Tree tensor network approximations to conformal field theories

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Benasque 2018

Ambitious goal

$$\text{tensor networks} \longleftarrow ? \longrightarrow \text{QFT}$$

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Many results for MPS, but few for layered networks, e.g. tree or MERA.

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Critical theories are interesting.

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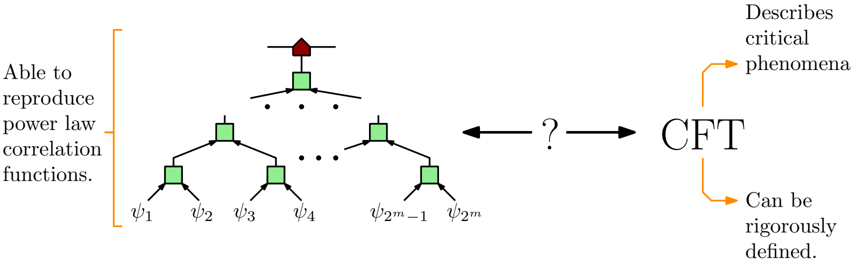
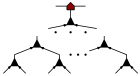
Should be rigorously defined.

Doable yet interesting goal

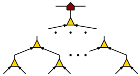


Outline

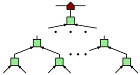
1. Build exact tree tensor network (TTN) for CFT correlation functions.



2. Truncate TTN to finite dimensional matrices and obtain error bounds.



3. Optimize TTN to get smaller bond dimensions.



CFT via vertex operator algebras

CFT can be defined via **vertex operator algebra**. Vertex operators $Y(\psi, z)$ give axiomatic footing for field operators and also normal ordered products and derivatives of field operators.

The space V of fields ψ is graded by L_0. Eigenvalues of L_0 are called **weights** or **energies**.

Useful relations

Vertex operators obey

$$q^{L_0} Y(\psi, z) q^{-L_0} = Y(q^{L_0}\psi, qz)$$

and

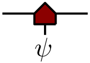
$$Y(\psi, x) Y(\phi, y) = Y(Y(\psi, x-y)\phi, y)$$

but can be **unbounded**

Scaled vertex operators

For a fixed space S, $\dim(S) < \infty$ **scaled vertex operator**

$$W_q(\psi, z) = q^{L_0/2} Y(q^{L_0/2}\psi, z) q^{L_0/2} \quad \equiv$$



is bounded

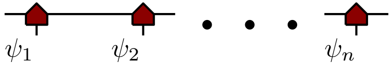
$$\|W_q(\psi, z)\| \leq \vartheta_S(q, z)\|\psi\| \ \forall \psi \in S \text{ and } 0 < q < \min\{|z|^2, 1/|z|^2\}.$$

Transfer operator

Calculation of correlation functions on equidistant lattice ($q = e^{-\text{spacing}}$) is equivalent to calculating expectation values of the **transfer operator**

$$T_l(\{\psi_i\}_{i=1}^n; z, q) = q^{lL_0} W_q(\psi_1, z) \circ W_q(\psi_2, z) \circ \ldots \circ W_q(\psi_n, z) q^{-lL_0}$$

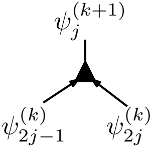
$$\simeq$$



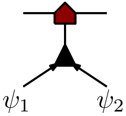
Conditions for constituent tensor

What we **want** for the constituent tensor $\epsilon_{z,q} : V \otimes V \to V$

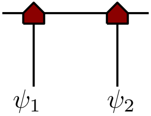
$$\epsilon_{z^{(k)},q^{(k)}}\left(\psi^{(k)}_{2j-1} \otimes \psi^{(k)}_{2j}\right) \quad \equiv$$



- Renormalized and initial fields should yield the same physics.



$$\simeq$$



$$\Updownarrow$$

$$W_{q^{(1)}}(\epsilon_{z,q}(\psi_1 \otimes \psi_2), z^{(1)}) \quad = \quad q^{\alpha L_0} W_q(\psi_1, z) \circ W_q(\psi_2, z) q^{-\alpha L_0}$$

- The map $\epsilon_{q,z}(\psi_1 \otimes \psi_2)$ should be bounded if ψ_1 and ψ_2 have only finite weight components in the weight decomposition (i.e. in eigenbasis of L_0).

The exact TTN

The map

$$\epsilon_{z,q} : \psi \otimes \phi \to W\left(\psi, (1-q)z\right) q^{L_0} \phi$$

and the map

$$z \to q^{5/2} z, \quad q \to q^2$$

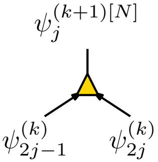
for parameters z and q such that
$0 < q < \min\left\{|(1-q)z|^2, \frac{1}{|(1-q)z|^2}\right\}$ does the job!

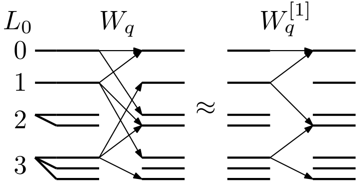
Uniqueness

- The TTN constituent tensor $\epsilon_{z,q}$ is not unique.
- There is at least a one-parameter family $\epsilon^{\alpha}_{z,q},\ \alpha > 1$

Truncation

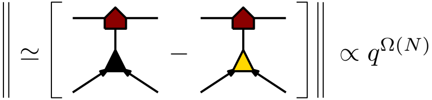
Truncated scaled vertex operator $W_q^{[N]}(\psi, z)$ does not change the weight of ψ by more than N. Using this object, we can introduce truncated TTN.





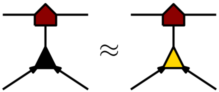
$$
\begin{array}{rcl}
\epsilon_{z,q}^{[N]} : \psi \otimes \phi & \rightarrow & W^{[N]}(\psi, (1-q)z)\, q^{L_0} \phi \\
z & \rightarrow & q^{5/2} z, \\
q & \rightarrow & q^2
\end{array}
$$

Truncation



If function $w \to \vartheta_w(q, z)$ grows sub-exponentially for z, q fixed, truncated tree tensor is a good approximation for a certain region of z, l and q. For WZW models this assumption holds with a great margin.

Truncation



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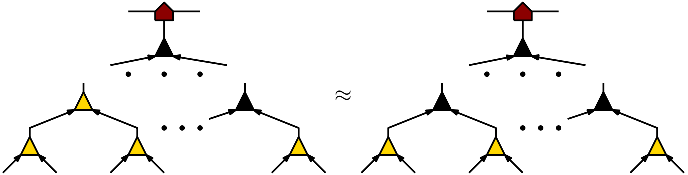
Truncating full tree

Intuition: Truncating just one tensor more does not change the TTN much!



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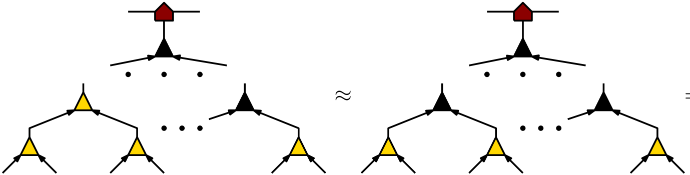
$A_i =$



$= A_{i+1}$

$$|A_0 - A_m\| \leq \sum_{i=1}^{m} \|A_{i-1} - A_i\|$$

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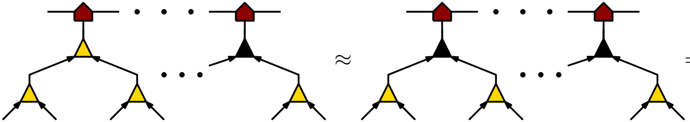

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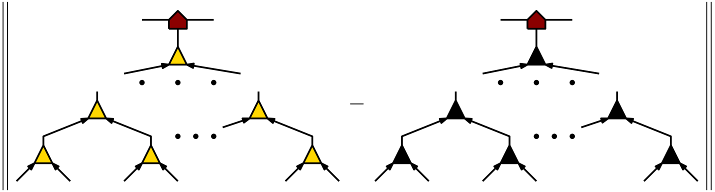
$$|A_0 - A_m\| \leq \sum_{i=1}^{m} \|A_{i-1} - A_i\|$$

$$\|CD\| \leq \|C\| \cdot \|D\|$$

Truncated tree approximates CFT correlation functions well



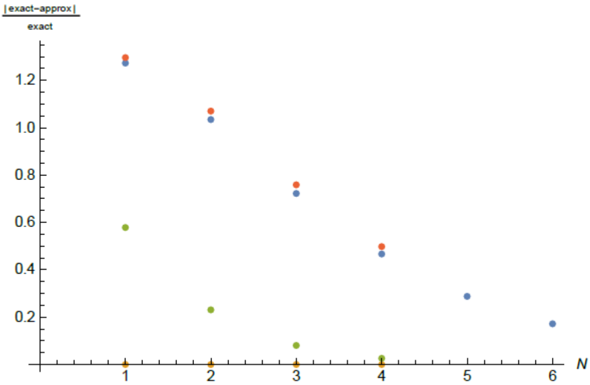
$\propto q^{\Omega(N)}$

Range of validity

Scaled vertex operators are bounded only for certain range of z and q.

For n initial fields this approximation holds on every level of the tree if $1 - \frac{1}{2n} < q < 1$ and "gauge" parameters z and l are chosen appropriately.

Example: free boson



$$\text{exact} = \frac{q^n}{(1-q^{n-1})^2 z^2}$$

$$\text{distance} = q^{n-1}$$

n

- 3
- 4
- 5
- 6

Idea for optimisation

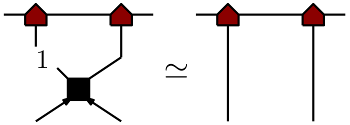
Observation: In our approximation scheme the resulting maximal weight of renormalized field depends only on the truncation parameter N and the right input - ϕ, but **not** on the left input ψ.

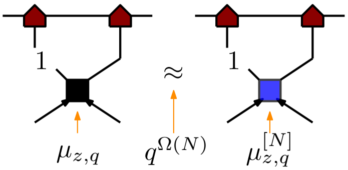
$$\epsilon_{z,q}^{[N]} : \psi \otimes \phi \to W^{[N]}(\psi, (1-q)z)\, q^{L_0} \phi$$

Idea: Add a new "super-disentangler" $\mu_{z,q} : V \otimes V \to \mathbb{C} \otimes V$ that rearranges the weights in such a way, that the in next layer of truncated tree tensors all the right inputs have weight 0.

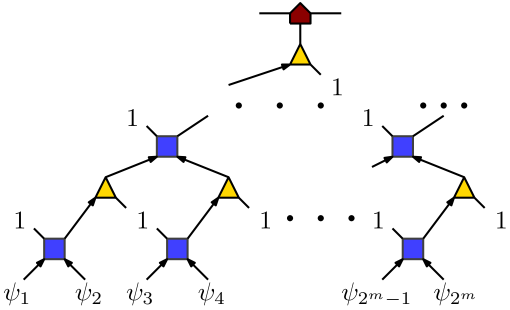
Properties of "super-disentangler"

Just like the tree tensor, the "super-disentangler" can be found by demanding the same physics on each level and boundedness. It can be truncated analogously to the tree tensor.





Efficient tree



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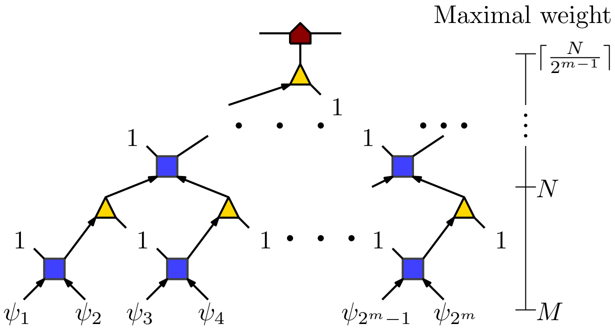

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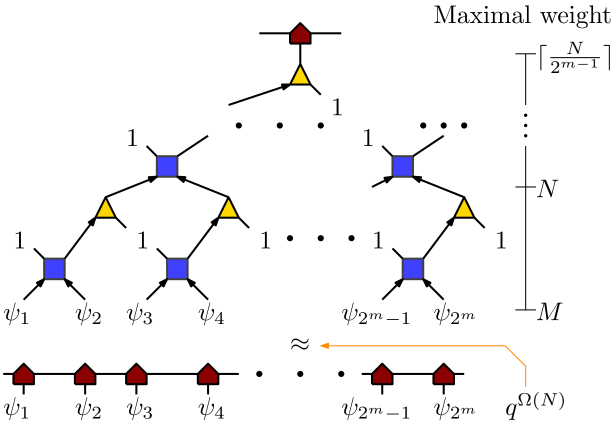

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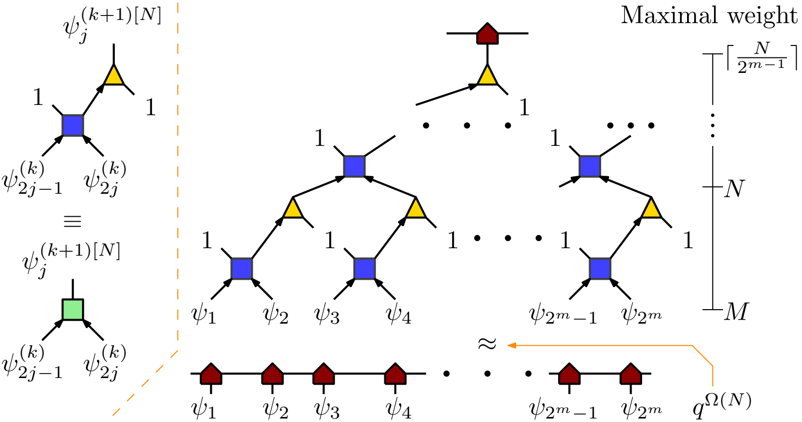

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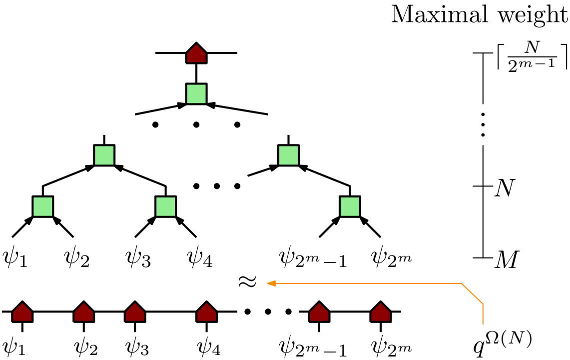

Theorem: efficiency of approximation

Let us denote by $d(w)$ the number of vectors of weight not more than w. We have proven that by using bond dimension not greater than D

$$D = d\left(\min\left\{\max\left\{\overset{\text{TTN}}{-\frac{N}{\log(q)}}, \overset{\text{TTN+MPS}}{nN}\right\}\right\}, \overset{\text{MPS}}{-\frac{nN}{a\log(q)}}\right)$$

one can approximate transfer operators of n insertions, and thus - n-point correlation functions of a given CFT with error $\leq e^{-bN} f(q,n)$ for a, b of order 1 if $w \to \vartheta_w(q,z)$ grows sub-exponentially (it is true for WZW).

Outlook

- We have constructed a tree tensor network that well approximates CFT correlation functions and (for WZW) converges to exact result with growth of bond dimension.
- We have established bounds on sufficient bond dimension.
- (Preliminary): entropy considerations may suggest that the construction is optimal.

- Question: what are the extra conditions for MERA?
- Question: Can we generalise the construction to other rigorously defined theories?