Quantization of Magnetic Poisson Structures

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cost Action MP 1405

Quantum Structure of Spacetime

HORIZON 2020

QSPACE Training School III

Centro de Ciencias de Benasque Pedro Pascual

September 28, 2018

Outline

- Magnetic Poisson structures
- Classical & quantum dynamics in fields of magnetic charge
- Deformation quantization
- Symplectic realization
- Higher geometric quantization

Magnetic Poisson Structures

- $M = \mathbb{R}^d$ configuration space x^i, M^* momentum space p_i, $\mathcal{M} = T^*M = M \times M^*$ 'phase space' $X^I = (x^i, p_i)$, with canonical symplectic form $\sigma_0(X, X') = p \cdot x' - p' \cdot x$

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$$\{x^i, x^j\}_\rho = 0 \quad , \quad \{x^i, p_j\}_\rho = \delta^i{}_j \quad , \quad \{p_i, p_j\}_\rho = -\rho_{ij}(x)$$

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- H-twisted Poisson structure on $\mathcal{M}$ with $H = \mathrm{d}\rho$ 'magnetic charge' $[\theta_\rho, \theta_\rho]_{\mathrm{S}} = \bigwedge^3 \theta_\rho^\sharp(\mathrm{d}\sigma_\rho)$ gives nonassociative algebra with Jacobiators $\{f, g, h\}_\rho = [\theta_\rho, \theta_\rho]_{\mathrm{S}}(\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h)$:

$$\{p_i, p_j, p_k\}_\rho = -H_{ijk}(x)$$

(Günaydin & Zumino '85)

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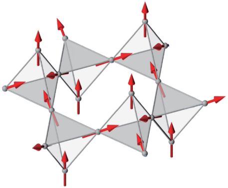
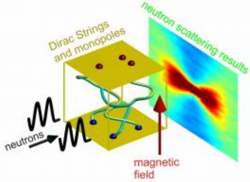
- In the lab: Neutron scattering off spin ice pyrochlore lattices

(Castelnovo, Moessner & Sondhi '08; Morris *et al.* '09; ...)





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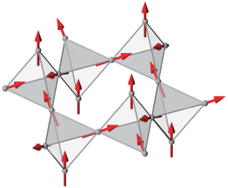

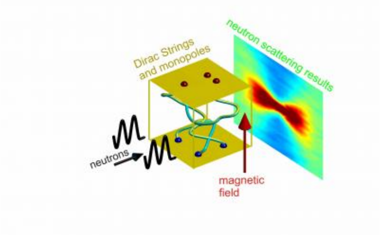

- Smooth $H = \mathrm{d}\rho \neq 0$ gives smooth distributions $\vec{\nabla}\cdot\vec{B} \neq 0$ of magnetic charge

Locally Non-Geometric Fluxes

- Born reciprocity $(x,p)\longmapsto(p,-x)$ preserves σ_0, maps $\rho\in\Omega^2(M)\ \longmapsto\ \beta\in\Omega^2(M^*)$ with twisted Poisson brackets:

$$\{x^i,x^j\}_\beta=-\beta^{ij}(p)\quad,\quad\{x^i,p_j\}_\beta=\delta^i{}_j\quad,\quad\{p_i,p_j\}_\beta=0$$

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- R-flux model: Phase space of closed strings propagating in 'locally non-geometric' R-flux backgrounds (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & Sz '12; Freidel, Leigh & Minic '17; ...)

Classical Motion in Fields of Magnetic Charge

- For $d = 3$, motion in magnetic field $\vec{B}$ (with or without sources) governed by Lorentz force

$$\dot{\vec{p}} \ = \ \frac{e}{m}\,\vec{p}\times\vec{B} \quad , \quad \vec{p} \ = \ m\,\dot{\vec{x}}$$

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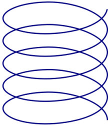
- $\vec{B} \;=\;$ constant:

  Motion follows helical trajectory with uniform velocity along $\vec{B}$-direction



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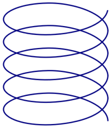
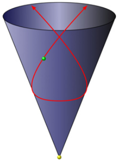


- Dirac monopole field $\vec{B} \ = \ \vec{B}_{\rm D}$: (Bakas & Lüst '13)

Conservation of Poincaré vector $\vec{K}$ confines motion to surface of cone, electric charge never reaches magnetic monopole and nonassociativity plays no role



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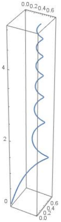

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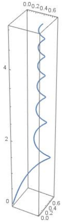

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- $\vec{B} = \frac{1}{3}\,H\,\vec{x}$, constant magnetic charge H: Motion is no longer integrable or confined, equivalent to motion in Dirac monopole field $\vec{B}_{\mathrm{D}}$ with additional frictional forces (Bakas & Lüst '13)

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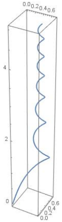

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- **Questions:**
  - What substitutes for canonical quantization of locally non-geometric closed strings?
  - Is there a sensible nonassociative quantum mechanics?

Quantization of Magnetic Poisson Structures

- Quantization Linear map $f \longmapsto \mathcal{O}_f$ for $f \in C^\infty(\mathcal{M})$:

$$[\mathcal{O}_f, \mathcal{O}_g] = \mathrm{i}\,\hbar\,\mathcal{O}_{\{f,g\}_\rho} + O(\hbar^2)$$

$$[\mathcal{O}_{x^i}, \mathcal{O}_{x^j}] = 0 \quad , \quad [\mathcal{O}_{x^i}, \mathcal{O}_{p_j}] = \mathrm{i}\,\hbar\,\delta^i{}_j\,\mathbb{1} \quad , \quad [\mathcal{O}_{p_i}, \mathcal{O}_{p_j}] = -\,\mathrm{i}\,\hbar\,\rho_{ij}(\mathcal{O}_x)$$

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$$\mathcal{P}_v^{-1}\,\mathcal{O}_{x^i}\,\mathcal{P}_v = \mathcal{O}_{x^i+v^i}$$

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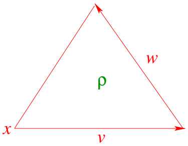
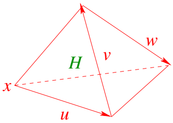
- Representation of translation group $\mathbb{R}^d$? (Jackiw '85)

$$\mathcal{P}_w\,\mathcal{P}_v = \mathrm{e}^{\,\mathrm{i}\,\Phi_2(x;v,w)}\,\mathcal{P}_{v+w} \quad , \quad \mathcal{P}_w\,(\mathcal{P}_v\,\mathcal{P}_u) = \mathrm{e}^{\,\mathrm{i}\,\Phi_3(x;u,v,w)}\,(\mathcal{P}_w\,\mathcal{P}_v)\,\mathcal{P}_u$$





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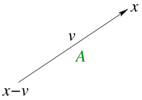

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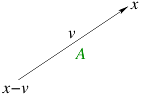

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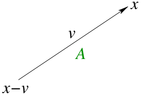

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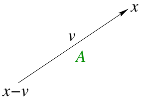

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- $\omega_{v,w}(x-u)\,\omega^{-1}_{u+v,w}(x)\,\omega_{u,v+w}(x)\,\omega^{-1}_{v,w}(x) = 1$
  2-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\, p\cdot x}\, \mathrm{e}^{\,-\,\mathrm{i}\, p\cdot y}\, (\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \left(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\sigma_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \right) W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

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- Magnetic Moyal–Weyl star product $\mathcal{O}_{f\star_\rho g} = \mathcal{O}_f\,\mathcal{O}_g$:

$$(f\star_\rho g)(X) = \frac{1}{(\pi\,\hbar)^d} \int_{\mathcal{M}} \int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\sigma_0(Y,Z)}\,\omega_{x+y-z,x-y+z}(x-y-z)\, f(X-Y)\, g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z$$

Quantization with $\mathrm{d}\rho = 0$

- Magnetic Weyl transform $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{-\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

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- Geometric quantization (canonical quantum mechanics)
  $\Longrightarrow$ deformation quantization (phase space quantum mechanics):
  - Observables/states: (real) functions on phase space
  - Operator product: star product , Traces: integration
  - State function (density matrix): $S \geqslant 0$, $\int_{\mathcal{M}} S = 1$
  - Expectation values: $\langle \mathcal{O} \rangle = \int_{\mathcal{M}} \mathcal{O} \star_\rho S$ ...

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  - Magnetic Poisson algebra is associative on $M^\circ = \mathbb{R}^3 \setminus \{0\}$, $\rho = \mathrm{d}A_{\rm D}$ locally
  - Quantum Hilbert space is $\mathcal{H} = \mathrm{L}^2(M^\circ, L)$ for a non-trivial line bundle $L \longrightarrow M^\circ$ iff Dirac charge quantization: $\dfrac{2\,e\,g}{\hbar} \in \mathbb{Z}$ (Wu & Yang '76)
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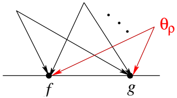
- For generic smooth distributions $H \in \Omega^3(M)$, standard geometric quantization breaks down

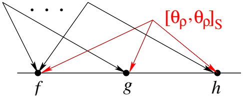
Deformation Quantization

- For any $H = \mathrm{d}\rho \in \Omega^3(M)$, Kontsevich formality provides noncommutative and nonassociative star product on $C^\infty(\mathcal{M})[[\hbar]]$:

$$f \star_H g = f\,g + \frac{\mathrm{i}\,\hbar}{2}\,\{f,g\}_\rho + \sum_{n\geqslant 2} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{b}_n(f,g)$$



$$[f,g,h]_{\star_H} = -\hbar^2\,\{f,g,h\}_\rho + \sum_{n\geqslant 3} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{t}_n(f,g,h)$$



where $\mathtt{b}_n = U_n(\theta_\rho,\dots,\theta_\rho)$ and $\mathtt{t}_n = U_{n+1}([\theta_\rho,\theta_\rho]_{\mathrm{S}},\theta_\rho,\dots,\theta_\rho)$ are bi-/tri-differential operators (Mylonas, Schupp & Sz '12)

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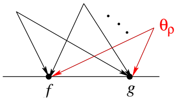

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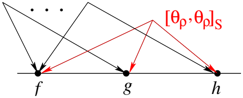

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- Nonassociative magnetic translations $\mathcal{P}_v := \mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\, p\cdot v}$ give 3-cocycle:

$$\mathcal{P}_v \star_H \mathcal{P}_w \;=\; \Pi_{v,w}(x)\; \mathcal{P}_{v+w}$$

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- **Problems:** Quantization formal in $\hbar$ for non-constant H, issues with dynamics, ...

Symplectic Realization

- Symplectic realization of a Poisson structure θ on M:
  Symplectic manifold (S, Ω) with surjective submersion $S \longrightarrow M$
  which is a Poisson map (Weinstein '83; Karasev '87; Coste, Dazord & Weinstein '87; Cattaneo & Xu '04)

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$$(f \circ_H g) \star_H \varphi := f \star_H (g \star_H \varphi)$$

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- $O(d,d)\times O(d,d)$-invariant Hamiltonian:

$$\mathcal{H} \;=\; \frac{1}{m}\, p_I\, \eta^{IJ}\, p_J \quad , \quad p_I \;=\; (p_i, \tilde{p}_i) \quad , \quad \eta \;=\; \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

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- **Higher structures:** Replace Hilbert spaces with 2-Hilbert spaces of sections of a suitable geometric object which encodes $H = {\rm d}\rho \neq 0$

Bundle Gerbes

- For $\pi : Y \longrightarrow M$ surjective submersion: $Y^{[p]} := Y \times_M \cdots \times_M Y$ forms a simplicial space with face maps $\pi_i : Y^{[p]} \longrightarrow Y^{[p-1]}$

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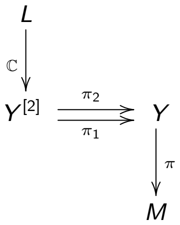

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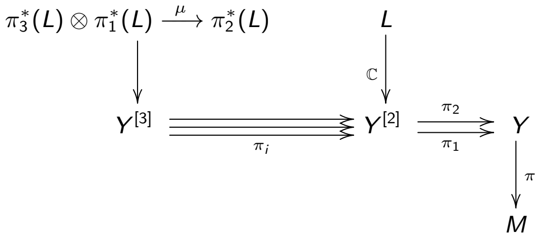

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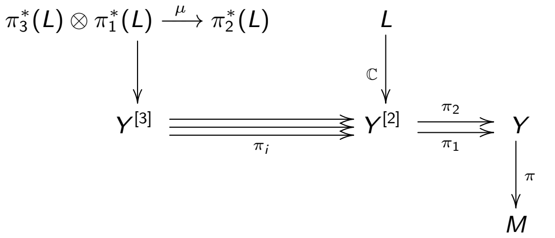

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- Connection: $\rho \in \Omega^2(Y)$ satisfying $\pi_2^*(\rho) - \pi_1^*(\rho) = F_{\nabla^L}$, curvature is $\pi^* H = \mathrm{d}\rho$, $H \in \Omega^3(M)$

Sections of Bundle Gerbes

- Bundle gerbes $\mathcal{G} = (L, Y) \Longrightarrow M$ are objects in a symmetric monoidal 2-category (Waldorf '07; Bunk, Sämann & Sz '16)

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  - Rig module category structure over rig category HVbdl(M)
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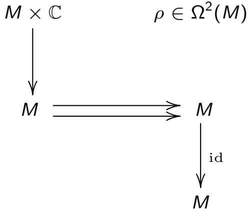
- Simple description on $M = \mathbb{R}^d$ for trivial bundle gerbes $\mathcal{G} = \mathcal{I}_\rho$:



  - Objects are vector bundles with connection: $\eta \in \Omega^1(M, u(n))$
  - Morphisms are parallel morphisms: $f : \eta \longrightarrow \eta'$ is a function $f : M \longrightarrow {\rm Mat}(n \times n')$ satisfying ${\rm i}\, \eta' f = {\rm i}\, f\, \eta - {\rm d}f$

Nonassociative Magnetic Translations

- Parallel transport functor $\mathcal{P}_v : \Gamma(M, \mathcal{I}_\rho) \longrightarrow \Gamma(M, \mathcal{I}_\rho)$:

(Bunk, Müller & Sz '18)

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- **Open issues:**
  - Understand physical significance of 2-Hilbert space $\Gamma(M, \mathcal{I}_\rho)$: states, observables, ...
  - Develop "Higher magnetic Weyl transform" to bridge higher geometric quantization with deformation quantization