Hydrodynamics of quantum information from random circuits

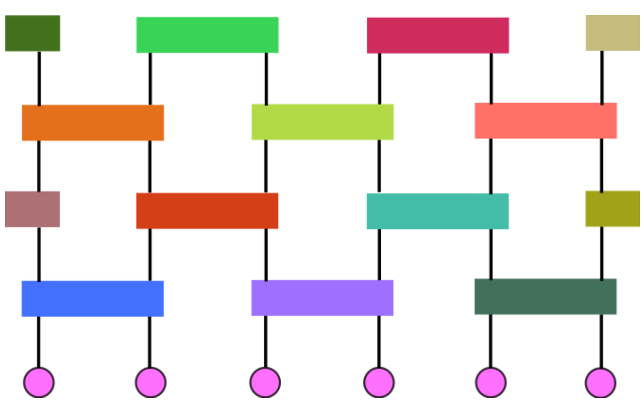
Tibor Rakovszky

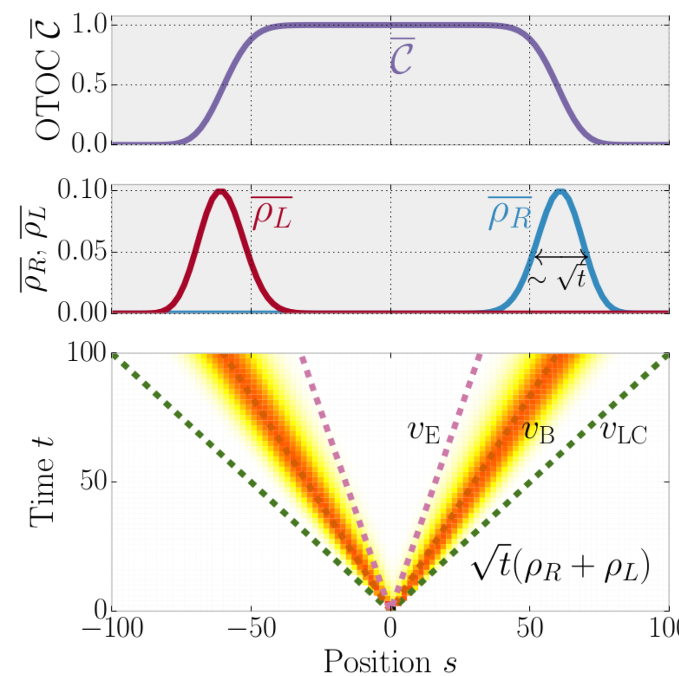
Fakultät für Physik, Technische Universität München

TECHNISCHE UNIVERSITÄT MÜNCHEN

Collaboration with C.V. Keyserlingk, F. Pollmann and S. L. Sondhi

Based on: ArXiv 1705.08910 (to appear in PRX) and 1710.09827





Benasque, 22.02.18

Outline

- **1. Motivation: Scrambling and chaos**

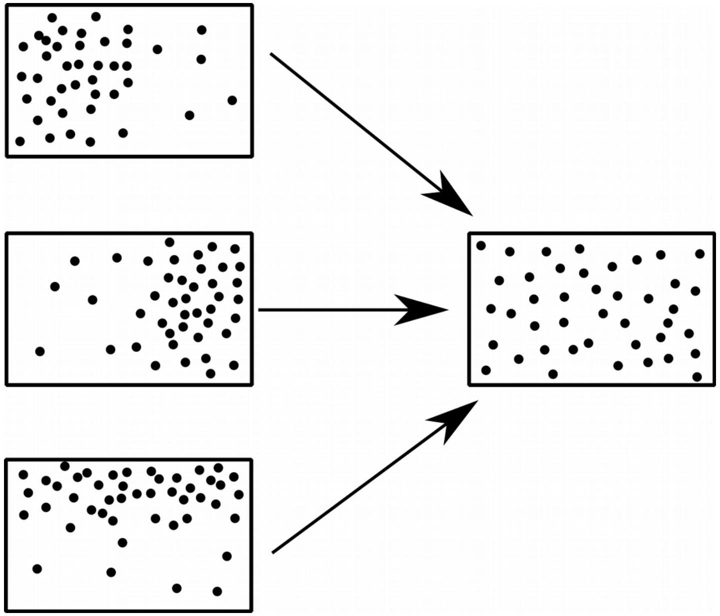
2. Operator hydrodynamics in random unitary circuits

Behavior of out-of-time-ordered correlators

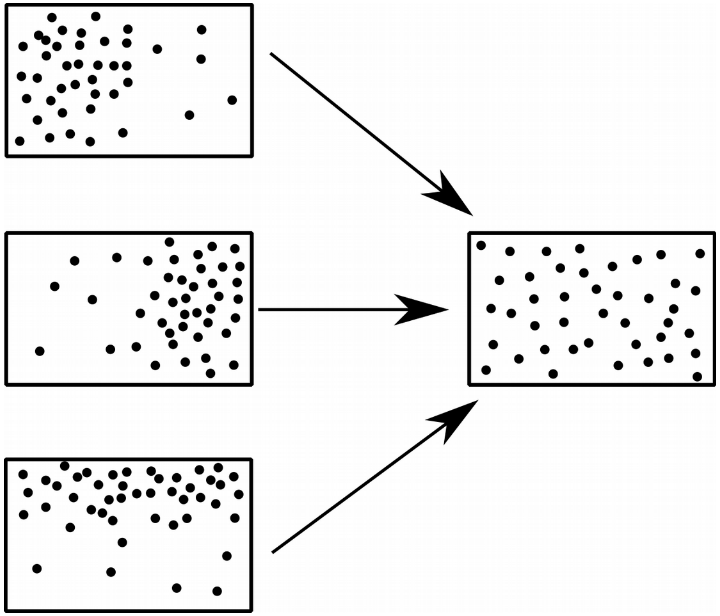
Entanglement growth

3. Coupling to a conserved charge

Thermalization: information of initial state is lost



Thermalization: information of initial state is lost locally



$$|\psi\rangle \to \rho^{(1)}(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$|\phi\rangle \to \rho^{(2)}(t) = |\phi(t)\rangle\langle\phi(t)|$$

$$\text{tr}|\rho_A^{(1)} - \rho_A^{(2)}| \approx 0$$

For all initial states Ψ, Φ and subsystem A

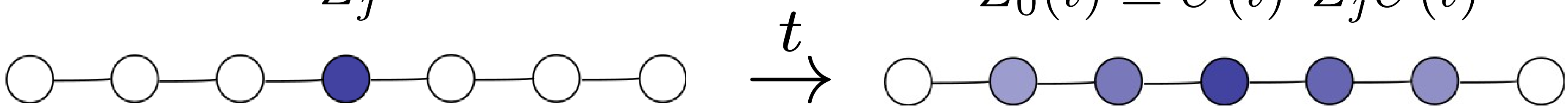
Lashkari et. al. JHEP (2013)

"Scrambling" of information → Requires signaling between subsystems

We can quantify scrambling via operator spreading

Spin ½ chain:

Z_j $\quad\quad$ $Z_0(t) \equiv \hat{U}(t)^\dagger Z_j \hat{U}(t)$



Pauli strings: $\sigma^{\vec{\mu}} = \sigma_1^{\mu_1} \sigma_2^{\mu_2} \ldots \sigma_L^{\mu_L}$ $\quad\quad$ $\mu_i = 0, 1, 2, 3$

$\text{e. g. } Z_1 X_2 \mathbb{1}_3 Z_4 \ldots$

$$\boxed{Z_j(t) = \sum_{\vec{\nu}} c_{\vec{\nu}}(t) \sigma^{\vec{\nu}}} \quad\quad \sum_{\vec{\nu}} |c_{\vec{\nu}}(t)|^2 = 1$$

Operators grow and get scrambled (look random within lightcone)

How to diagnose?

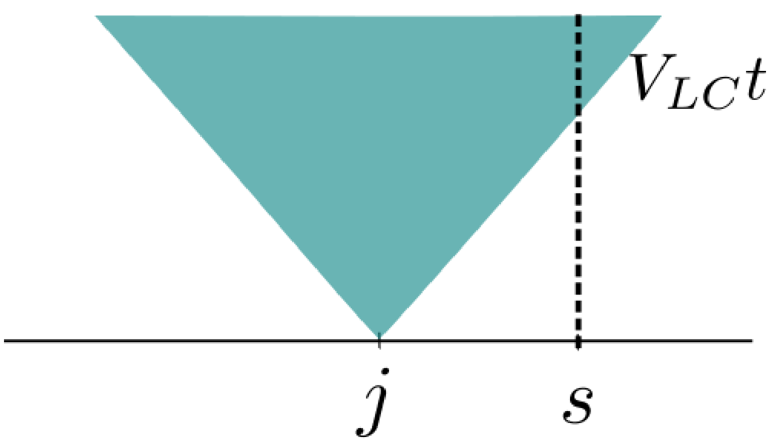
Motivation I: Out-of-time-ordered correlator measures the spreading of quantum information

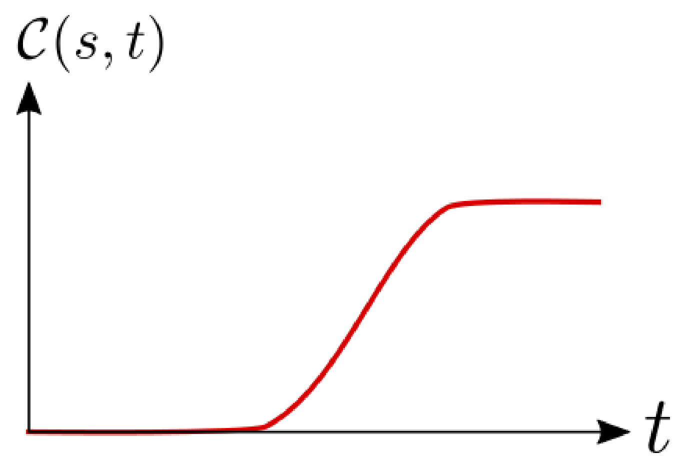
$$Z_j(t) = \sum_{\vec{\nu}} c_{\vec{\nu}}(t)\sigma^{\vec{\nu}}$$

Operators grow and get scrambled

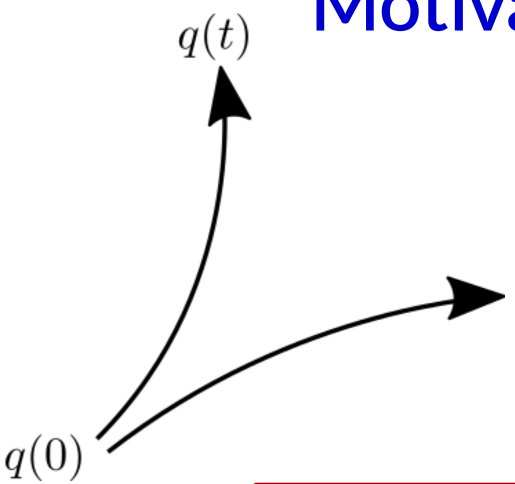
How to diagnose? → Out-of-time-ordered correlator (OTOC):

$$\mathcal{C}(s,t) = \frac{1}{2}\left\langle \left| [Z_j(t),\ Z_s] \right|^2 \right\rangle_{\beta=0} = \sum_{\substack{\vec{\nu} \\ [\sigma^{\vec{\nu}}, Z_s]\neq 0}} |c_{\vec{\nu}}(t)|^2$$





Motivation II: Many-body quantum chaos



Classical chaos: $\left(\frac{\partial q(t)}{\partial q(0)}\right)^2 = (\{q(t), p(0)\})^2 \propto e^{2\lambda_L t}$

λ measures how fast information spreads
(Kolmogorov-Sinai entropy)

$$\mathcal{C}(t) \equiv \left\langle \Big| [A(t),\, B] \Big|^2 \right\rangle_\beta = -\mathrm{Re}\,\langle A(t) B A(t) B \rangle_\beta + \ldots$$

$\mathcal{C} \propto e^{2\lambda_L t}$ in weakly coupled field theories, SYK model

Larkin, Ovchinnikov JETP 28 (1969); Maldacena et. al. JHEP (2015); Maldacena, Stanford PRD 94 (2016), etc.

What about local lattice systems?

- Exponential growth?
- Universal features?
- Relationship to entanglement growth?

Numerical studies, e.g.
A. Bohrdt et. al. NJP 19 (2017)
D. Luitz, Y. Bar Lev: PRB (2017)

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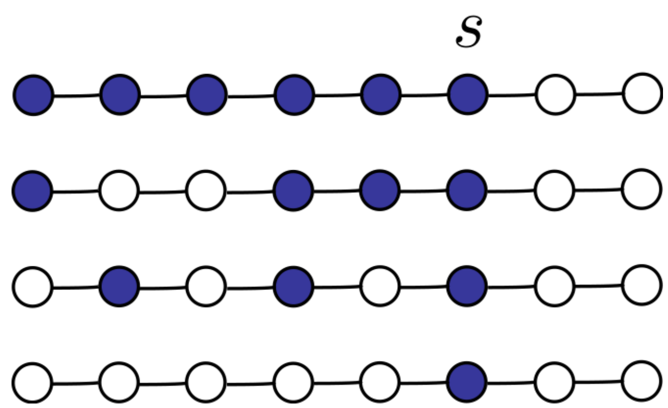
Operator spreading in 1D has a hidden conservation law

Local operator density (of right endpoints):

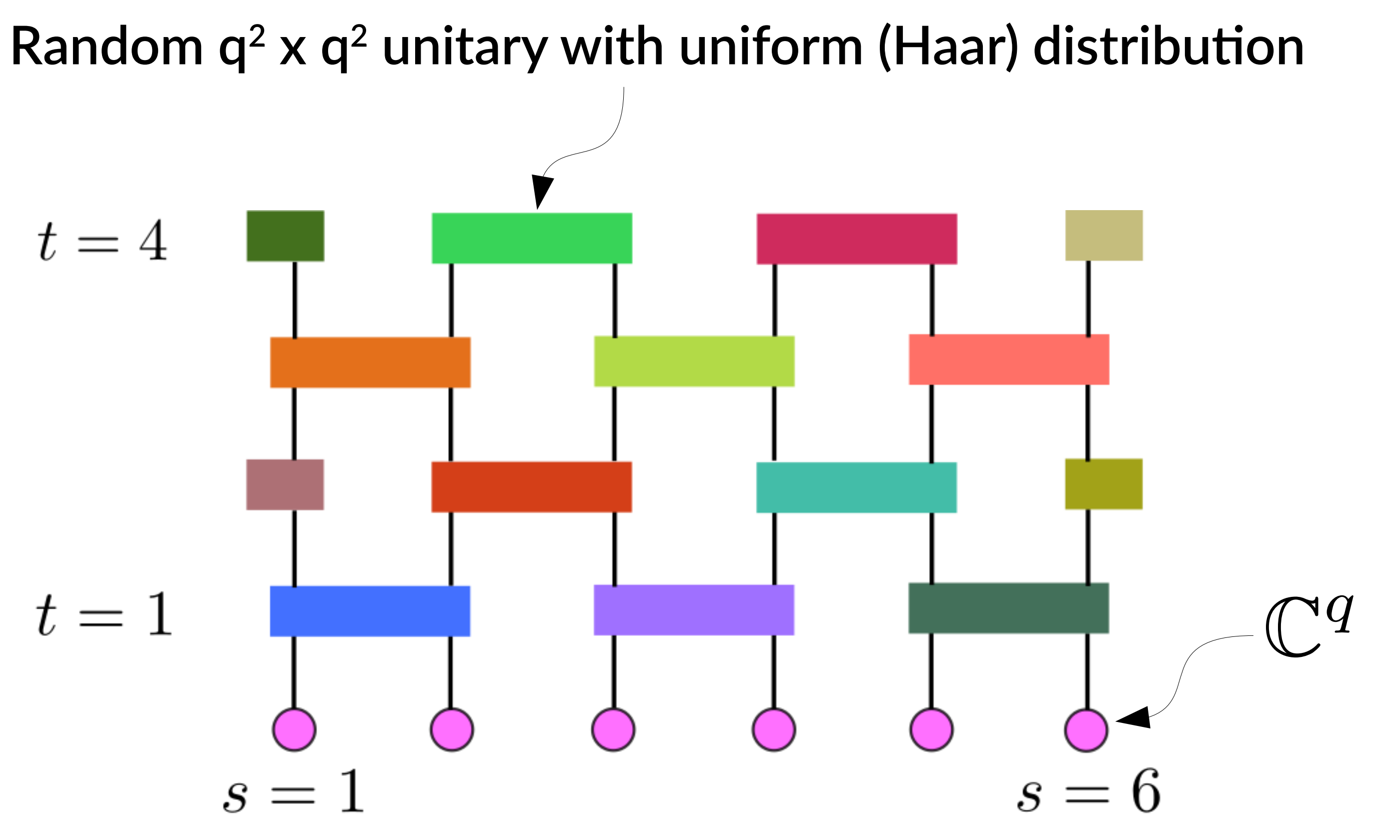
$$\rho_R(s,t) = \sum_{\vec{\nu}} |c_{\vec{\nu}}(t)|^2 \ \delta(\sigma_s^{\nu_s} \neq \mathbb{1} \text{ and } \sigma_{r>s}^{\nu_r} = \mathbb{1})$$



Conserved during time evolution: $$\sum_s \rho_R(s,t) = 1$$

Initial condition: $\rho_R(s,0) = \delta(s-j)$

Random unitary circuits are a toy model for chaotic systems



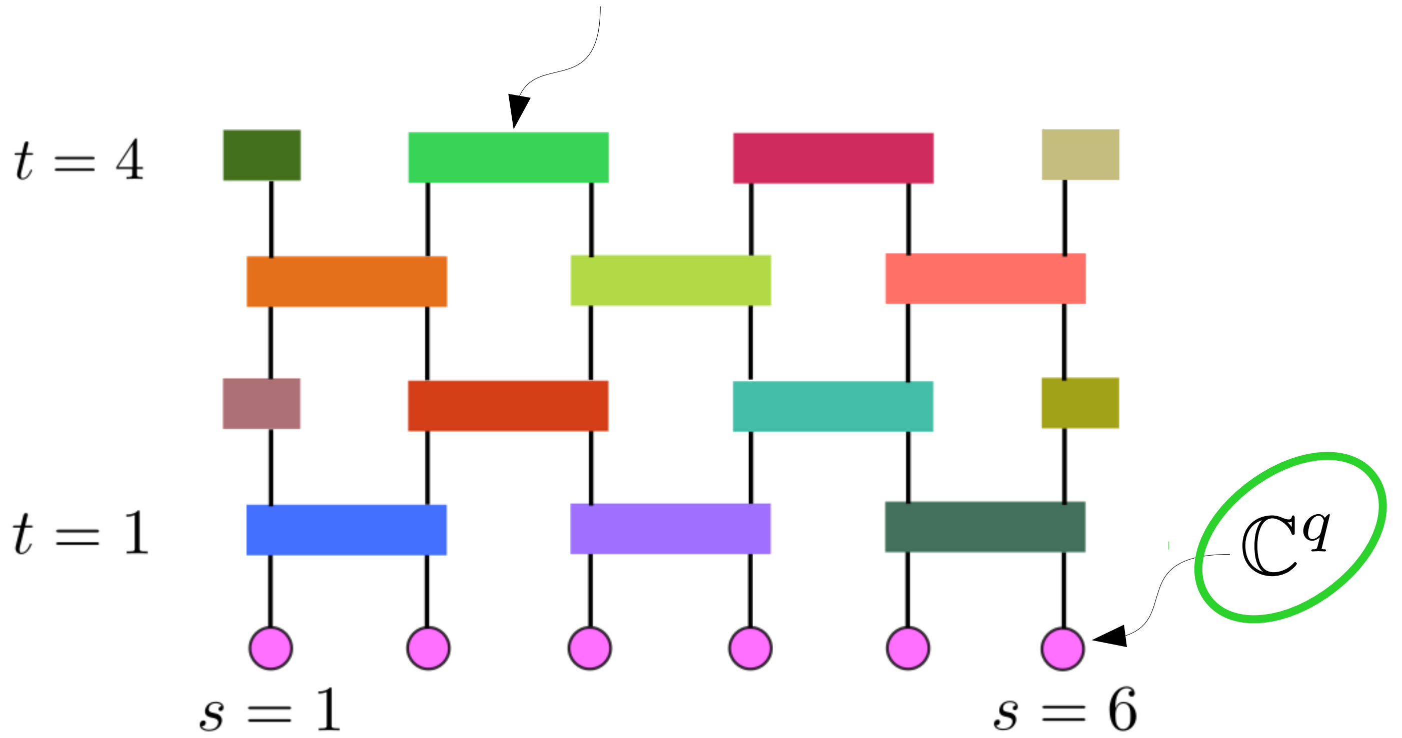
Light cone velocity: $v_{\mathrm{LC}} = \dfrac{\Delta s}{\Delta t} = 1$

After averaging: $\overline{c_{\vec{\nu}}} = 0 \qquad \overline{c_{\vec{\nu}} c_{\vec{\nu}'}} = \delta_{\vec{\nu}\vec{\nu}'} \overline{|c_{\vec{\nu}}|^2}$

No local conserved quantities, only constraint is locality + unitarity

Random unitary circuits are a toy model for chaotic systems

Random q^2 x q^2 unitary with uniform (Haar) distribution



Light cone velocity: $v_{\text{LC}} = \dfrac{\Delta s}{\Delta t} = 1$

After averaging: $\overline{c_{\vec{\nu}}} = 0 \qquad \overline{c_{\vec{\nu}} c_{\vec{\nu}'}} = \delta_{\vec{\nu}\vec{\nu}'} \; \overline{|c_{\vec{\nu}}|^2}$

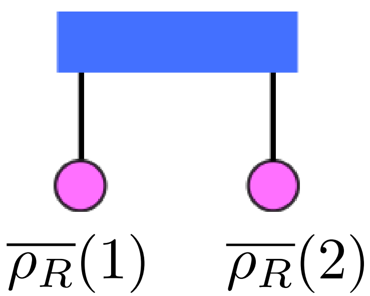
No local conserved quantities, only constraint is locality + unitarity

Average operator density obeys biased diffusion equation

Density of <u>right</u> endpoints:

$$\rho_R(s,t) = \sum_{\vec{\nu}} |c_{\vec{\nu}}(t)|^2 \;\; \delta(\sigma_s^{\nu_s} \neq \mathbb{1} \text{ and } \sigma_{r>s}^{\nu_r} = \mathbb{1})$$

Applying 2-site unitary:

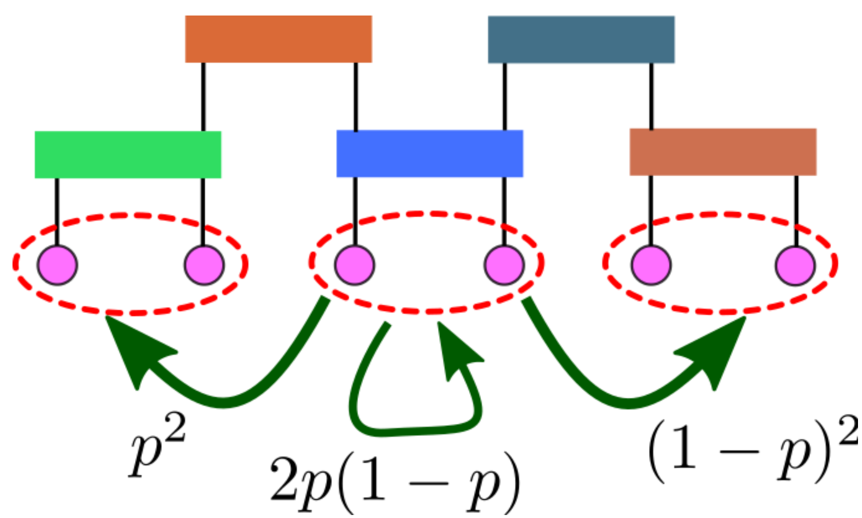


$$\overline{\rho_R}(1,t+1) = p\,[\overline{\rho_R}(1,t) + \overline{\rho_R}(2,t)]$$

$$\overline{\rho_R}(2,t+1) = (1-p)[\overline{\rho_R}(1,t) + \overline{\rho_R}(2,t)]$$

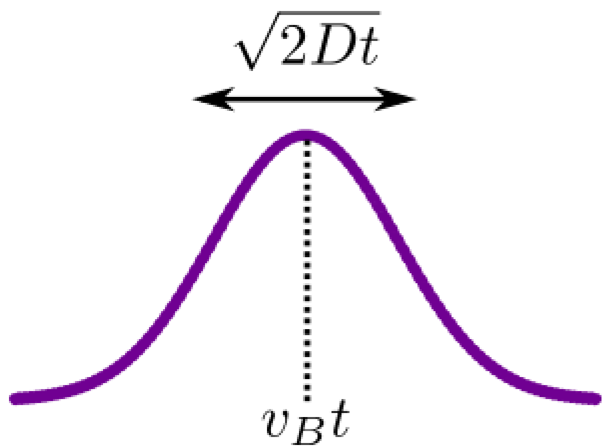
$$p = \frac{q^2-1}{q^4-1}$$

After two layers:



Biased diffusion determines the OTOC

$$\partial_t \rho_R = v_B \partial_x \rho_R + D_\rho \partial_x^2 \rho_R$$

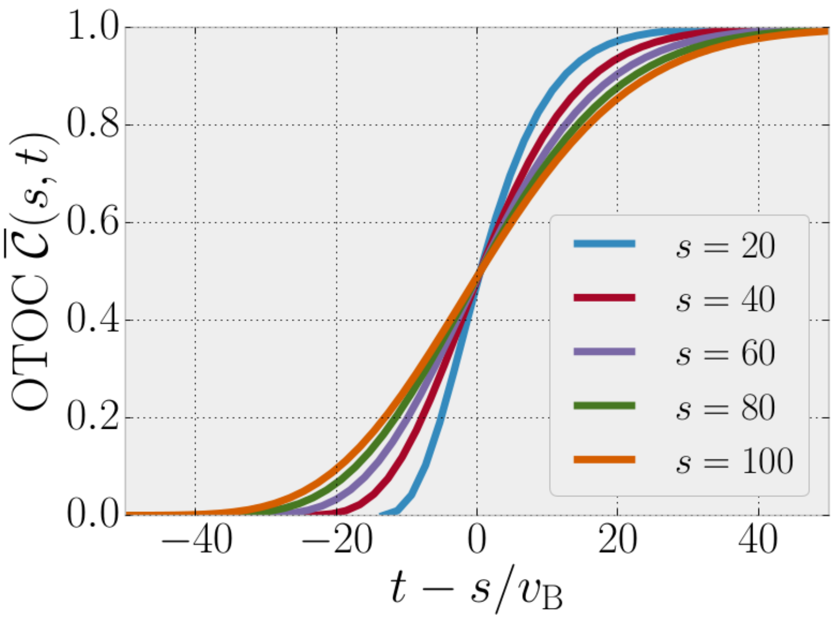


Drift (butterfly) velocity: $v_B = \frac{q^2 - 1}{q^2 + 1} < v_{LC}$

Diffusion constant: $D_\rho = \frac{q}{1 + q^2}$

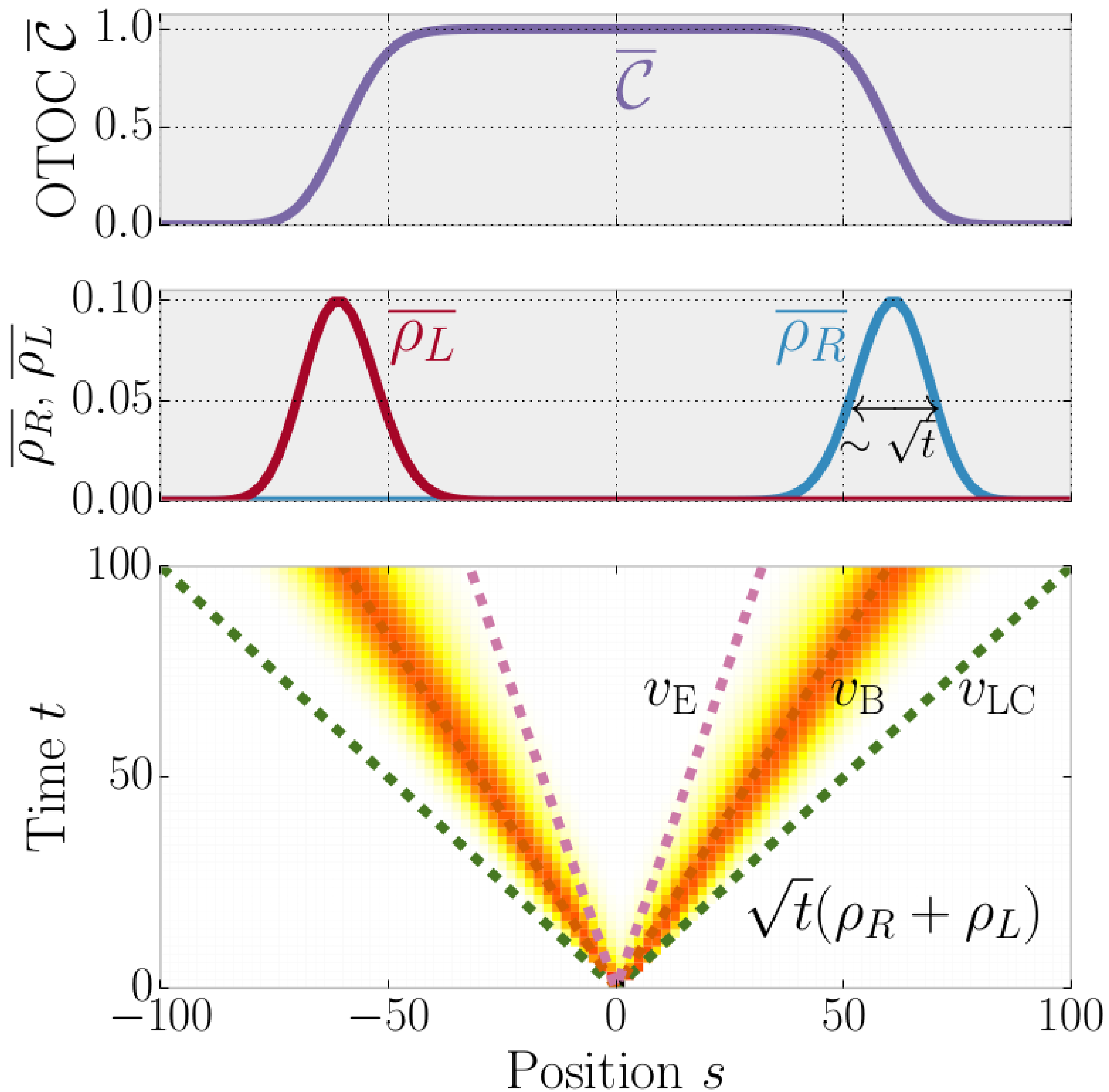
OTOC: $$\overline{\mathcal{C}}(s,t) \approx 1 - \sum_{r \leq s} \overline{\rho_R}(r,t)$$

$(+ \text{ terms exponentially small in } s,\ t)$



All operators are equally probable
Once we reach site s

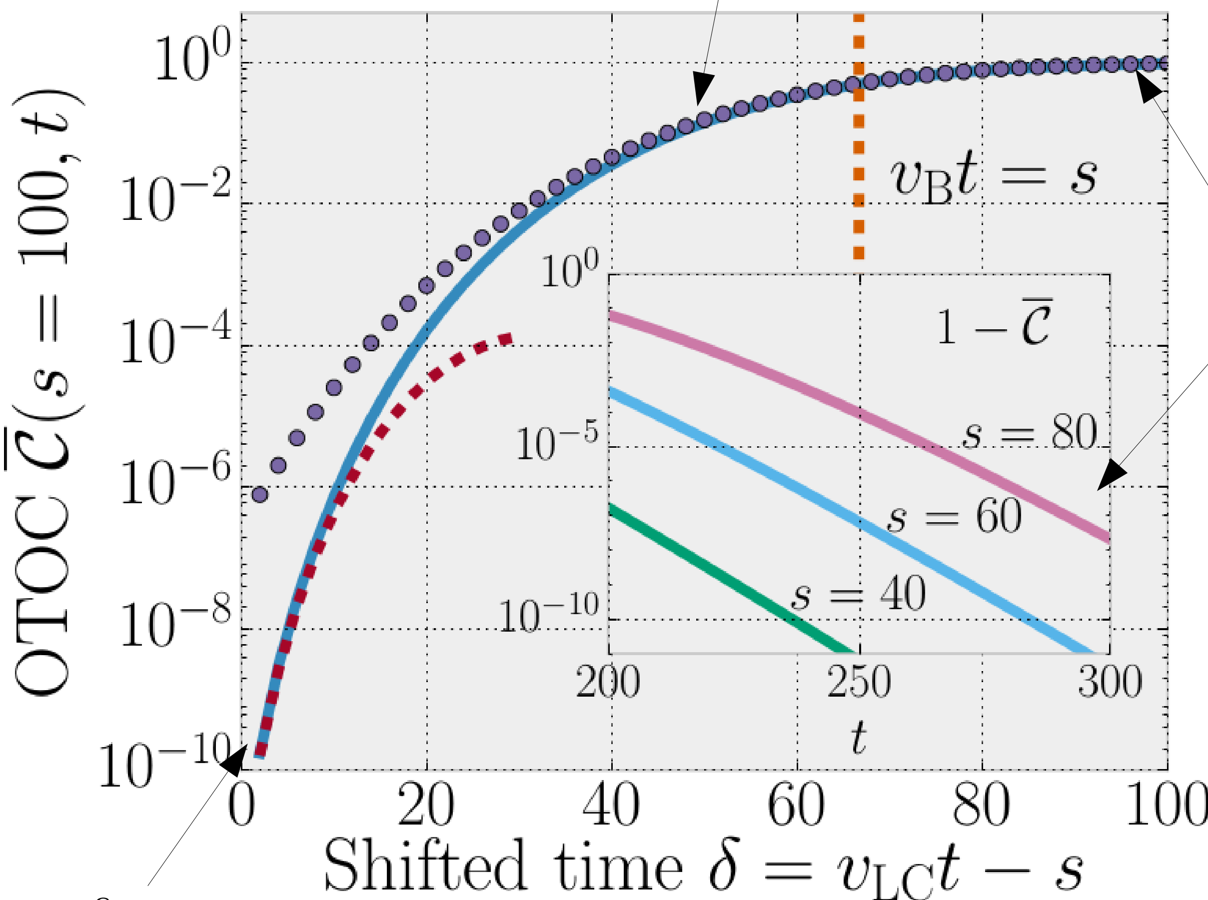
Operator spreading is described by biased diffusion



Out-of-time-order correlator has 3 distinct regimes

$$\mathcal{C}(s,t) = \frac{1}{2}\left\langle |[Z_j(t),\, Z_s]|^2 \right\rangle_{\beta=0}$$

$$\overline{\mathcal{C}} \approx \frac{1}{2}\mathrm{erfc}\left(\frac{s - v_B t}{\sqrt{2t(1-v_B^2)}}\right)$$



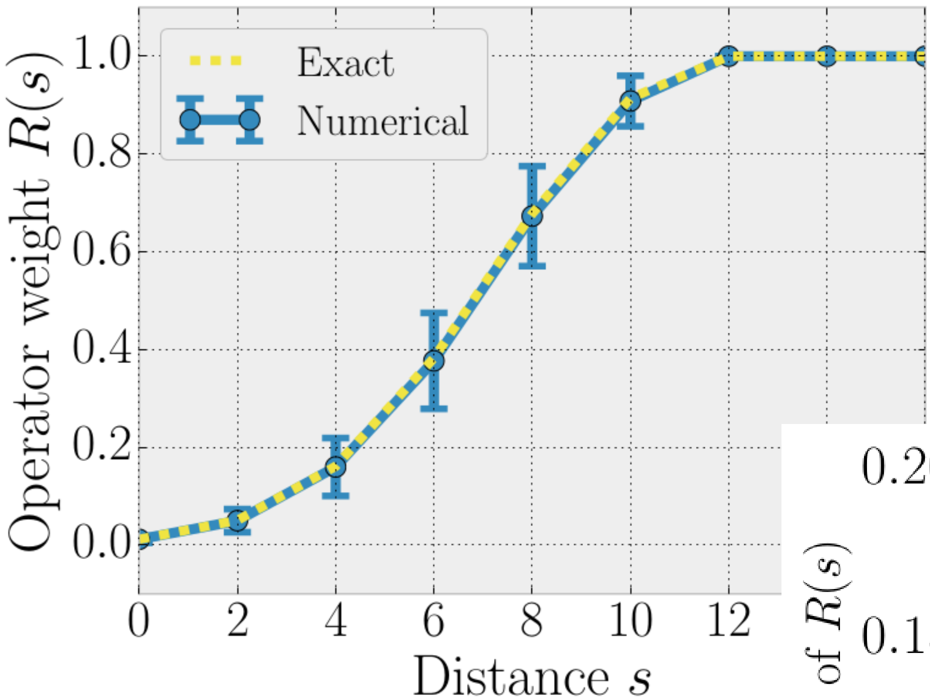
$$1 - \overline{\mathcal{C}} \sim \left(\frac{2q}{1+q^2}\right)^t$$

$$\overline{\mathcal{C}} \sim \left(\frac{q^2}{1+q^2}\right)^s e^{\frac{\delta}{2}\log\frac{\gamma s}{\delta}}$$

$$\gamma = e(1-v_B^2)/2$$

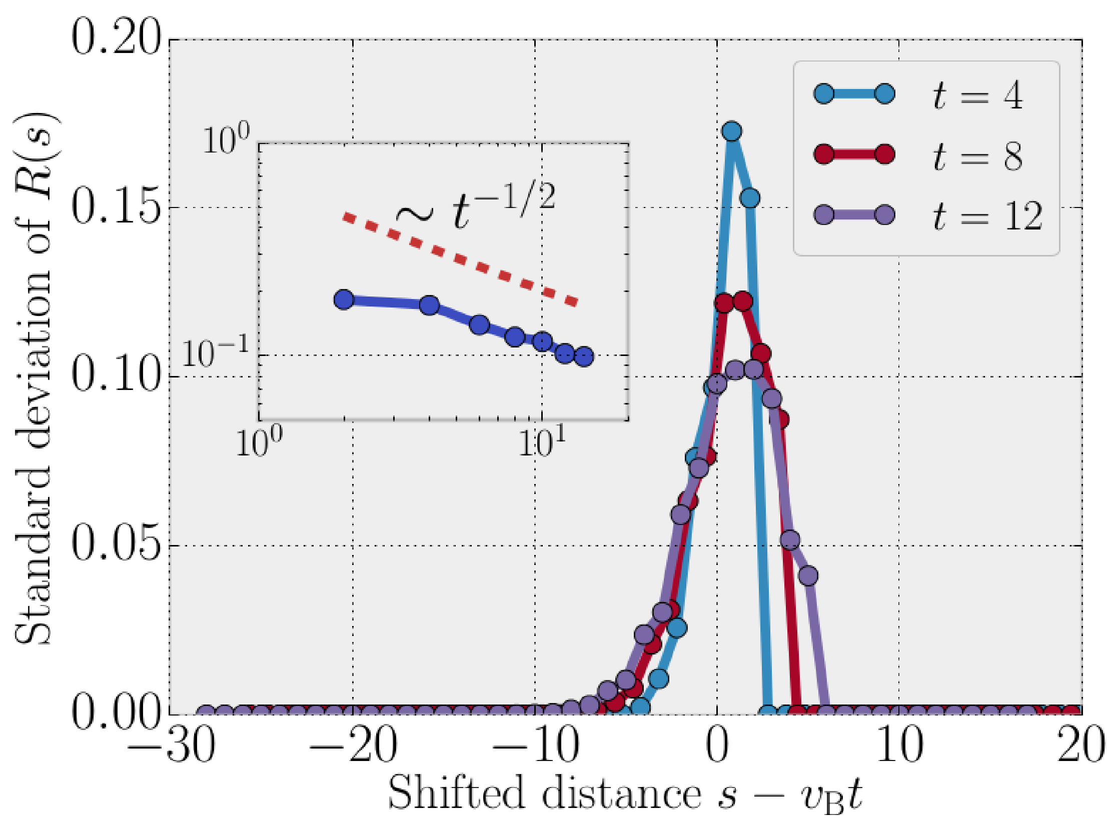
- Initial exponential increase, exponent depends on s
- Saturates exponentially

Fluctuations decrease algebraically in time



$$R(s) = \sum_{r \leq s} \rho_R(r) \approx 1 - \mathcal{C}(s)$$

Time evolving the matrix product operator using TEBD



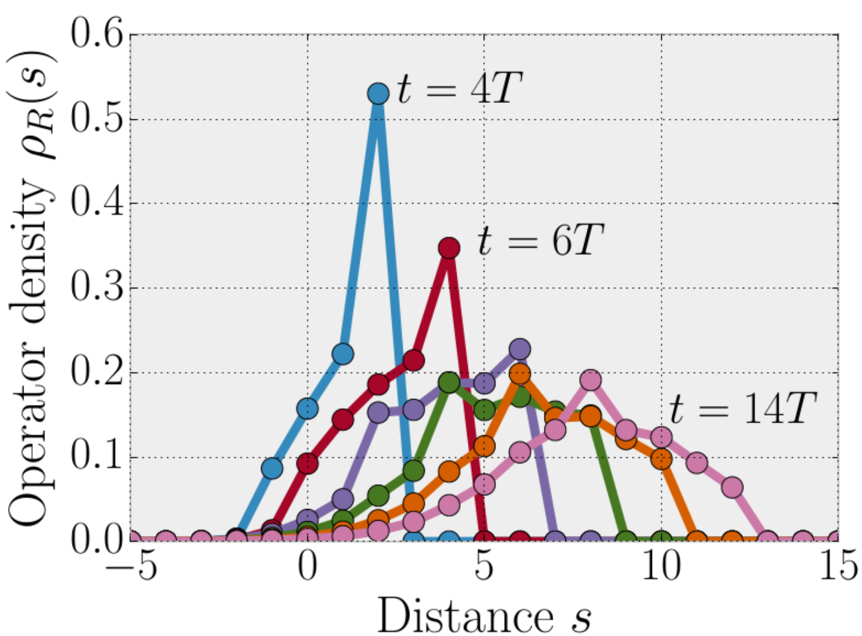
Diffusive broadening appears also in clean driven spin chain

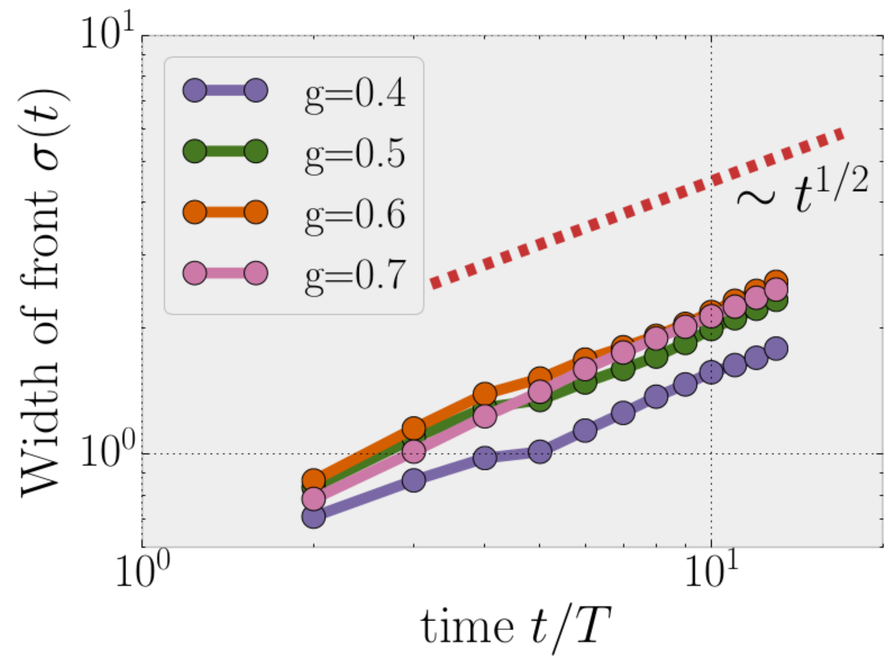
Kicked Ising model:

$$\hat{U} = e^{-i\frac{T}{2} g \sum_s X_s} e^{-i\frac{T}{2} \sum_s Z_s Z_{s+1} + h Z_s}$$

$$T = 0.8 \qquad h = 0.809$$

$$\sigma(t) = \sqrt{\sum_s s^2 \rho_R(s) - \left(\sum_s s \rho_R(s)\right)^2}$$





More recently: static tilted field Ising model; Leviatan et. al. ArXiv 1702.08894

Entanglement grows when an operator leaves the subsystem

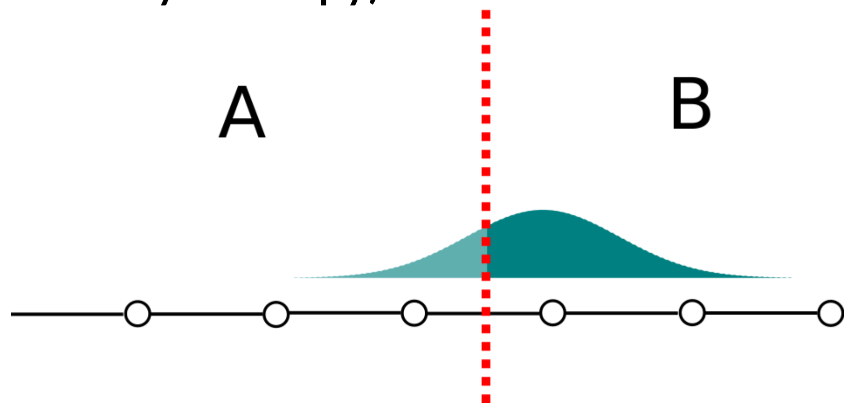
Start from 'ferromagnetic' product state:

$$\hat{\omega}(t=0) = |00\ldots0\rangle\langle 00\ldots0| = \frac{1}{q^L} \sum_{\vec{\mu}\in \text{Z-strings}} \sigma^{\vec{\mu}}$$

Initial right endpoint

$$\overline{e^{-S_A^{(2)}(t)}} \equiv \overline{\operatorname{tr}\hat{\omega}_A^2(t)} = \frac{1}{q^{L_A}} + \frac{q^2-1}{q^2} \sum_{s_0=1}^{L} \sum_{s=1}^{L_A} \frac{\overline{\rho_R}(s,t;s_0)}{q^{L_A-s_0}}$$

(Purity / exponentiated 2$^{\text{nd}}$ Rényi entropy)



Entanglement:
- only sensitive to operator growth (not scrambling)
- average behavior of many operators

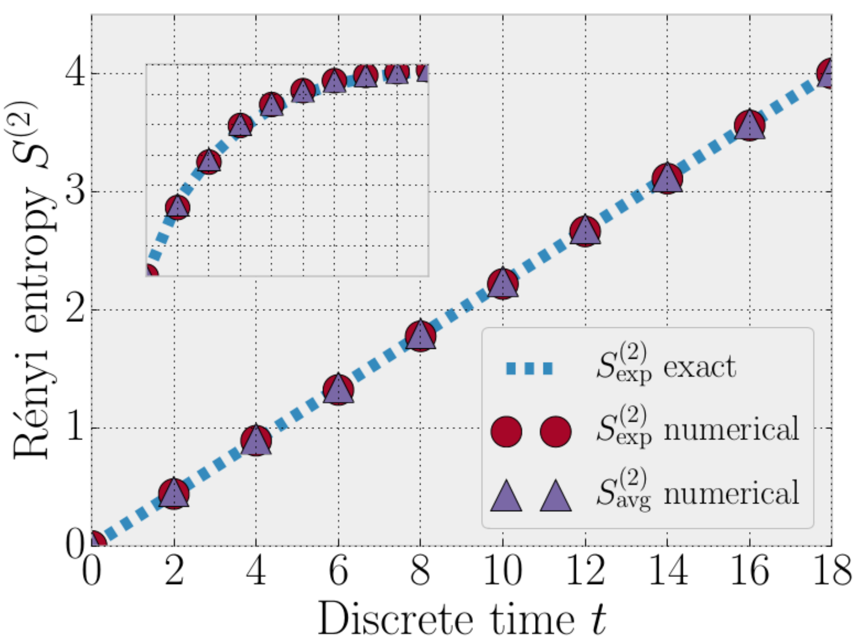
Diffusion leads to slower entanglement growth

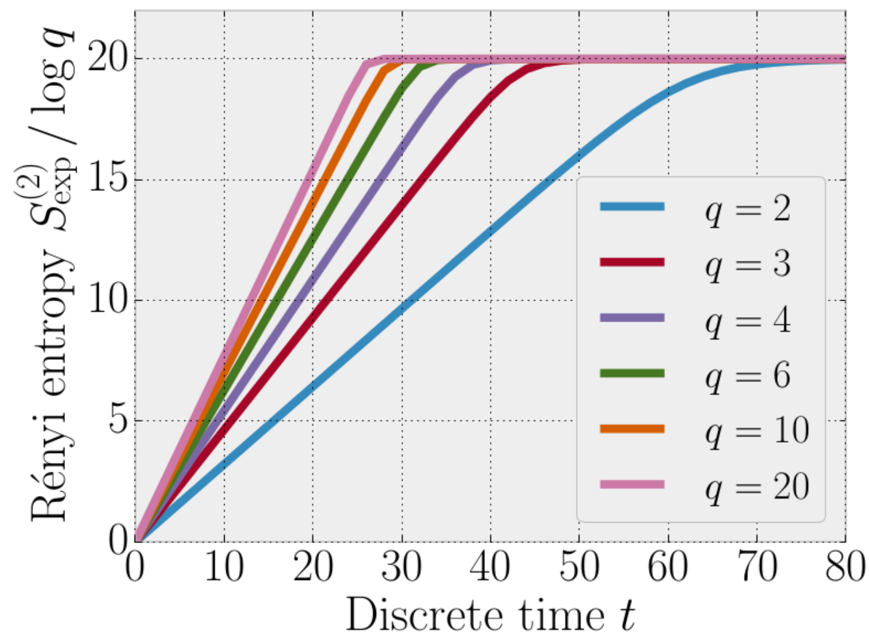
$$\text{if } t < L_A \;\rightarrow\; \overline{e^{-S_A^{(2)}(t)}} \approx \left(\frac{2q}{1+q^2}\right)^t \equiv q^{-v_E t}$$

Entanglement velocity:

$$v_E = 1 - \frac{\log 2}{\log q} + \frac{\log\left(1+q^{-2}\right)}{\log q} < v_B$$

$$S^{(2)}_{\text{avg}} \equiv \overline{S^{(2)}} \qquad S^{(2)}_{\text{exp}} \equiv -\log \overline{e^{-S^{(2)}}}$$





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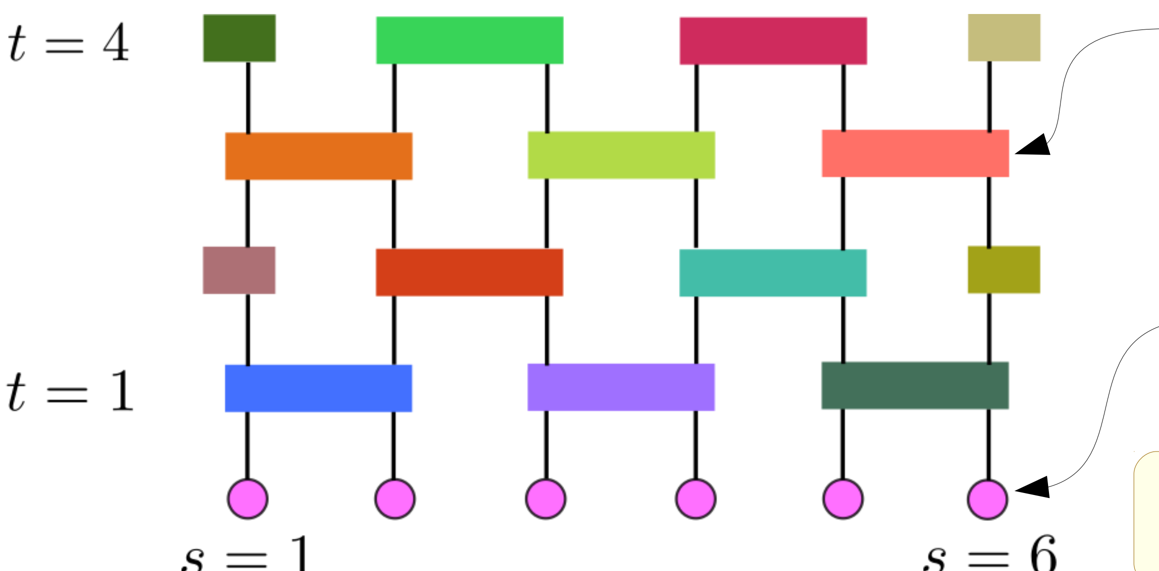
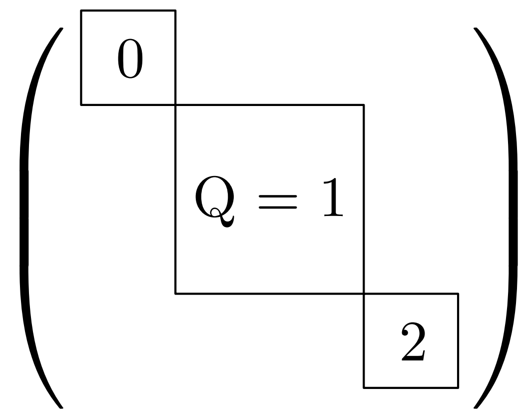
Hydrodynamic approach: conserved quantities are essential

Random circuit:
- Only conserved quantity is ρ_R
- Within lightcone all operators are equally probable
- OTOC measures probability of having reached site s

Most systems have more structure: conserved energy, charge etc.

→ Consider modified circuit with conserved charge Q





$$\mathbb{C}^2 \sim |0\rangle,\ |1\rangle$$

Local op. basis: $\mathbb{1},\ Z,\ \sigma^+,\ \sigma^-$

Charge diffusion leads to slow relaxation for the OTOC

Gate on sites r, r+1: $\overline{\hat{Q}_r}(t+1) = \frac{1}{2}(\hat{Q}_r + \hat{Q}_{r+1}) \rightarrow$

$$\partial_t Q = D_Q \partial_x^2 Q$$

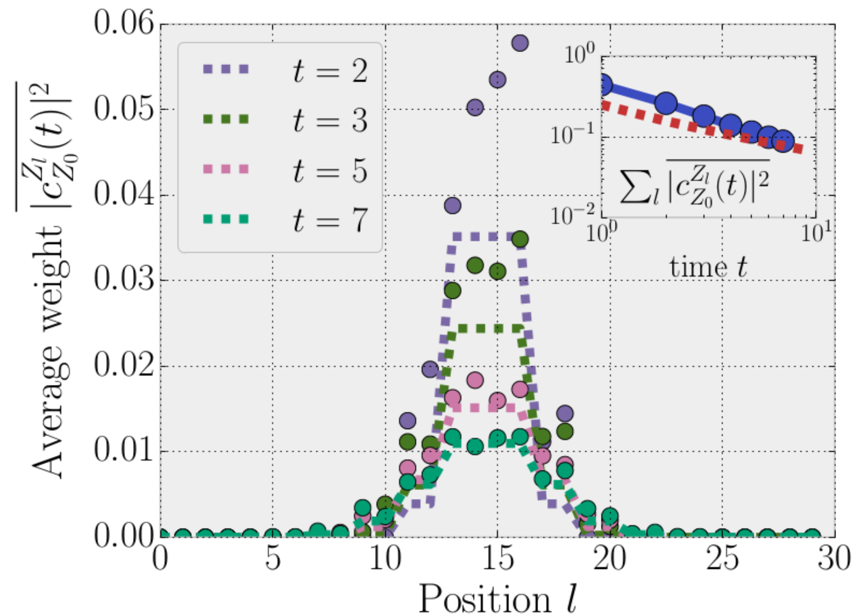
$$\overline{|c_{Z_0 \to Z_l}|} \sim 1/\sqrt{t}$$

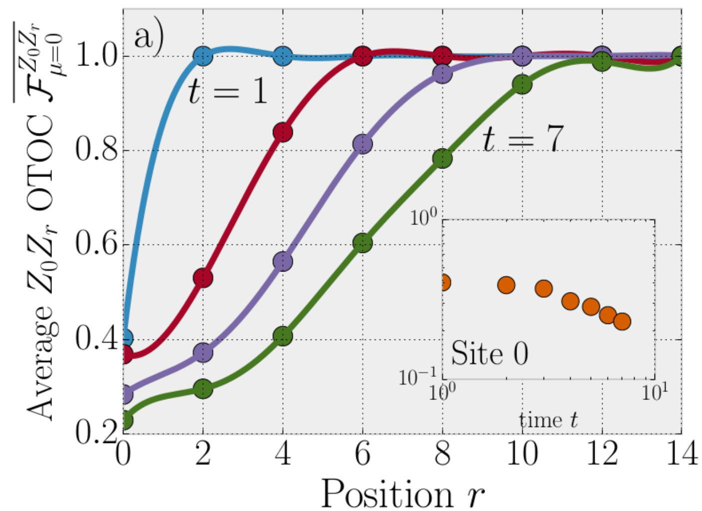
$$\sum_l \overline{|c_{Z_0 \to Z_l}|^2} > \sum_l \overline{|c_{Z_0 \to Z_l}|}^2 \sim \frac{1}{\sqrt{t}}$$

OTOC: $\mathcal{F} \sim 1 - \mathcal{C} \sim \frac{1}{\sqrt{t}}$

$$\mathcal{F}_\mu^{VW}(t) \equiv \mathrm{Re}\langle V^\dagger(t) W^\dagger V(t) W \rangle_\mu$$

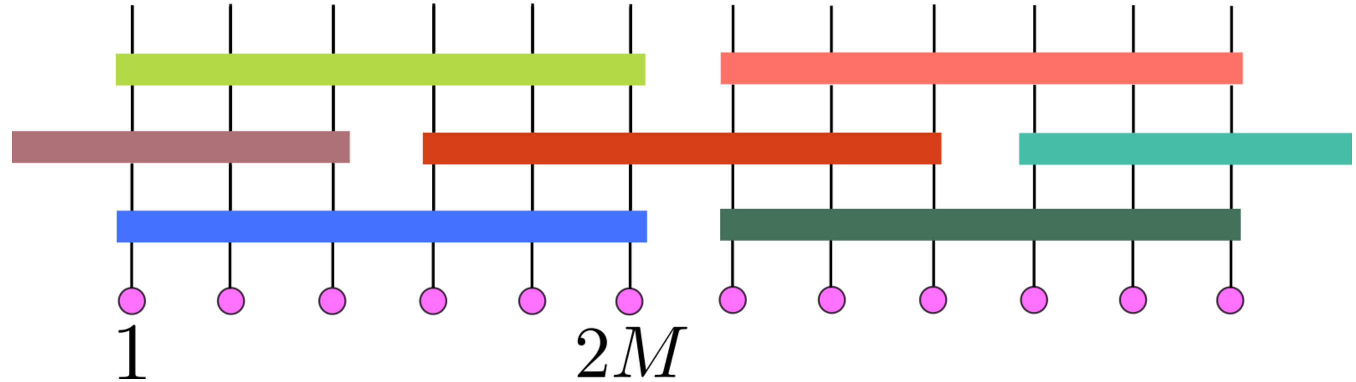
$$\langle O \rangle_\mu \equiv \mathrm{tr}(O e^{-\mu \sum_l \hat{Q}_l}) / \mathrm{tr}(e^{-\mu \sum_l \hat{Q}_l})$$



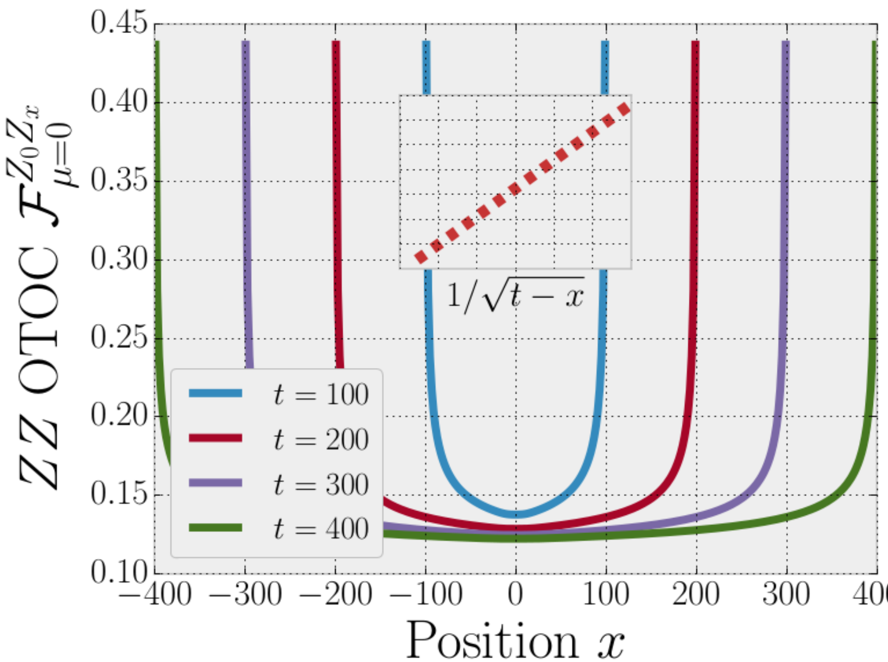


The OTOC develops a power law tail behind the front

Coarse-graining:



$$\mathcal{F} \sim \frac{1}{\sqrt{t-x}}$$



Physical picture: in each step there is a conversion from "conserved" to "non-conserved" Pauli strings

$$\rho_R(s,t) = \overbrace{\rho_R^{\mathrm{c}}(s,t)}^{|c_{Z_0\to Z_s}(t)|^2} + \rho_R^{\mathrm{nc}}(s,t)$$

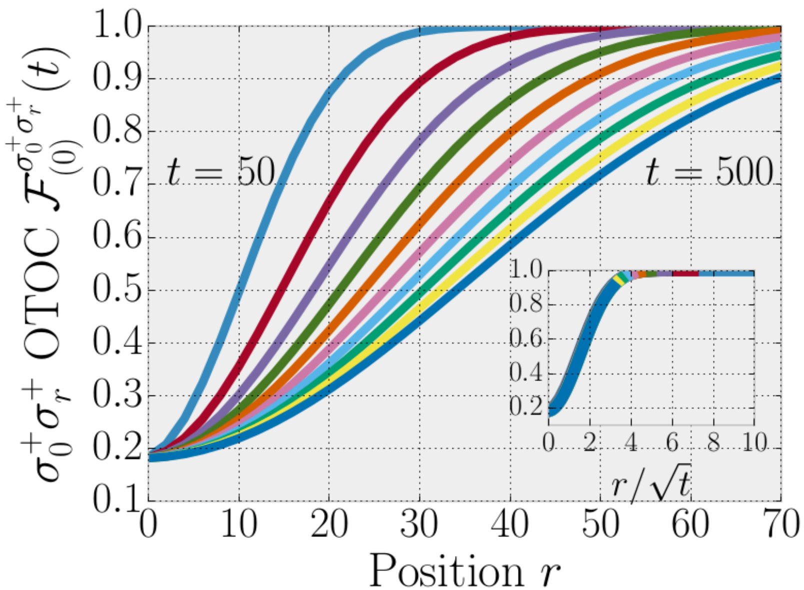
$$\partial_t \rho_R = v_B \partial_x \rho_R^{\mathrm{nc}} + D_\rho \partial_x^2 \rho_R^{\mathrm{nc}} + D_Q \partial_x^2 \rho_R^{\mathrm{c}}$$

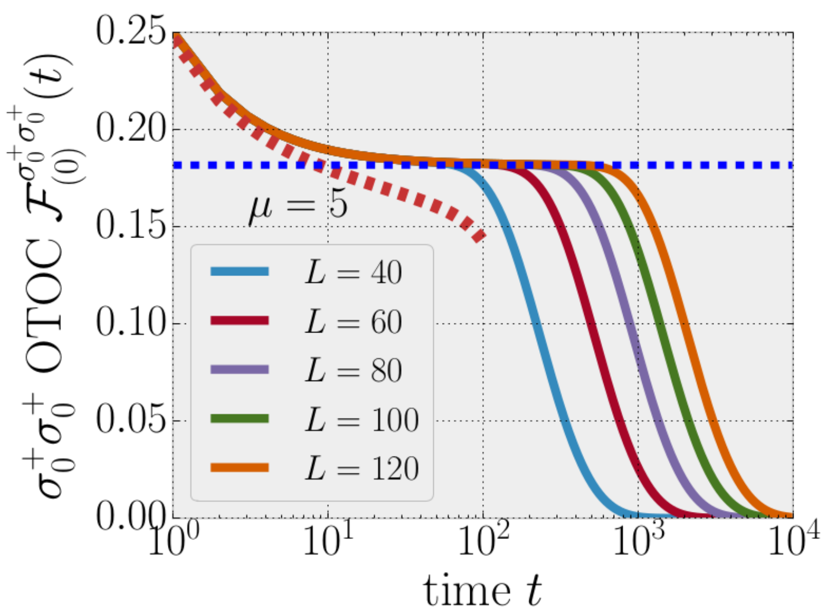
See: Khemani et. al. ArXiv 1710.09835

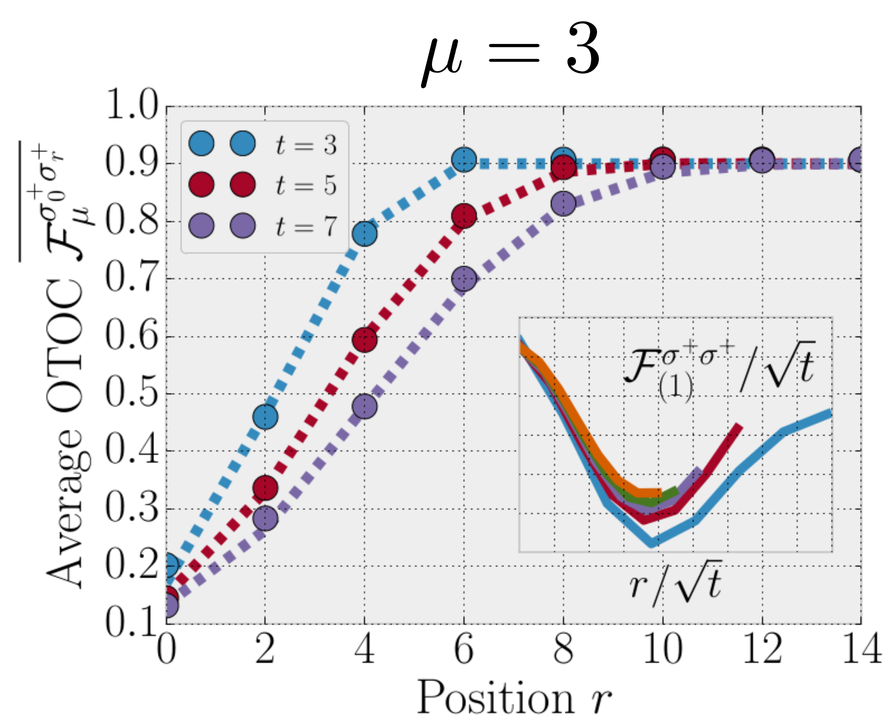
At low filling, the ballistic front can only develop at long times

$$\mu \gg 1 \longrightarrow \mathcal{F}_{\mu}^{VW}(t) \equiv \mathrm{Re}\langle V^{\dagger}(t) W^{\dagger} V(t) W \rangle_{\mu} \approx \sum_{N} e^{-N\mu} \mathcal{F}_{(N)}^{VW}(t)$$

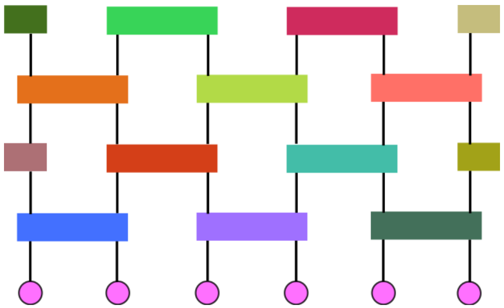
diffusion of $2N + Q_V + Q_W$ interacting particles



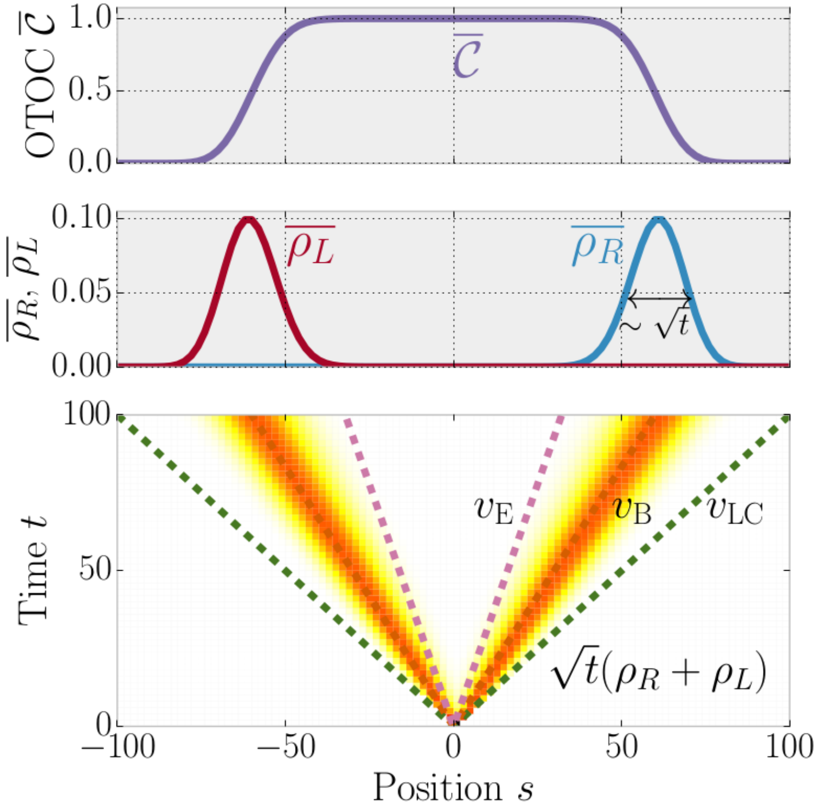




Conclusions, open questions



- Random circuits provide a toy model for chaotic dynamics
- One dimension → 'hidden' conserved density → obeys biased diffusion
- Diffusion → slower entanglement growth
- Conserved charge → hydrodynamic tails



Open questions:
- Other universality classes?
- Many-body localized phase?
- Comparison with field theories?

Details: ArXiv 1705.08910 and 1710.09827

Related work: Nahum et. al: PRX (2017); ArXiv 1705.08975 and 1705.10364
Khemani et. al: ArXiv 1710.09835

Acknowledgement: DPG FOR1807 Grant