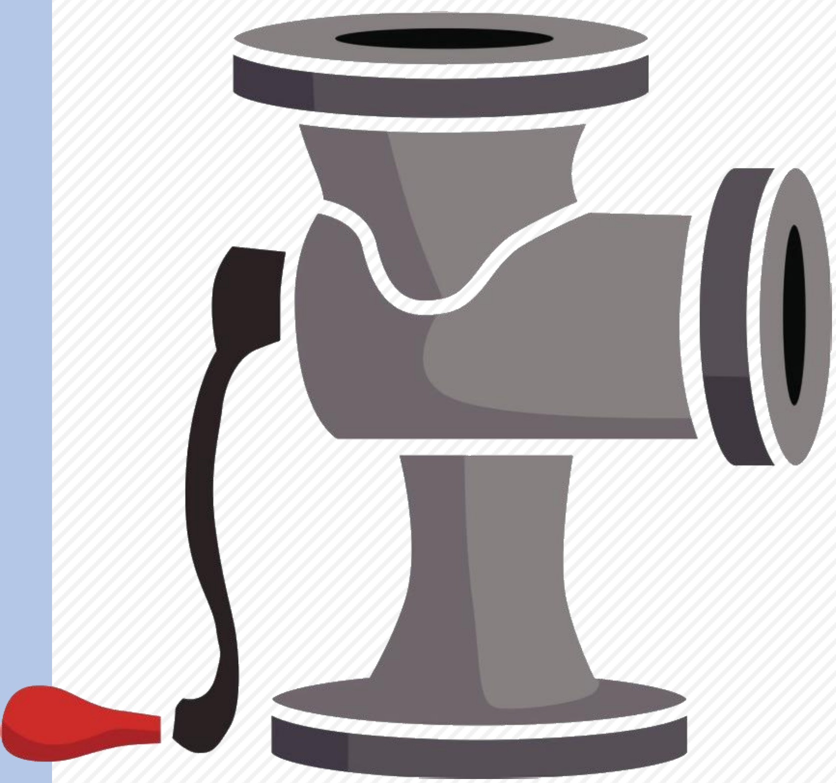
Computational power of symmetry-protected topological phases



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Part 0: Motivation

Exploiting the algebraic structure of quantum phases

“Zoo” of quantum phases

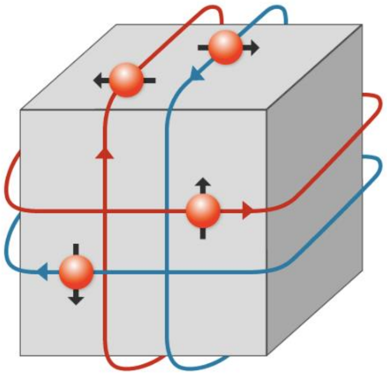
Symmetry breaking

- Crystals, ferromagnets
- Local order parameters
- Group Theory

Topological

- FQHE, Toric code
- Patterns of long range entanglement
- Category Theory

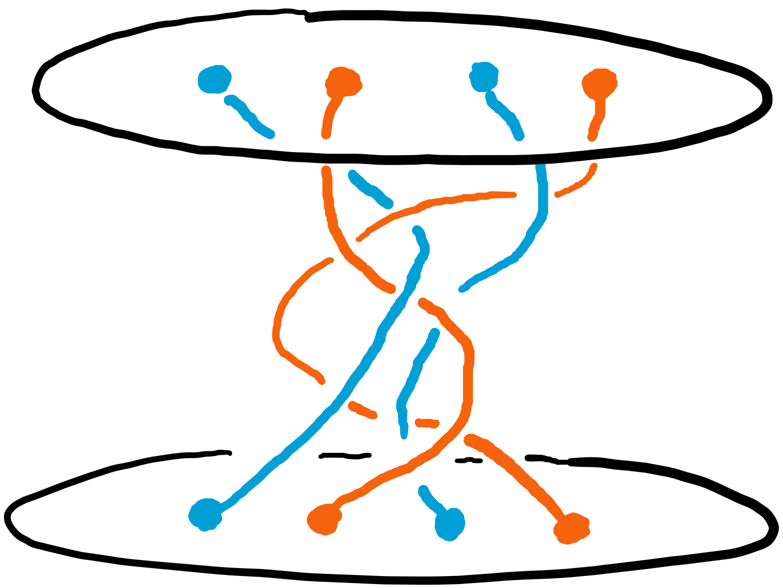
Symmetry-protected



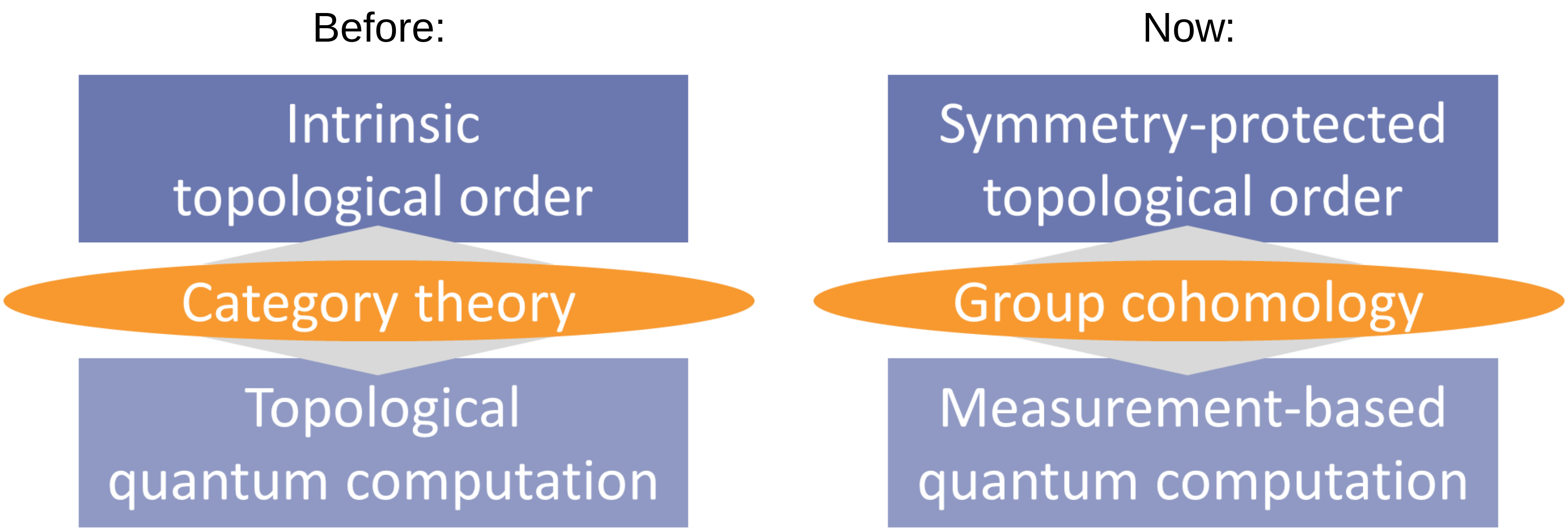
- Topological insulator, Haldane phase
- Patterns of short range, symmetric entanglement
- Group cohomology

- Quantum phases of matter have a rich algebraic classification.
- Can we take advantage of this in application?

Example: Topological QC



- Anyonic excitations can be braided, fused to achieve fault-tolerant quantum computation
- Braiding, fusion rules etc. are described by language of category theory
- Category theory classifies possible topological orders, *and identifies which ones allow universal TQC!*



Who cares?

QI: Classify MBQC resource states, understand source of quantum computational power + “robust” computation

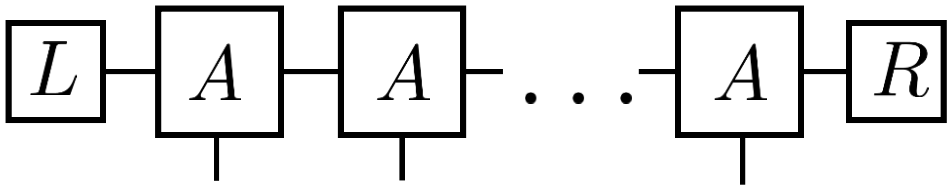
CM: Apply algebraic structure of quantum phases to other fields, classify phases by their computational capability → new insights

Part 1: Background

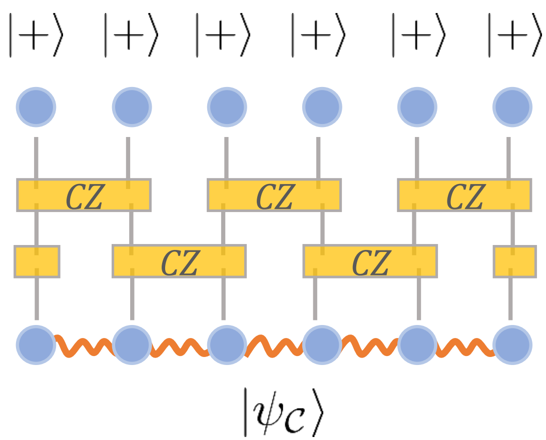
Tensor networks as a unifying framework for SPT order and MBQC

Matrix product states

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} \langle R|\, A^{i_n} A^{i_{n-1}} \ldots A^{i_1} |L\rangle \, |i_1 \ldots i_n\rangle$$



Example: Cluster state



$$A^0 = |0\rangle\langle +|$$

$$A^1 = |1\rangle\langle -|$$

$$A^{++} = I$$

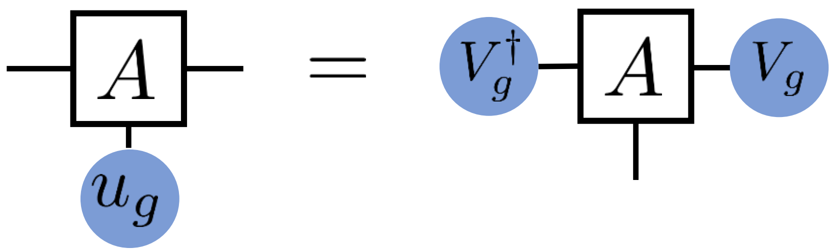
$$A^{+-} = Z$$

$$A^{-+} = X$$

$$A^{--} = XZ$$

Classification of SPTO with MPS

On-site symmetry group *G*: $u(g)^{\otimes n}|\psi\rangle = |\psi\rangle$

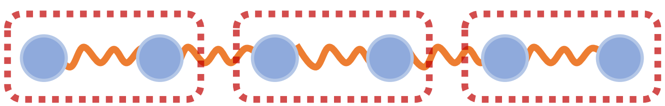


$$V_g V_h = \omega(g,h) V_{gh}$$
$$[\omega] \in H^2(G, U(1))$$

SPT phases classified by **group cohomology**

Chen *et. al.* 2011
Schuch *et. al.* 2011

Example: Cluster state $G = Z_2 \times Z_2$



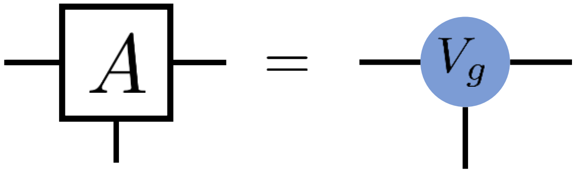
$u_{x_1} = X \quad X \quad X$

$u_{x_2} = \quad X \quad X \quad X$

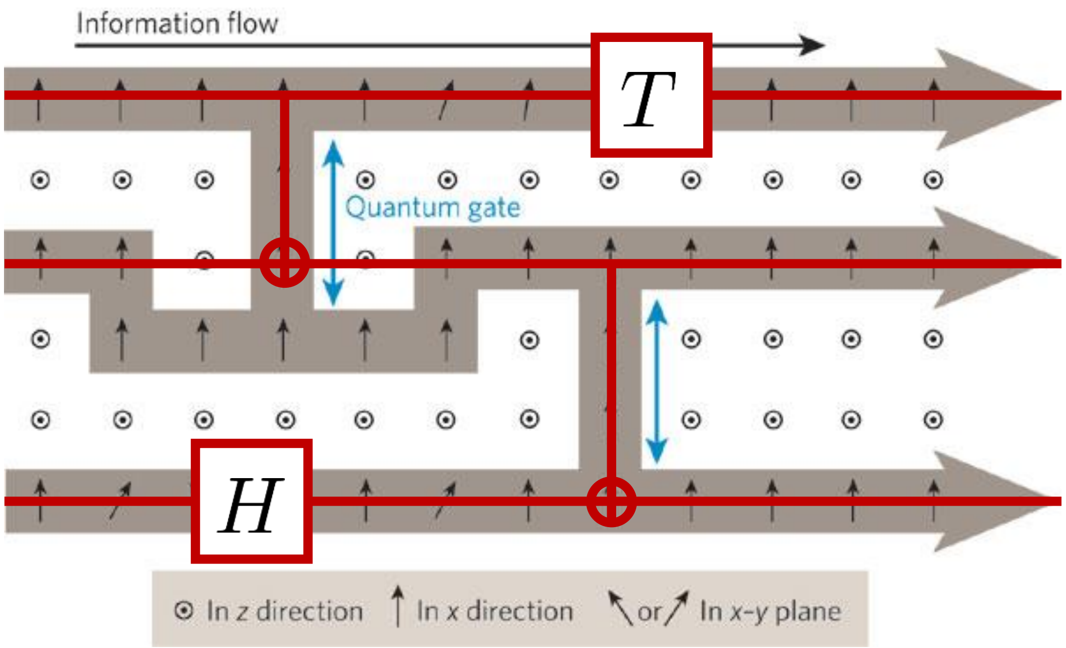
$V_e = I \qquad V_{x_2} = Z$

$V_{x_1} = X \qquad V_{x_1 x_2} = XZ$

1D cluster state has SPT order.

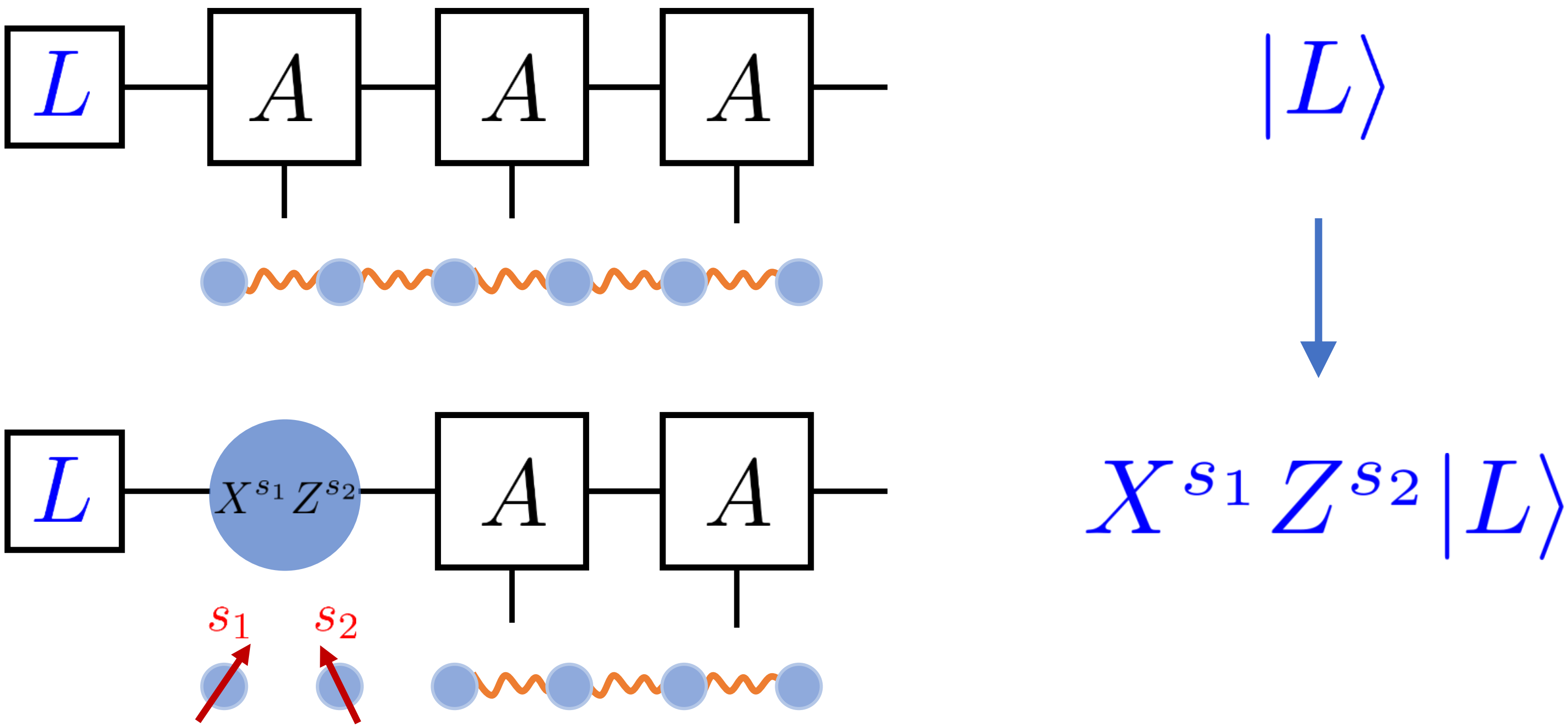


Measurement-based QC



- Measurement on entangled state → computation (universal?)
- MBQC defines the **computational power** of a quantum state
- What features of a state make it a useful resource?

MBQC in tensor networks

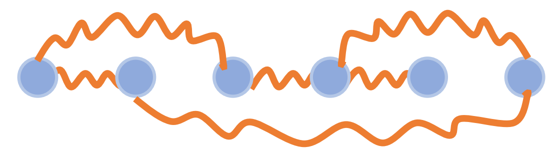


- Measurement → Unitary evolution in virtual space

Tensor networks capture both SPT order and computation

Entanglement “sweet spot”

Which states are useful as MBQC resources?
Goldilocks problem:



Product state
-Too trivial!

SPT states
-Just right!

Most states
-Too entangled!

Gross *et. al.* 2009

Phases are related by patterns of entanglement
→ SPT phases are useful?*

SPT-Entanglement in MPS

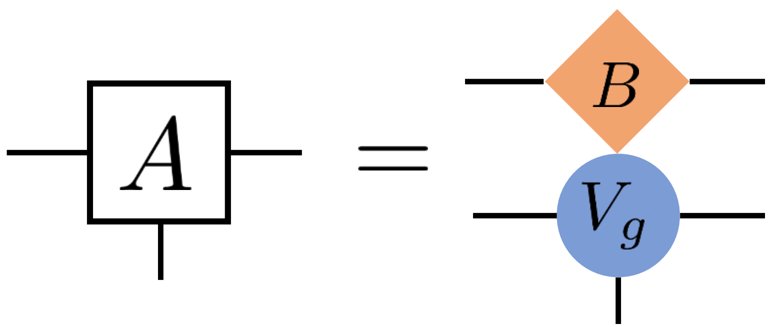
- For certain SPT phases, MPS matrices have the form:

Else et. al. 2012

$$A^i = B^i \otimes V_{g^i}$$

Wire basis:

$$\mathcal{B} = \{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$$



Junk subspace
contains microscopic details

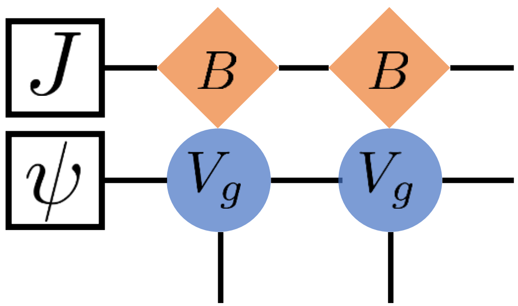
Logical subspace
uniform throughout phase

- *Theorem: For general abelian phases:*

$$A^i = \bigoplus_{\alpha} \left(B^i_\alpha \otimes V_{g^i_\alpha}\right) \mathcal{P}^i$$

Computation in SPT phases

- Encode information in logical subspace: $|L\rangle = |J\rangle \otimes |\psi\rangle$



$$\mathcal{B} = \{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$$

- Measure within wire basis: $|L'\rangle = B_i|J\rangle \otimes V_{g_i}|\psi\rangle$

Else et. al. 2012

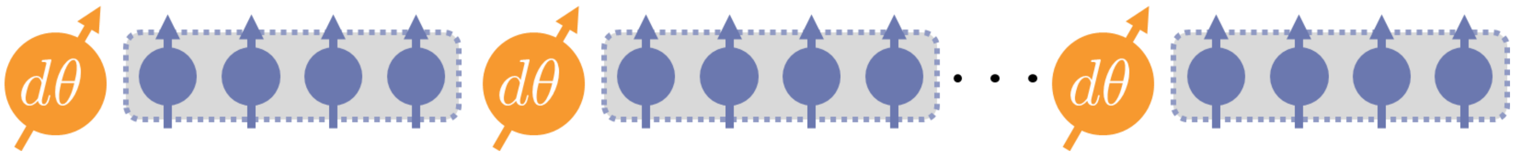
  - ➢ Perfect “identity gate” throughout phase!
- Measure outside wire basis: $|L'\rangle = B_i|J\rangle \otimes V_{g_i}|\psi\rangle + B_j|J\rangle \otimes V_{g_j}|\psi\rangle$
  - ➢ Microscopic details become entangled with logical state…

Need a way to get non-trivial gates in *all* phases

Part 2: 1D Results

Extracting the useful entanglement from 1D SPT phases

The solution…



Two simple ingredients allow computation throughout the phase:

1. Non-trivial gates are infinitesimal rotations.
2. Gates are separated by a distance >> correlation length.

Idea: slowly accumulate unitary rotation while keeping junk subspace separated from logical subspace.

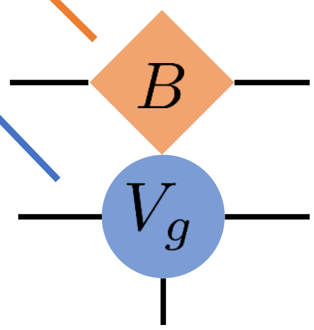
What gates can we do?

Primitive gates are infinitesimal rotations:

$$\mathcal{T}(i,j;d\alpha) = e^{d\alpha\left(\nu_{ij} V_{g_i}^{\dagger} V_{g_j} - \nu_{ji} V_{g_j}^{\dagger} V_{g_i}\right)}$$

$$\nu_{ij} = \operatorname{Tr} B_i B_j^{\dagger}$$

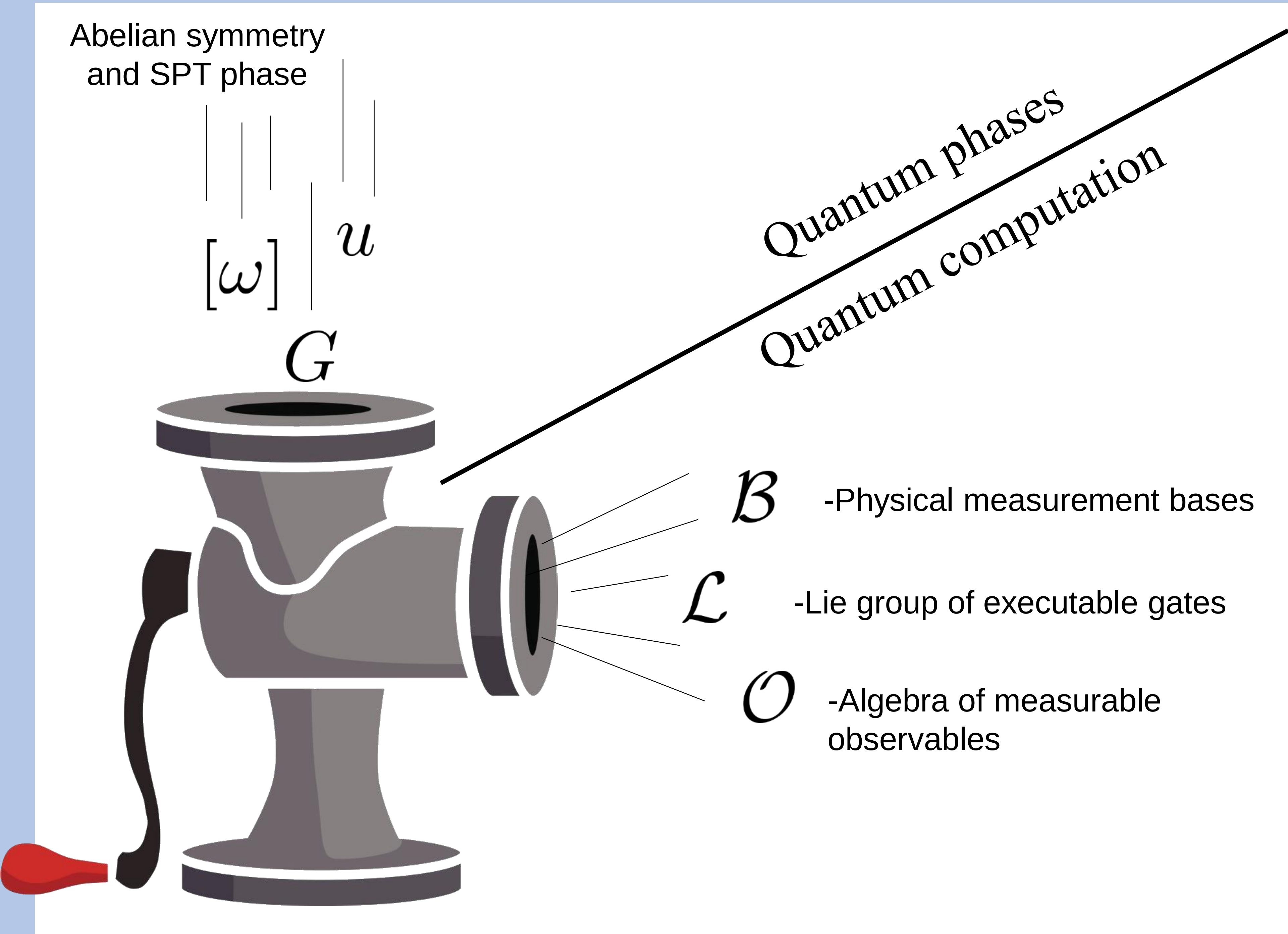
Microscopic details



$$\mathcal{L}\left(G, u, [\omega]\right)$$

Lie group of executable gates

Computational power is uniform within 1D SPT phases


Abelian symmetry
and SPT phase
$[\omega]$
G
u
Quantum phases
Quantum computation
$\mathcal{B}$ -Physical measurement bases
$\mathcal{L}$ -Lie group of executable gates
$\mathcal{O}$ -Algebra of measurable observables

Determining computational power

Theorem: If G is a finite abelian (sub)group, and [ω] is maximally non-commutative, then:

$$\mathcal{L}\left(G, u, [\omega]\right) \supset SU\left(p^n\right)$$

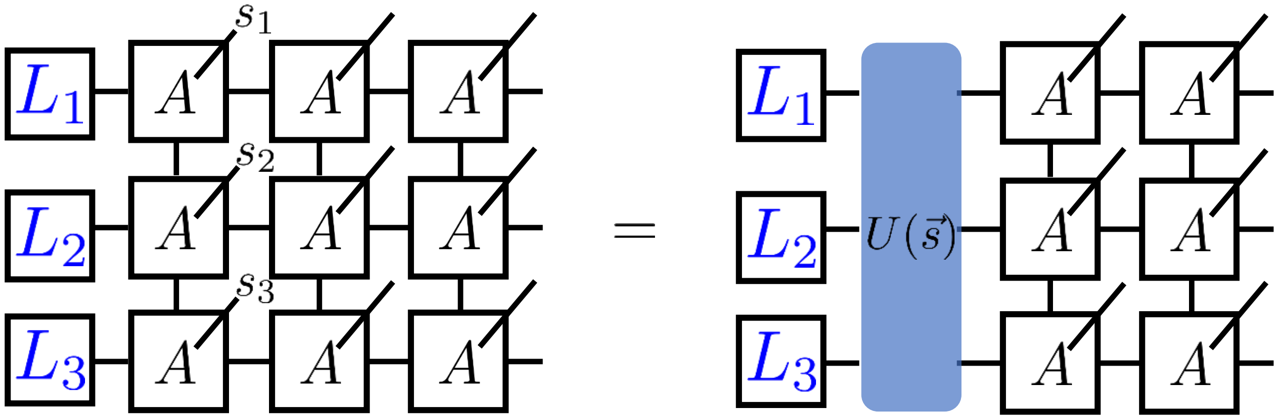
Symmetry Group G	**Group of Gates $\mathcal{L}$**
$SU(N), Z_N \times Z_N$	$SU(N)$
$SO(N), Sp(2N)$	$SU(2)$
D_{2n}, A_4, S_4	$SU(2)$
$(Z_2)^4$	$SU(4)$ *or* $SU(2) \times SU(2)$
$Z_4 \times Z_2$	$SU(2)$ *or* $U(1)$
$\vdots$	$\vdots$

Algebraic structure of SPT order can be used to identify universal resources (like TQC)

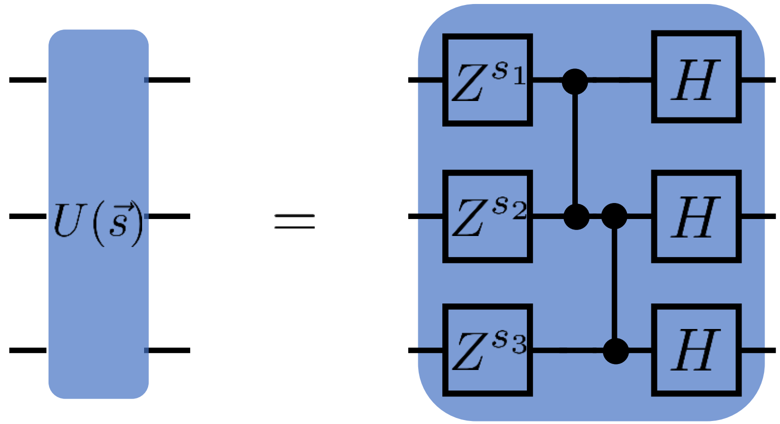
Part 3: 2D Results

A computationally universal phase of matter

MBQC with PEPS



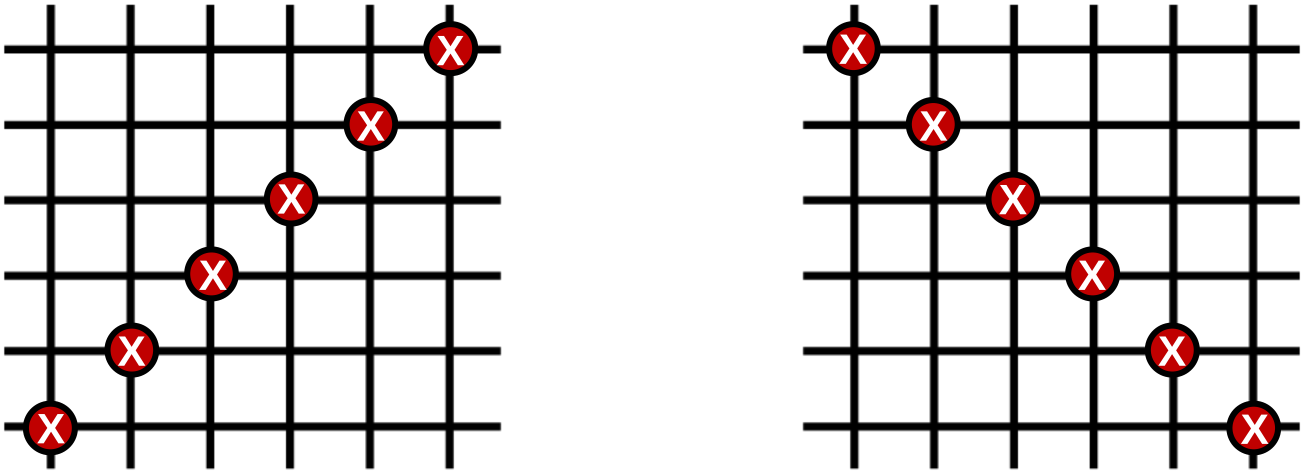
Example: 2D cluster state



This quantum cellular automata structure is enough to prove universality of the 2D cluster state

“Cluster Phase”

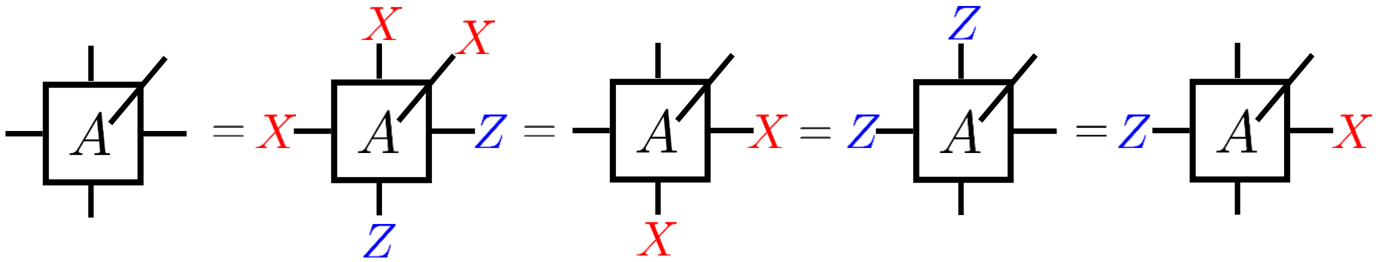
- Define the cluster phase as the gapped quantum phase which contains the 2D cluster state *and* respects the “line-like” symmetries of the cluster state:



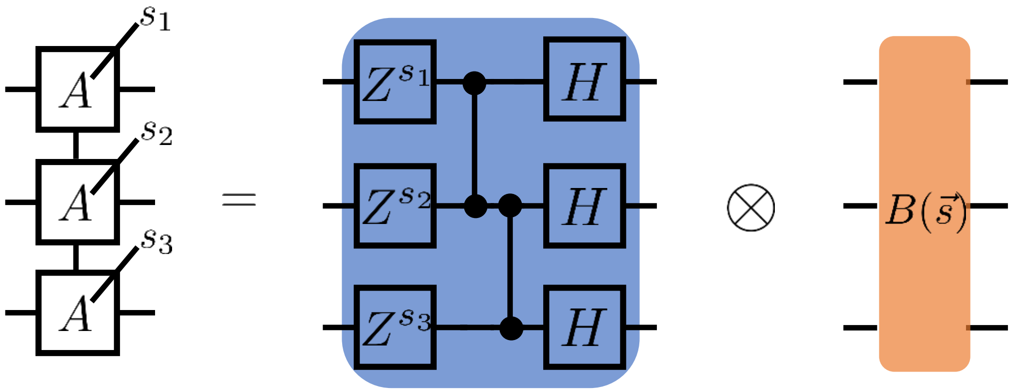
- This is a non-conventional SPT phase.

Cluster phase computation

- For every state in the cluster phase:



- These symmetries imply cluster QCA exists on virtual subspace:



Combine with 1D methods → cluster phase is a *universal* computational phase of matter!

Summary

- **Result 1:** 1D SPT phases are ubiquitously useful for quantum computation. The algebraic structure of SPT phases can be used to classify their computational power.
  [PRL 119, 010504, '17], [PRA 96, 012302, '17]

- **Result 2:** There exists a computationally universal phase of matter, the "cluster phase". It is a non-conventional 2D SPT phase.
  [arXiv tomorrow]

What's Next

- Extend 2D results to other cluster-like phases and uncover any underlying algebraic structure.
- Does conventional SPT order lead to similar results in 2D?
- Understand physics of 2D cluster phase. How to characterize SPT phases with non-global symmetries?
- Extend general idea to other phases of matter (symmetry enriched order, fracton order, etc.).

Can we classify phases of matter by their computational power?

Thank you!