Reaching infinity in numerical simulations

Free hyperboloidal evolution in spherical symmetry

Alex Vano-Vinuales

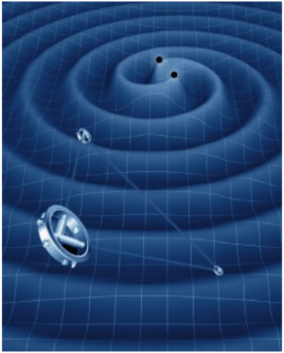
Cardiff University

NR beyond GR, Benasque - 5th June 2018

AVV, S. Husa & D. Hilditch, CQG 32 (2015) 175010, gr-qc/1412.3827.

AVV & S. Husa: CQG 35 (2018) 045014, gr-qc/1705.06298; 1412.4801, 1601.04079.

Reaching future lightlike infinity



Gravitational waves are only well defined at future null infinity ($\mathscr{I}^+$), where observers of astrophysical events are located.

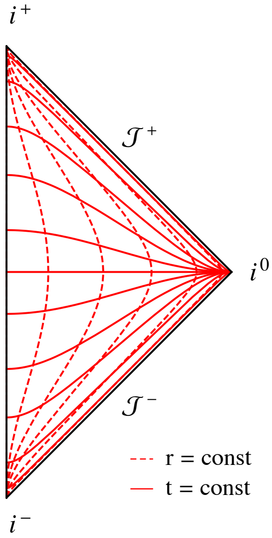
The study of spacetimes' global properties can also benefit from including $\mathscr{I}^+$.

A possible approach to this problem is Penrose's conformal compactification: we conformally rescale the physical metric $\tilde{g}_{\mu\nu}$

$$g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu}, \qquad (1)$$

so that $\Omega|_{\mathscr{I}^+} = 0$ at the appropriate order to keep $g_{\mu\nu}$ finite there.

Reaching future lightlike infinity



Gravitational waves are only well defined at future null infinity ($\mathscr{I}^+$), where observers of astrophysical events are located.

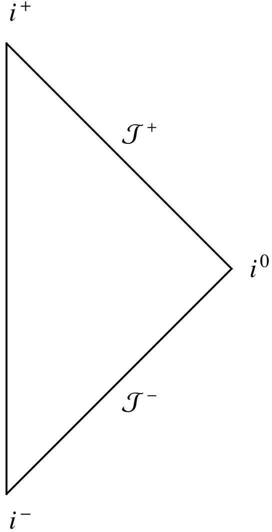
The study of spacetimes' global properties can also benefit from including $\mathscr{I}^+$.

A possible approach to this problem is Penrose's conformal compactification: we conformally rescale the physical metric $\tilde{g}_{\mu\nu}$

$$g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu}, \tag{1}$$

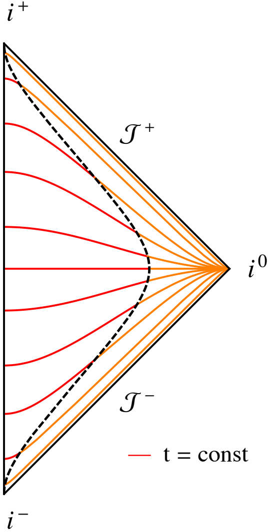
so that $\Omega|_{\mathscr{I}^+} = 0$ at the appropriate order to keep $g_{\mu\nu}$ finite there.

Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

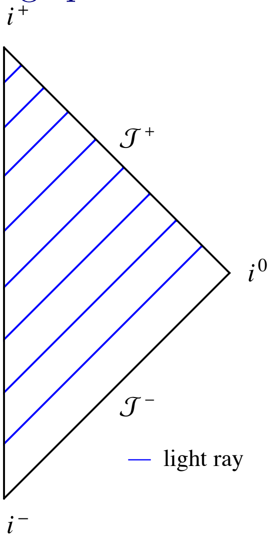
Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices

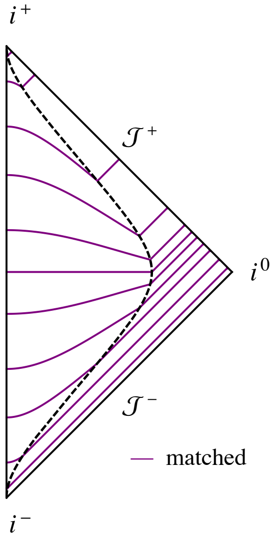
Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices

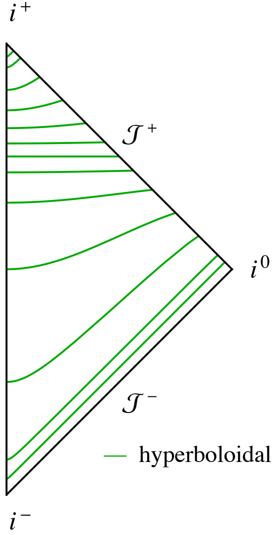
Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices
- Cauchy-Characteristic matching / extraction

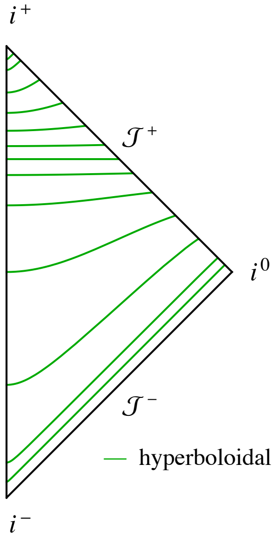
Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices
- Cauchy-Characteristic matching / extraction
- Hyperboloidal slices

Slicing spacetime



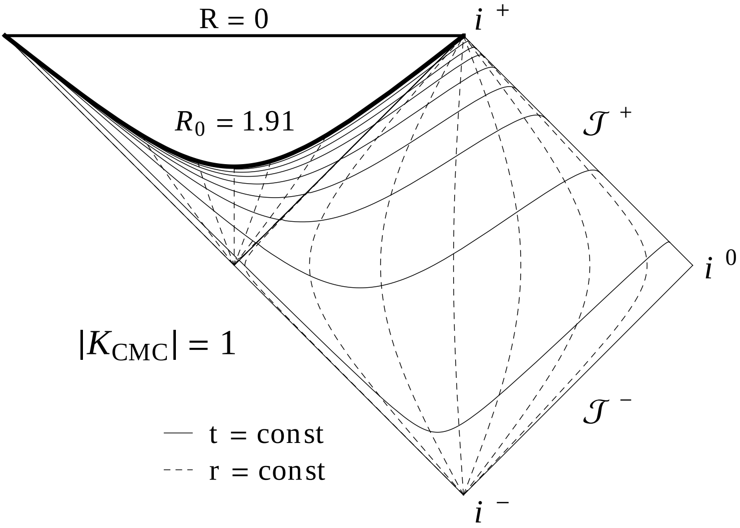
Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices
- Cauchy-Characteristic matching / extraction
- Hyperboloidal slices

Advantages of the hyperboloidal approach:

- Extraction at $\mathscr{I}^+$, no approximations.
- Slices spacelike & smooth everywhere.
- More resolution for the central part.

Schwarzschild trumpet CMC foliation



Brief history of the numerical hyperboloidal IVP

- Conformal Field Equations by Friedrich: generality maintained and regularity manifestly shown.
- Numerical implementations by Hübner (tested by Husa, continuum instabilities found) and by Frauendiener.
- Free evolution (generalized harmonic) and a fixed conformal factor by Zenginoğlu: Schwarzschild in spherical symmetry.
- Moncrief and Rinne's constrained axisymmetric code.
- Tetrad formalism by Bardeen, Sarbach and Buchman.
- Wave equation tests in dual foliation formalism by Hilditch.

Brief history of the numerical hyperboloidal IVP

- Conformal Field Equations by Friedrich: generality maintained and regularity manifestly shown.
- Numerical implementations by Hübner (tested by Husa, continuum instabilities found) and by Frauendiener.
- Free evolution (generalized harmonic) and a fixed conformal factor by Zenginoğlu: Schwarzschild in spherical symmetry.
- Moncrief and Rinne's constrained axisymmetric code.
- Tetrad formalism by Bardeen, Sarbach and Buchman.
- Wave equation tests in dual foliation formalism by Hilditch.

Main difficulties of the numerical implementation:

- Extra formally divergent terms at $\mathscr{I}^+$ appear in the equations:

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu} - \frac{2}{\Omega}\left(\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\nabla^\gamma\nabla_\gamma\Omega\right) - \frac{3}{\Omega^2} g_{\mu\nu}(\nabla_\gamma\Omega)\nabla^\gamma\Omega. \quad (2)$$

- Non-trivial background ($\tilde{K} \neq 0$), unlike with Cauchy slices.

Basic approach

Formulation:

- Free evolution: BSSN, Z4
- Time-independent $\Omega = |K_{CMC}| \frac{r_{\mathscr{I}}^2 - r^2}{6 r_{\mathscr{I}}}$
- Spherical symmetry
- + Massless scalar field

Hyperboloidal initial data:

- Height function approach
- Compactified slice
- Minkowski spacetime
- Schwarzschild trumpet

perturbed by a scalar field.

Hyperbolic gauge conditions:

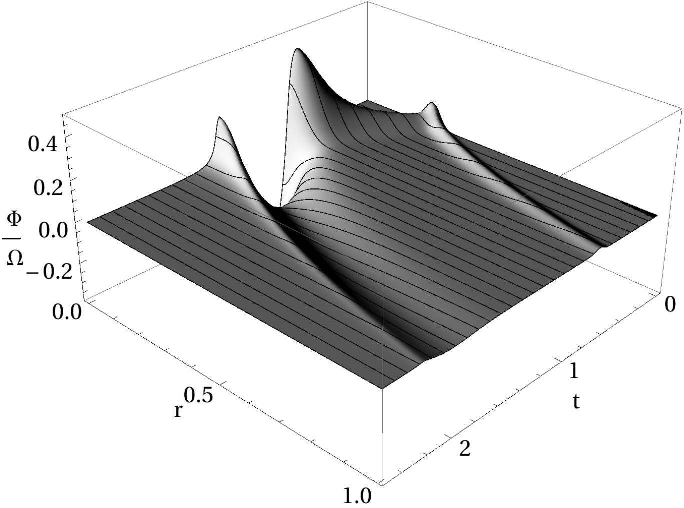
- Slicing (α): Bona-Massó $\dot{\alpha} = \beta^r \alpha' - f(\alpha)\, \alpha^2\, (K - K_0) + L_{\alpha 0}$
- Shift (β^a): Gamma-driver $\dot{\beta}^r = \beta^r \beta^{r\prime} + \lambda\, \Lambda^r - \eta\, \beta^r + L_{\beta^r 0}$
- Preferred conformal gauge

with scri-fixing condition.

Numerical implementation:

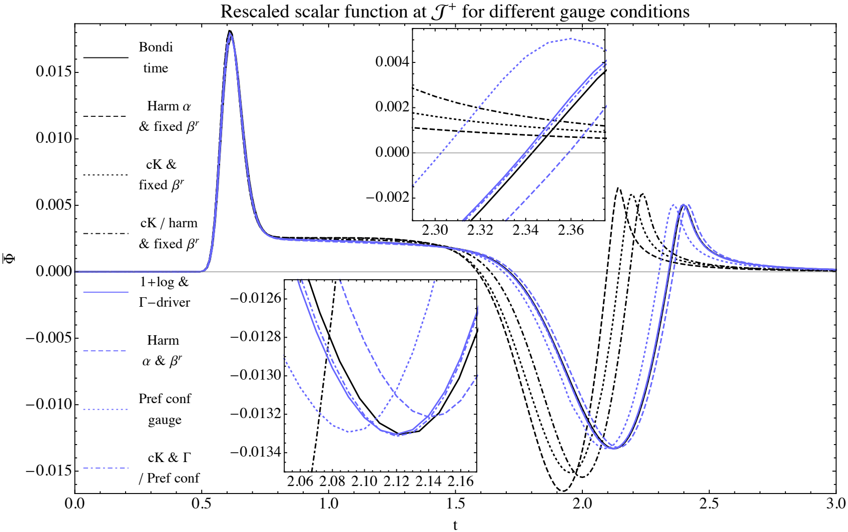
- Method Of Lines
  - Finite differences
  - Runge-Kutta 4th order
- Kreiss-Oliger dissipation
- (Non-)staggered grid.

Scalar field



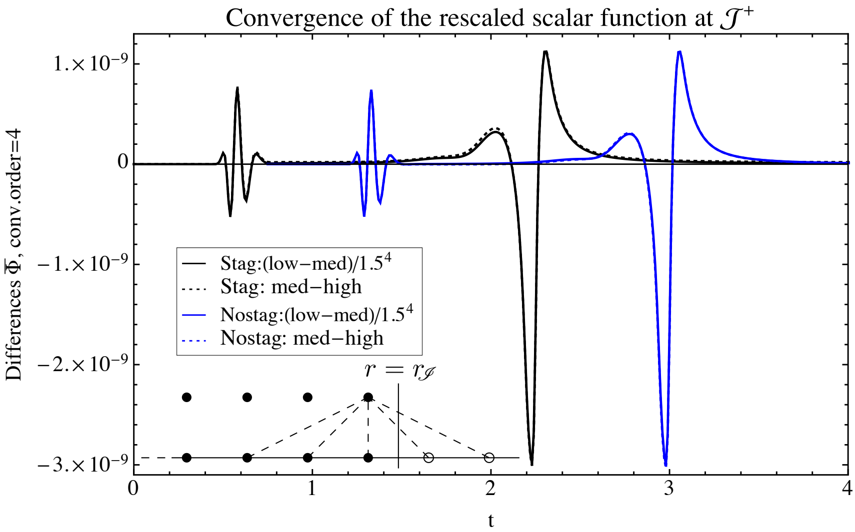
AVV, S. Husa and D. Hilditch, arXiv:1412.3827 [gr-qc]

Scalar field signal at $\mathscr{I}^+$

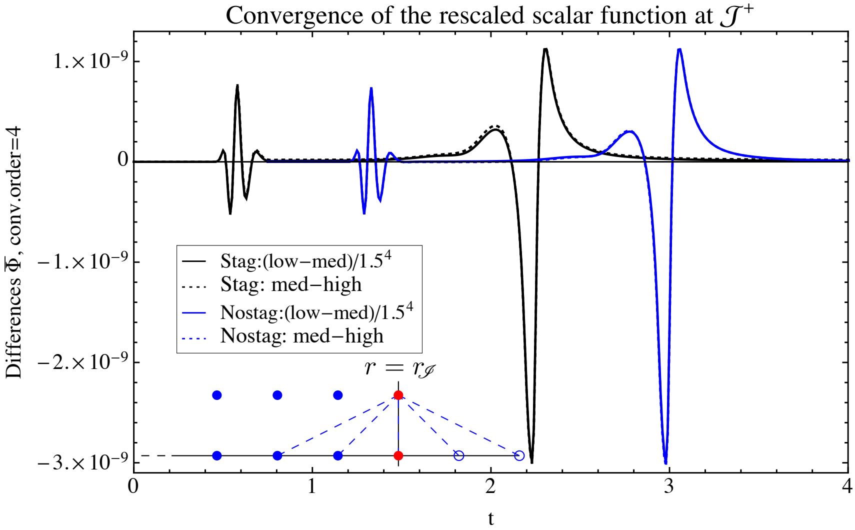


AVV and S. Husa, arXiv:1705.06298 [gr-qc]

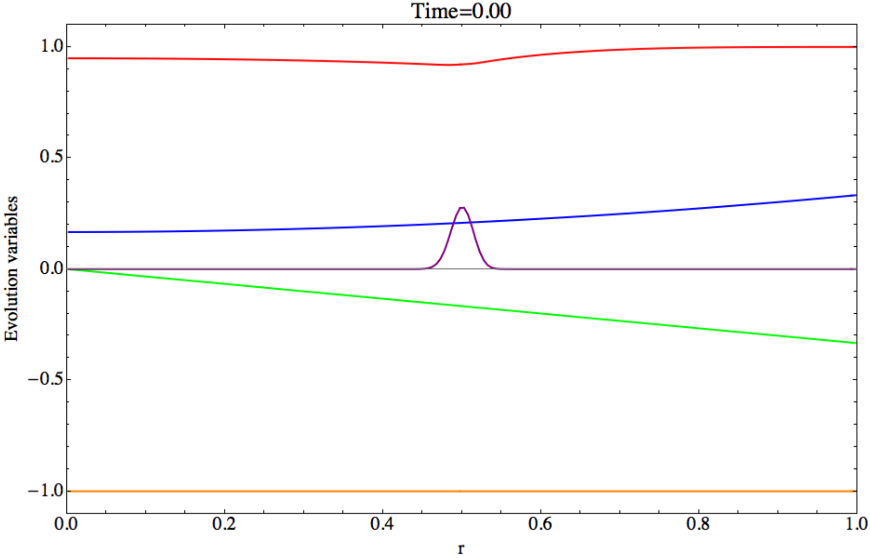
Scalar field - convergence at $\mathscr{I}^+$



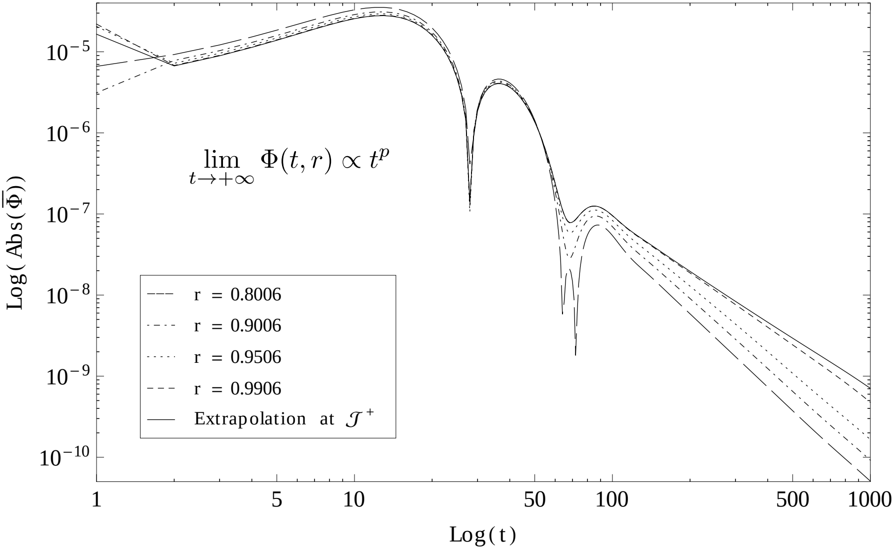
Scalar field - convergence at $\mathscr{I}^+$



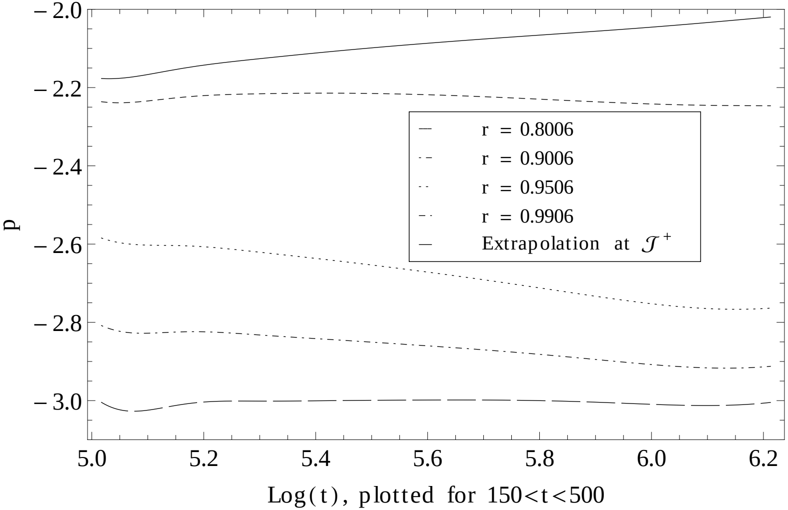
Evolution: χ, $\tilde{K}$, α, β^r, Φ/Ω



Power-law decay tails of the scalar field



Slopes of the decay tails



$f(R)$ + hyperboloidal

Consider the equations of motion for $f(R)$

$$F(\tilde{R})\tilde{R}_{\mu\nu}-\frac{1}{2}f(\tilde{R})\tilde{g}_{\mu\nu}+\left[\tilde{g}_{\mu\nu}\tilde{\Box}-\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right]F(\tilde{R})=8\pi T_{\mu\nu}, \quad F(\tilde{R})=\frac{df(\tilde{R})}{d\tilde{R}},$$

for the case $f(\tilde{R})=\tilde{R}+\alpha\tilde{R}^2$.

Express it in terms of the rescaled metric $g_{\mu\nu}\equiv\Omega^2\tilde{g}_{\mu\nu}$:

$f(R)$ + hyperboloidal

$$R[\nabla]_{ab} - \frac{1}{2} g4_{ab} R[\nabla] + 2\alpha R[\nabla]_{ab} R[\nabla] \Omega^2 - \frac{1}{2}\alpha g4_{ab} R[\nabla]^2 \Omega^2 - 6\alpha \Omega \nabla_a\Omega \nabla_b R[\nabla] - 6\alpha\Omega \nabla_a R[\nabla] \nabla_b\Omega -$$

$$12\alpha R[\nabla] \nabla_a\Omega \nabla_b\Omega - 24\alpha\, g4^{cd} \nabla_a\nabla_d\nabla_c\Omega \nabla_b\Omega - 2\alpha\Omega^2 \nabla_b\nabla_a R[\nabla] + \frac{2\nabla_b\nabla_a\Omega}{\Omega} - 12\alpha\, g4^{cd}\, \Omega \nabla_b\nabla_a\nabla_d\nabla_c\Omega - 24\alpha\, g4^{cd} \nabla_a\Omega \nabla_b\nabla_d\nabla_c\Omega -$$

$$\frac{4\alpha\, g4_{ab} \nabla_c\Omega \nabla^c R[\nabla]}{\Omega^3} + 2\alpha\, g4_{ab}\, \Omega \nabla_c\Omega \nabla^c R[\nabla] + 48\alpha \nabla_b\nabla_c\nabla_a\Omega \nabla^c\Omega + 4\alpha\, g4_{ab} R[\nabla] \nabla_c\Omega \nabla^c\Omega - \frac{8\alpha\, g4_{ab} R[\nabla] \nabla_c\Omega \nabla^c\Omega}{\Omega^4} +$$

$$\frac{48\alpha \nabla_b\Omega \nabla_c\nabla_a\Omega \nabla^c\Omega}{\Omega} + \frac{48\alpha \nabla_a\Omega \nabla_c\nabla_b\Omega \nabla^c\Omega}{\Omega} + 12\alpha\, g4_{ab}\, g4^{de} \nabla_c\nabla_e\nabla_d\Omega \nabla^c\Omega + 48\alpha\, g4^{cd} \nabla_c\nabla_a\Omega \nabla_d\nabla_b\Omega + \frac{2\alpha\, g4_{ab}\, g4^{cd} \nabla_d\nabla_c R[\nabla]}{\Omega^2} +$$

$$\frac{4\alpha\, g4_{ab}\, g4^{cd} R[\nabla] \nabla_d\nabla_c\Omega}{\Omega^3} - \frac{2\, g4_{ab}\, g4^{cd} \nabla_d\nabla_c\Omega}{\Omega} + 12\alpha\, g4^{cd} R[\nabla]_{ab}\, \Omega \nabla_d\nabla_c\Omega - 4\alpha\, g4_{ab}\, g4^{cd} R[\nabla]\, \Omega \nabla_d\nabla_c\Omega -$$

$$\frac{24\alpha\, g4^{cd} \nabla_a\Omega \nabla_b\Omega \nabla_d\nabla_c\Omega}{\Omega} + 12\alpha\, g4^{cd} \nabla_b\nabla_a\Omega \nabla_d\nabla_c\Omega + \frac{8\alpha\, g4_{ab}\, g4_{cd} \nabla^c R[\nabla] \nabla^d\Omega}{\Omega^3} - 24\alpha\, g4_{cd} R[\nabla]_{ab} \nabla^c\Omega \nabla^d\Omega +$$

$$6\alpha\, g4_{ab}\, g4_{cd} R[\nabla] \nabla^c\Omega \nabla^d\Omega + \frac{4\alpha\, g4_{ab}\, g4_{cd} R[\nabla] \nabla^c\Omega \nabla^d\Omega}{\Omega^4} + \frac{3\, g4_{ab}\, g4_{cd} \nabla^c\Omega \nabla^d\Omega}{\Omega^2} - \frac{48\alpha\, g4_{cd} \nabla_b\nabla_a\Omega \nabla^c\Omega \nabla^d\Omega}{\Omega} +$$

$$\frac{96\alpha\, g4_{ab} \nabla^c\Omega \nabla_d\nabla_c\Omega \nabla^d\Omega}{\Omega^5} - \frac{48\alpha\, g4_{ab} \nabla^c\Omega \nabla_d\nabla_c\Omega \nabla^d\Omega}{\Omega} - \frac{24\alpha\, g4_{ab}\, g4^{de} \nabla_c\Omega \nabla^c\Omega \nabla_e\nabla_d\Omega}{\Omega^5} + \frac{12\alpha\, g4_{ab}\, g4^{de} \nabla_c\Omega \nabla^c\Omega \nabla_e\nabla_d\Omega}{\Omega} -$$

$$\frac{48\alpha\, g4_{ab}\, g4^{de} \nabla^c\Omega \nabla_e\nabla_d\nabla_c\Omega}{\Omega^4} - \frac{48\alpha\, g4_{ab}\, g4^{cd}\, g4^{ef} \nabla_e\nabla_c\Omega \nabla_f\nabla_d\Omega}{\Omega^4} - 6\alpha\, g4_{ab}\, g4^{cd}\, g4^{ef} \nabla_d\nabla_c\Omega \nabla_f\nabla_e\Omega +$$

$$\frac{12\alpha\, g4_{ab}\, g4^{cd}\, g4^{ef} \nabla_d\nabla_c\Omega \nabla_f\nabla_e\Omega}{\Omega^4} + \frac{12\alpha\, g4_{ab}\, g4_{cd}\, g4^{ef} \nabla^c\Omega \nabla^d\Omega \nabla_f\nabla_e\Omega}{\Omega} + \frac{12\alpha\, g4_{ab}\, g4^{cd}\, g4^{ef} \nabla_f\nabla_e\nabla_d\nabla_c\Omega}{\Omega^3} == 8\pi T_{ab}$$

(2)

$f(R)$ + hyperboloidal

Consider the equations of motion for $f(R)$

$$F(\tilde{R})\tilde{R}_{\mu\nu}-\frac{1}{2}f(\tilde{R})\tilde{g}_{\mu\nu}+\left[\tilde{g}_{\mu\nu}\tilde{\Box}-\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right]F(\tilde{R})=8\pi T_{\mu\nu}, \quad F(\tilde{R})=\frac{df(\tilde{R})}{d\tilde{R}},$$

for the case $f(\tilde{R})=\tilde{R}+\alpha\tilde{R}^2$.

Express it in terms of the rescaled metric $g_{\mu\nu}\equiv\Omega^2\tilde{g}_{\mu\nu}$:

$$G_{\mu\nu}=8\pi\ T_{\mu\nu}+g_0+\frac{g_1}{\Omega}+\frac{g_2}{\Omega^2}+\frac{g_3}{\Omega^3}+\frac{g_4}{\Omega^4}+\frac{g_5}{\Omega^5}, \quad g_i=g_i(R,\Omega,g_{\mu\nu}).$$

Much more complicated than the General Relativity case.
Need to check that formally divergent terms cancel appropriately (+ well-posedness, ...).

Summary

Hyperboloidal initial value problem:

- promising and efficient approach for numerical simulations,
- allows the study of global properties and extraction of signals.
- To our knowledge, this is the first stable free evolution with a standard formulation.
- Far-field infrastructure, potentially useful for testing alternative theories of gravity?

Getting ready for further work:

- Simulations in AdS ($\mathscr{I}^+$ is timelike $\rightarrow$ boundary conditions).
- 3-dimensional code and initial data $\rightarrow$ binary systems.

AVV, S. Husa & D. Hilditch, CQG 32 (2015) 175010, gr-qc/1412.3827.

AVV & S. Husa: CQG 35 (2018) 045014, gr-qc/1705.06298; 1412.4801, 1601.04079.

Thank you for your attention!

Questions?