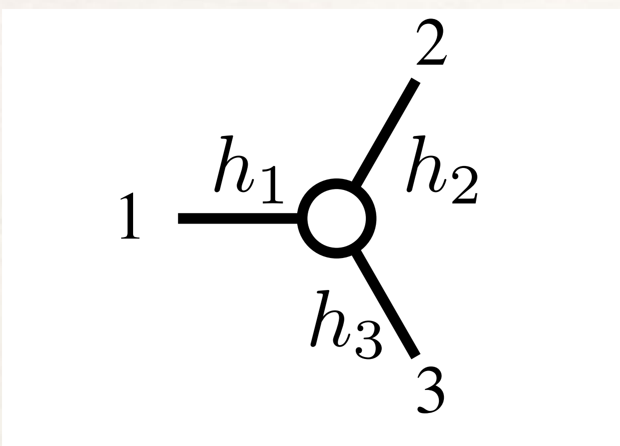
$$A_3(1^+, 2^+, 3^-) = \frac{[12]^3}{[13][23]}$$

This is true to all orders: just kinematics

<u>They exist only for complex momenta</u>

General 3pt amplitudes

- Two solutions for 3pt kinematics

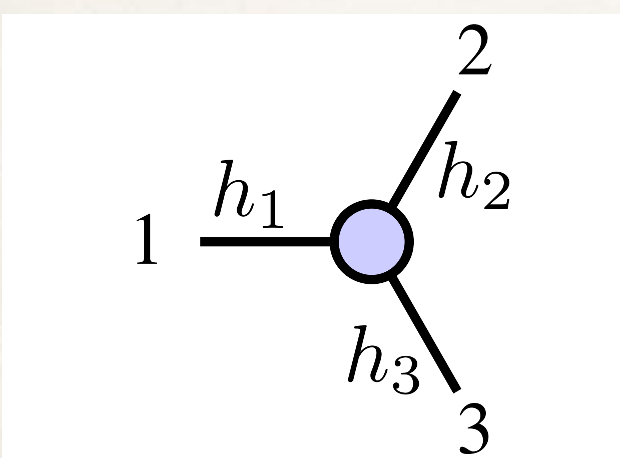


$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$

Under the little group rescaling:

$$A_3(t\lambda_j, t^{-1}\tilde{\lambda}_j) \sim t^{2h_j} \cdot A_3$$

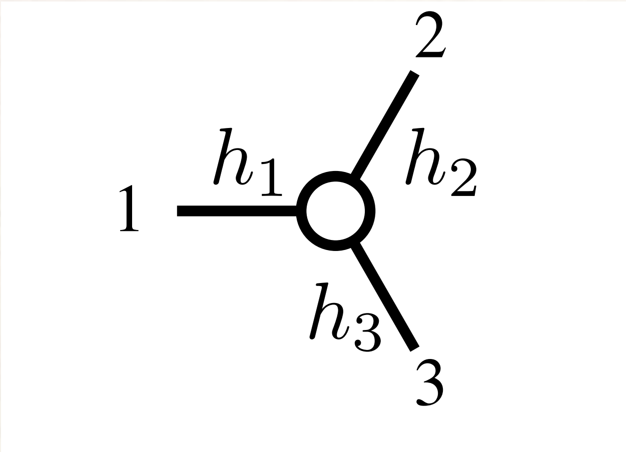
Solve the system of equations



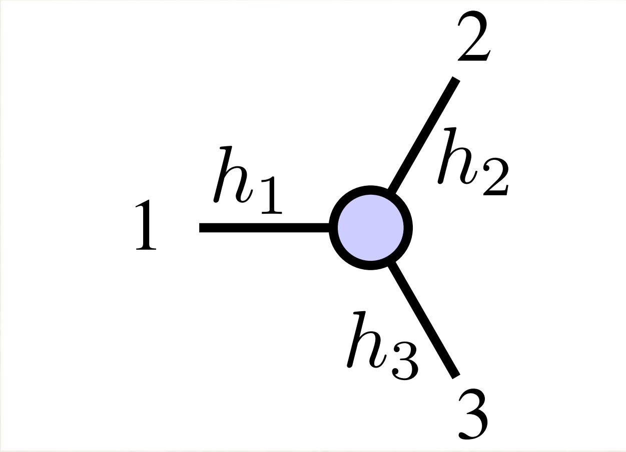
$$\tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3$$

General 3pt amplitudes

- Two solutions for 3pt amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

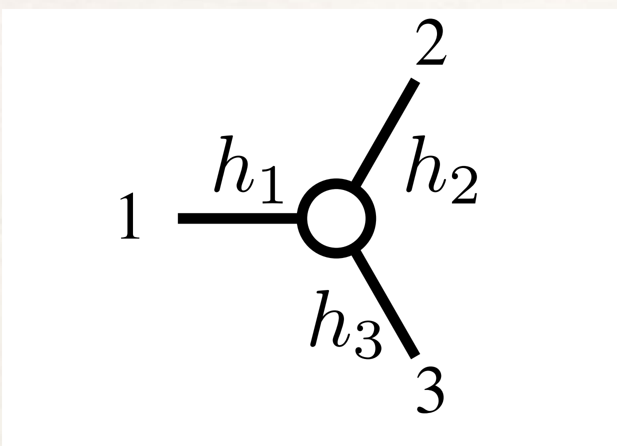


$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

Which one is correct?

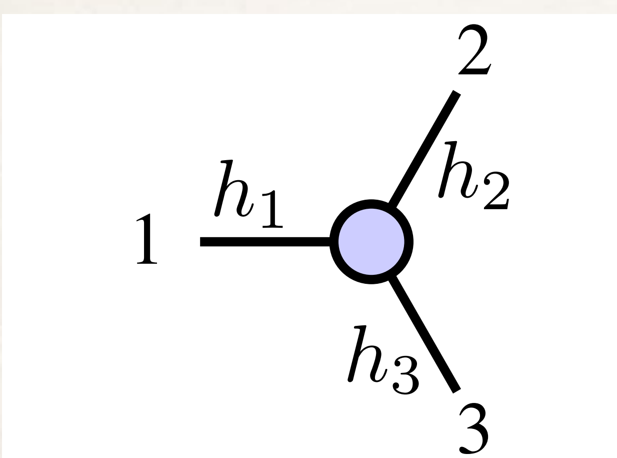
General 3pt amplitudes

- Two solutions for 3pt amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \leq 0$$



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$$h_1 + h_2 + h_3 \geq 0$$

Mass dimension must be positive!

All spins allowed

- Note that these formulas are valid for any spins
- For example for amplitude $A_3(1^0, 2^{1^+}, 3^{2^+})$

$$A_3 = \frac{\langle 23\rangle\langle 31\rangle^3}{\langle 12\rangle^3}$$

- But we can also do higher spins $A_3(1^{3^+}, 2^{5^+}, 3^{12^-})$

$$A_3 = \frac{\langle 23\rangle^{10}\langle 31\rangle^{14}}{\langle 12\rangle^{20}}$$

- Completely fixed just by kinematics!

Tree-level amplitudes

Locality and unitarity

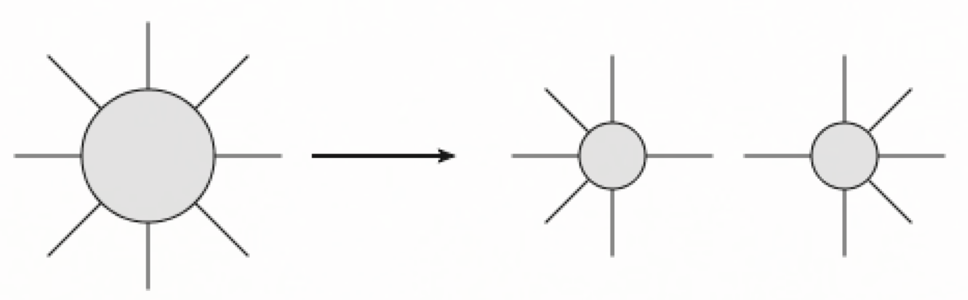
- Higher point amplitudes: not completely fixed by Lorentz symmetry, but they satisfy powerful constraints

- Only poles: Feynman propagators

Locality $\frac{1}{P^2}$ where $P = \sum_{k \in \mathcal{P}} p_k$

- On the pole

Unitarity



Feynman diagrams recombine on both sides into amplitudes

$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

Three point of spin S

- I will discuss amplitudes of single spin S particle
- For 3pt amplitudes we get

$$A_3 = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^S \qquad A_3 = \left(\frac{[12]^3}{[23][31]}\right)^S$$

$(- - +)$ minimal powercounting $(+ + -)$

- There exist also non-minimal amplitudes

$$A_3 = (\langle 12\rangle\langle 23\rangle\langle 31\rangle)^S \qquad A_3 = ([12][23][31])^S$$

$(- - -)$ $(+ + +)$

Four point amplitude

- Let us consider a 4pt amplitude of particular helicities

$$A_4(--++)$$

- Mandelstam variables:

$$s = (p_1 + p_2)^2 = \langle 12\rangle[12] = \langle 34\rangle[34]$$
$$t = (p_1 + p_4)^2 = \langle 14\rangle[14] = \langle 23\rangle[23]$$
$$u = (p_1 + p_3)^2 = \langle 13\rangle[13] = \langle 24\rangle[24]$$

- One can show that the little group dictates:

$$A_4 = (\langle 12\rangle[34])^{2S} \cdot F(s,t)$$

- It must be consistent with factorizations

From 3pt to 4pt

- Three point amplitudes exist for all spins
- For 4pt amplitude: we have a powerful constraint

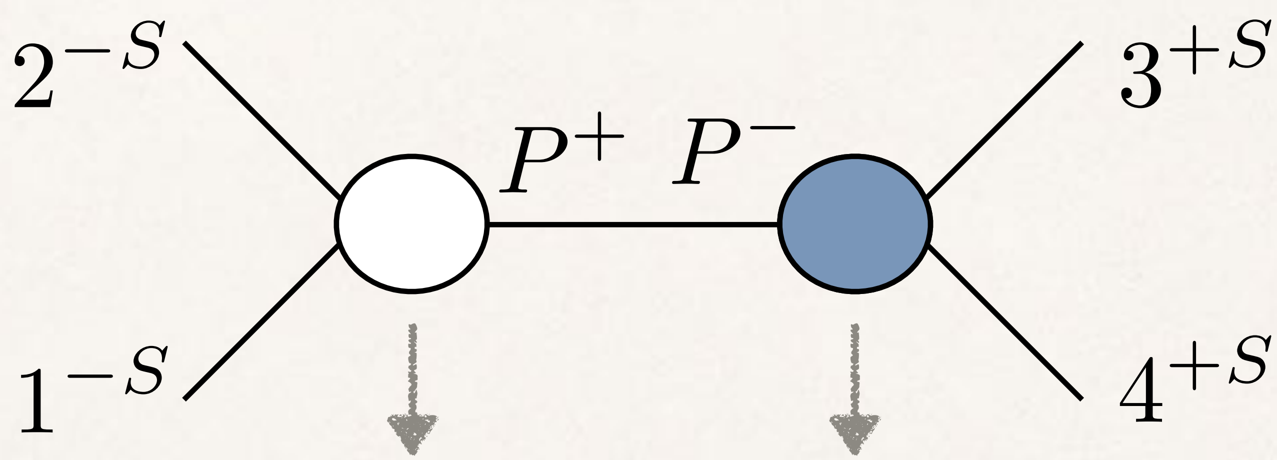
$$A_4 \xrightarrow[s=0]{} A_3 \frac{1}{s} A_3$$

This must be true on all channels

- This will immediately kill most of the possibilities
- We are left with spectrum of spins: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

s-channel constraint

- The s-channel factorization dictates $P = 1 + 2 = -3 - 4$



$$A_4 \to \left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S \quad \text{on s=0}$$

Note: $s = \langle 12\rangle[12] = \langle 34\rangle[34]$

s-channel constraint

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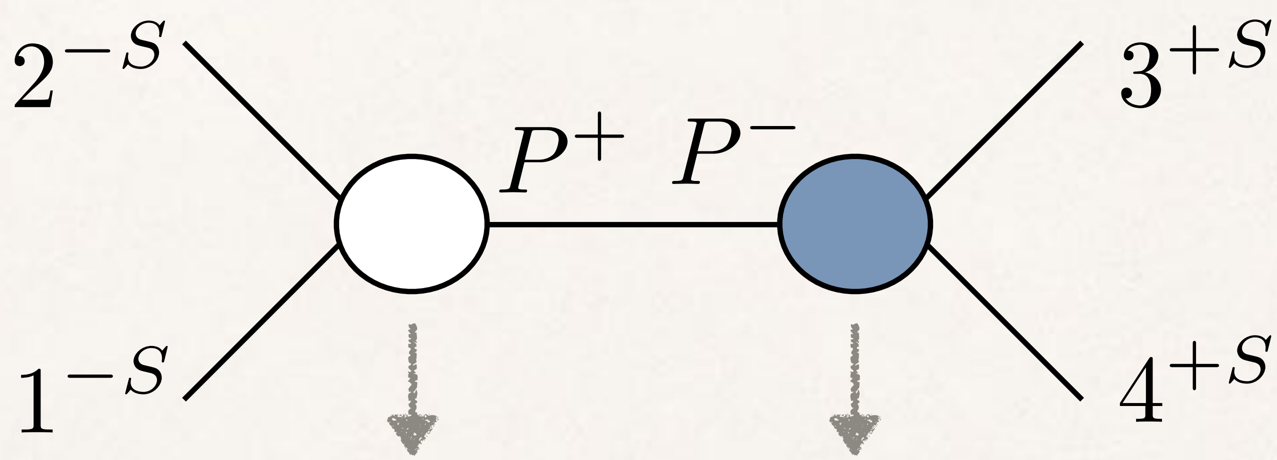

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s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^3}{\langle 1P\rangle[3P]\langle 2P\rangle[4P]}\right)^S$$

s-channel constraint

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- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^2}{\langle 23\rangle[23]}\right)^S$$

s-channel constraint

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- We get

$$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^2}{t}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

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$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get

$$(\langle 12\rangle[34])^{2S}\cdot\frac{1}{s\,t^S}$$

Note: $t = -u$

"Trivial" helicity factor

Important piece

Comparing channels

- On s-channel we got: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \frac{1}{s\,t^S}$
- On t-channel we get: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \frac{1}{t\,s^S}$
- Require simple poles: $\frac{1}{s}, \frac{1}{t}, \frac{1}{u}$ and search for $F(s,t,u)$

Comparing channels

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- On t-channel we get: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\frac{1}{t\,s^S}}$
- Require simple poles: $\frac{1}{s}, \frac{1}{t}, \frac{1}{u}$ and search for $F(s,t,u)$
- There are only two solutions:

$$F(s,t,u) = \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \qquad\qquad F(s,t,u) = \frac{1}{stu}$$

spin 0 (ϕ^3) spin 2 (GR)

Where are gluons (spin-1)?

- Need to consider multiplet of particles

$$A_3 = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^S f^{a_1a_2a_3} \qquad A_3 = \left(\frac{[12]^3}{[23][31]}\right)^S f^{a_1a_2a_3}$$

- The same check gives us S=1 and requires

$$f^{a_1a_2a_P}f^{a_3a_4a_P} + f^{a_1a_4a_P}f^{a_2a_3a_P} = f^{a_1a_3a_P}f^{a_2a_4a_P}$$

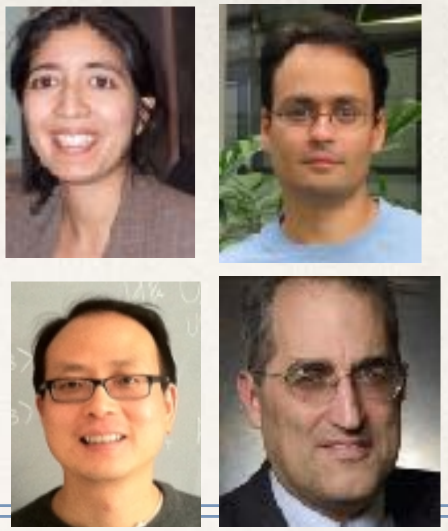
and the result corresponds to
SU(N) Yang-Mills theory

Power of 4pt check

- We can apply this check for cases with mixed particle content:
  - Spin >2 still not allowed
  - Spin 2 is special: only one particle and it couples universally to all other particles
  - We get various other constraints on interactions (of course all consistent with known theories)
- General principles very powerful
- Higher point amplitudes: recursive construction

BCFW recursion relations

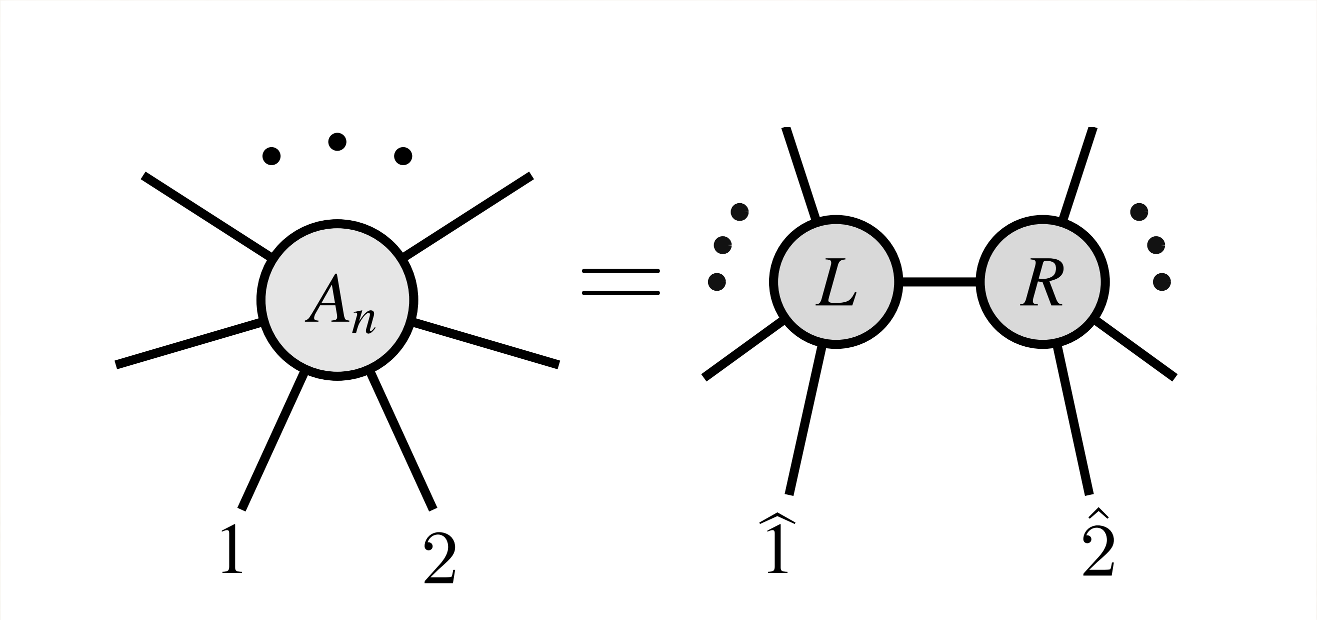
(Britto, Cachazo, Feng, Witten, 2005)



$$A_n = -\sum_j A_L(z_j)\frac{1}{P_j^2}A_R(z_j)$$

$$z_j = \frac{P_j^2}{2\langle 1|P_j|2]}$$

$$\lambda_1 \to \lambda_1 - z\lambda_2 \qquad \widetilde{\lambda}_2 \to \widetilde{\lambda}_2 + z\widetilde{\lambda}_1$$



Chosen such that internal line is on-shell

Sum over all distributions of legs keeping 1,2 on different sides

Thank you for attention!