Generalization of the Haldane conjecture to SU(3) spin chains

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26 February, 2018

Haldane:

The SU(2) symmetric Heisenberg spin chain behaves differently for integer and half-integer spins

based on a mapping of the low energy degrees of freedom into a
1+1 dimensional O(3) nonlinear sigma model

The SU(3) symmetric Heisenberg spin chain behaves differently for p=3m and p=3m±1 spins in the fully symmetric representation

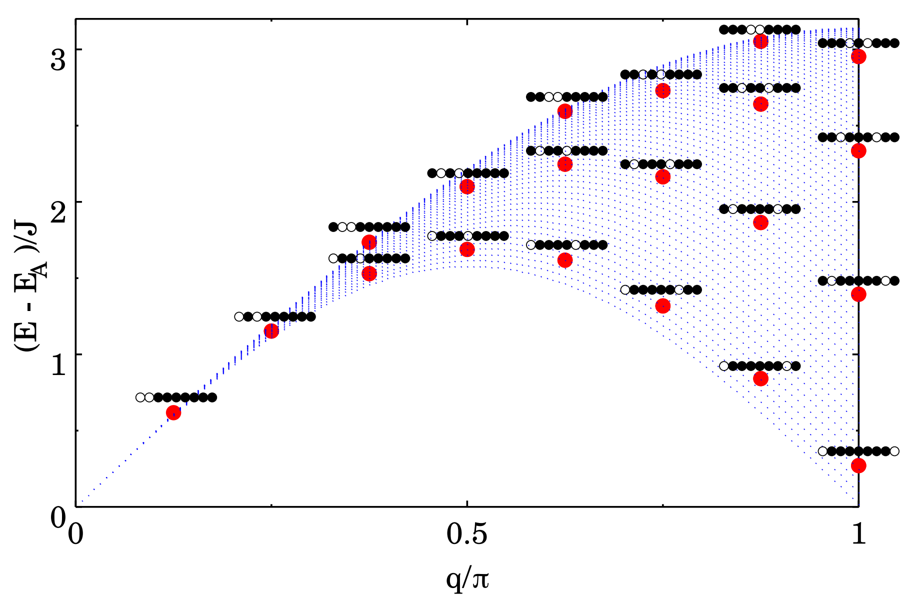
based on a mapping of the low energy degrees of freedom into a
1+1 dimensional SU(3)/(U(1) x U(1)) nonlinear sigma model

Half-integer spins

S=1/2

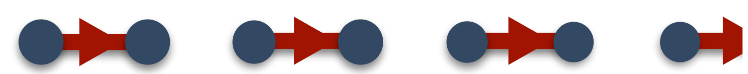
$$H = \sum_j J_1 \mathbf{S}_j \mathbf{S}_{j+1} + J_2 \mathbf{S}_j \mathbf{S}_{j+2}$$

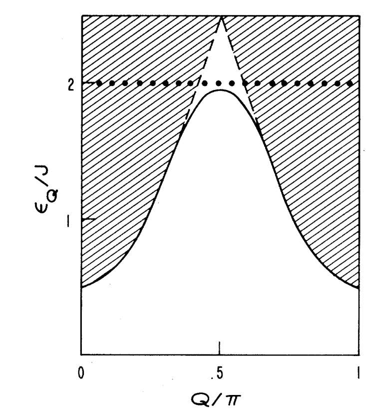
n.n Heisenberg chain
Bethe ansatz solvable,
gapless



J1-J2 Heisenberg chain dimerized,
gapped

$$H_{MG} = \sum_i \mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{2} \sum_i \mathbf{S}_i \mathbf{S}_{i+2}$$





gapless | 0.2411 | gapped and dimerized | J_2/J_1

Hulthen, Ark.Mat.Astron.FysikA,26 (11):106p., 1938.
Karbach et al., Comput. Phys. **12**, 565 (1998)

Majumdar Ghosh J. of Math. Phys., 10(8), 1399 (1969)
Shastry Sutherland, Phys. Rev. Lett., 47, 964 (1981)
Okamoto, Nomura, Phys. Lett. A **169**, 433 (1992)

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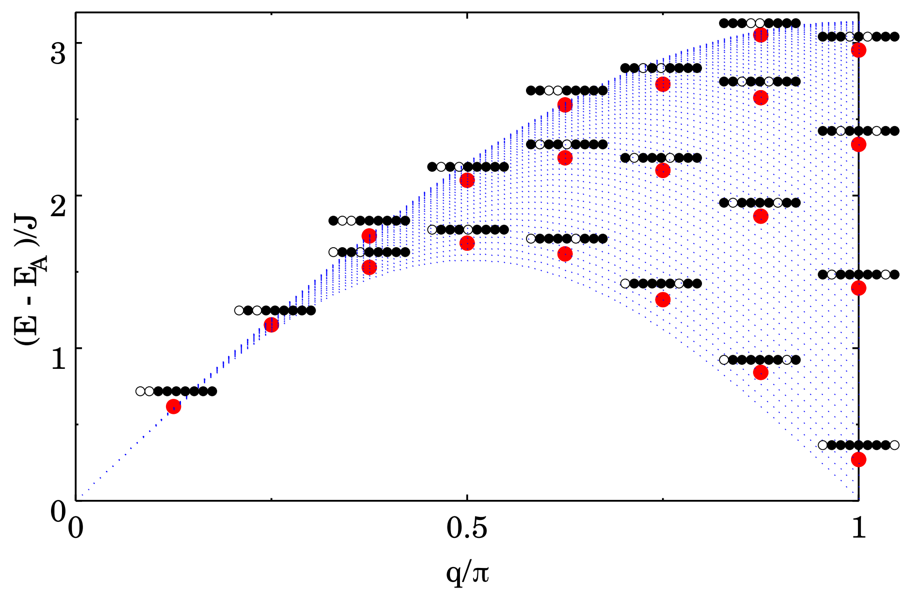

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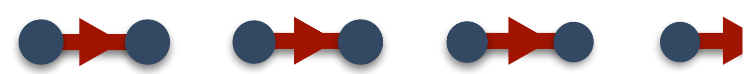

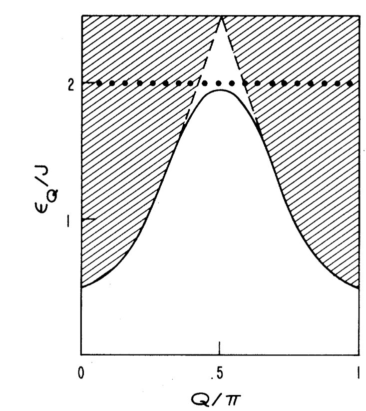

gaple

Lieb Schulz-Mattis-Affleck theorem:

half integer spin chain is either gapless or dimerized

dimerized

J_2/J_1

Lieb, Schultz, Mattis (1961)
Affleck, Lieb (1986)

Hulthen, Ark.Mat.Astron.FysikA,26 (11):106p., 1938.
Karbach et al., Comput. Phys. **12**, 565 (1998)

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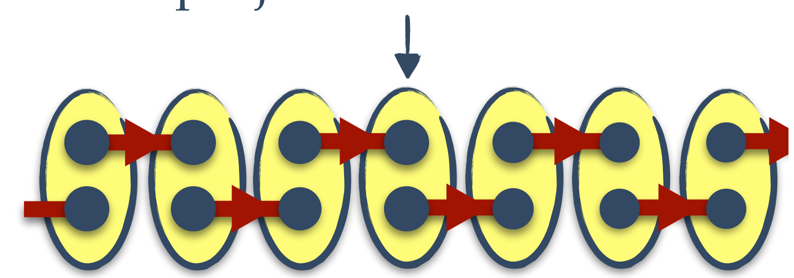
Integer spins

S=1

AKLT model

$$H_{\mathrm{AKLT}} = \sum_i \mathrm{Proj}\left(|\mathbf{S}_i + \mathbf{S}_{i+1}| = 2\right)$$
$$= \sum_i \frac{1}{3} + \frac{1}{2}\mathbf{S}_i\mathbf{S}_{i+1} + \frac{1}{6}\left(\mathbf{S}_i\mathbf{S}_{i+1}\right)^2$$

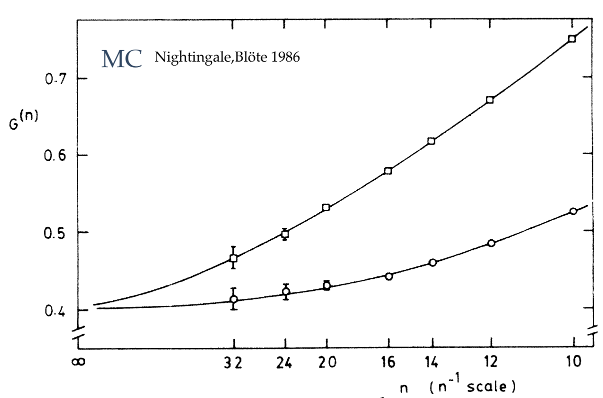
projection onto S=1

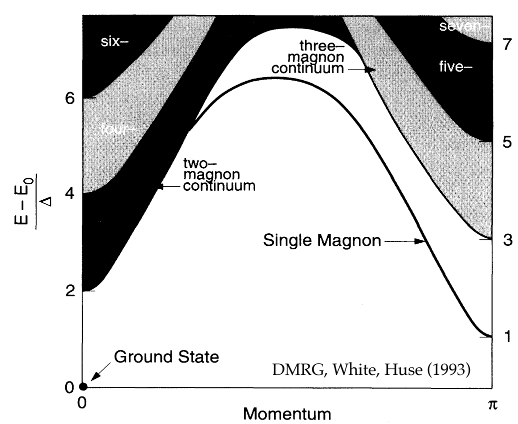


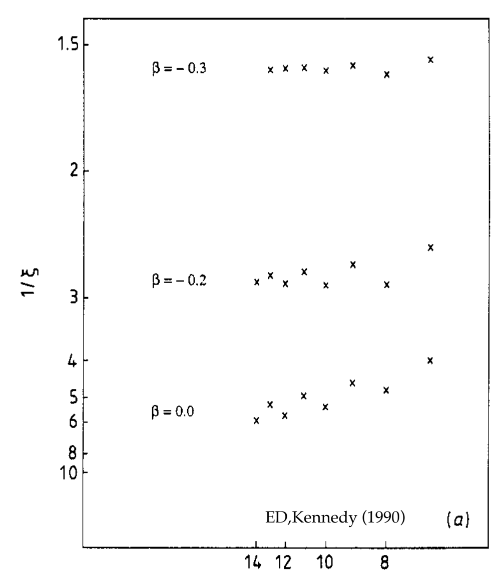
Unique gapped ground state

Affleck, Kennedy, Lieb, Tasaki (1988).

nearest neighbour Heisenberg model







Compound	Chemical formula	
	$CsNiCl_3$	Johnson (1979)
NENP	$Ni(C_2H_8N_2)_2NO_2ClO_4$	Meyer (1982)
NENF	$Ni(C_2H_8N_2)_2NO_2PF_6$	
NINO	$Ni(C_3H_{10}N_2)_2NO_2ClO_4$	Renald(1988)
NINAZ	$Ni(C_3H_{10}N_2)_2N_3ClO_4$	Renald(1990)
NDMAZ	$Ni(C_5H_{14}N_2)_2N_3ClO_4$	Yamashita(1995)
NDMAP	$Ni(C_5H_{14}N_2)_2N_3PF_6$	Momfort(1996)
TMNIN	$(CH_3)_4NNi(NO_2)_3$	Gadet(1991)
YBANO	Y_2BaNiO_5	Cheong (1992)
	$AgVP_2S_6$	Lee (1986)

Renald,et al. 2002

Path integral for SU(2) Heisenberg model

Haldane (1983)

$$\mathcal{Z} = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}\left(e^{-d\tau H} e^{-d\tau H} \dots e^{-d\tau H}\right)$$

inserting a complete set of states at every time step

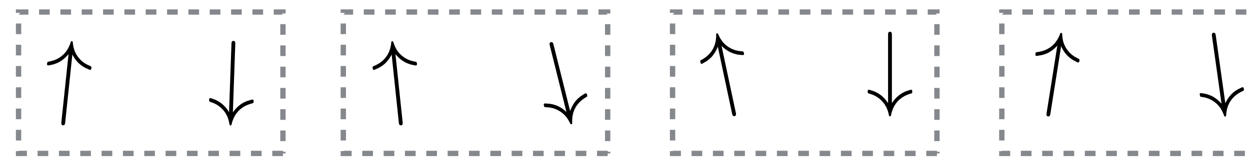
1D quantum model → 1+1D classical fields

$$\int \mathcal{D}[\mathbf{n}] \exp\left(-\int d\tau \left[JS^2 \sum_i \mathbf{n}_{i,\tau}\mathbf{n}_{i+1,\tau} + 2S\sum_i \mathbf{n}_{i,\tau}\partial_\tau \mathbf{n}_{i,\tau}\right]\right)$$

$$\int \mathcal{D}[\mathbf{n}] \exp\left(-\int d\tau \left[JS^2 \sum_i \mathbf{n}_{i,\tau}\mathbf{n}_{i+1,\tau} + 2S\sum_i \mathbf{n}_{i,\tau}\partial_\tau \mathbf{n}_{i,\tau}\right]\right)$$

classical energy   Berry phase (imaginary)

$$\mathbf{n}_{2j} = \frac{S}{\sqrt{1+\mathbf{l}^2}}(\ \ \mathbf{m}_j + \mathbf{l}_j)$$
$$\mathbf{n}_{2j+1} = \frac{S}{\sqrt{1+\mathbf{l}^2}}(-\mathbf{m}_j + \mathbf{l}_j)$$

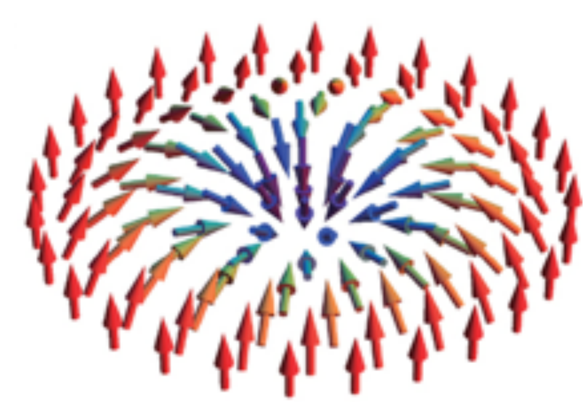


$\mathbf{l}_i$ terms can be integrated out (~ Legendre transformation)

$$S[\mathbf{m}] = \int dx dt \frac{1}{2g}\left(v(\partial_x \mathbf{m})^2 + \frac{1}{v}(\partial_t \mathbf{m})^2 + \boxed{\frac{i\theta}{8\pi}\epsilon_{ij}\mathbf{m}\cdot(\partial_i \mathbf{m}\times\partial_j \mathbf{m})}\right)$$
$$g = \frac{2}{S}$$

$\theta = 2\pi S$ the rest is an integer topological charge

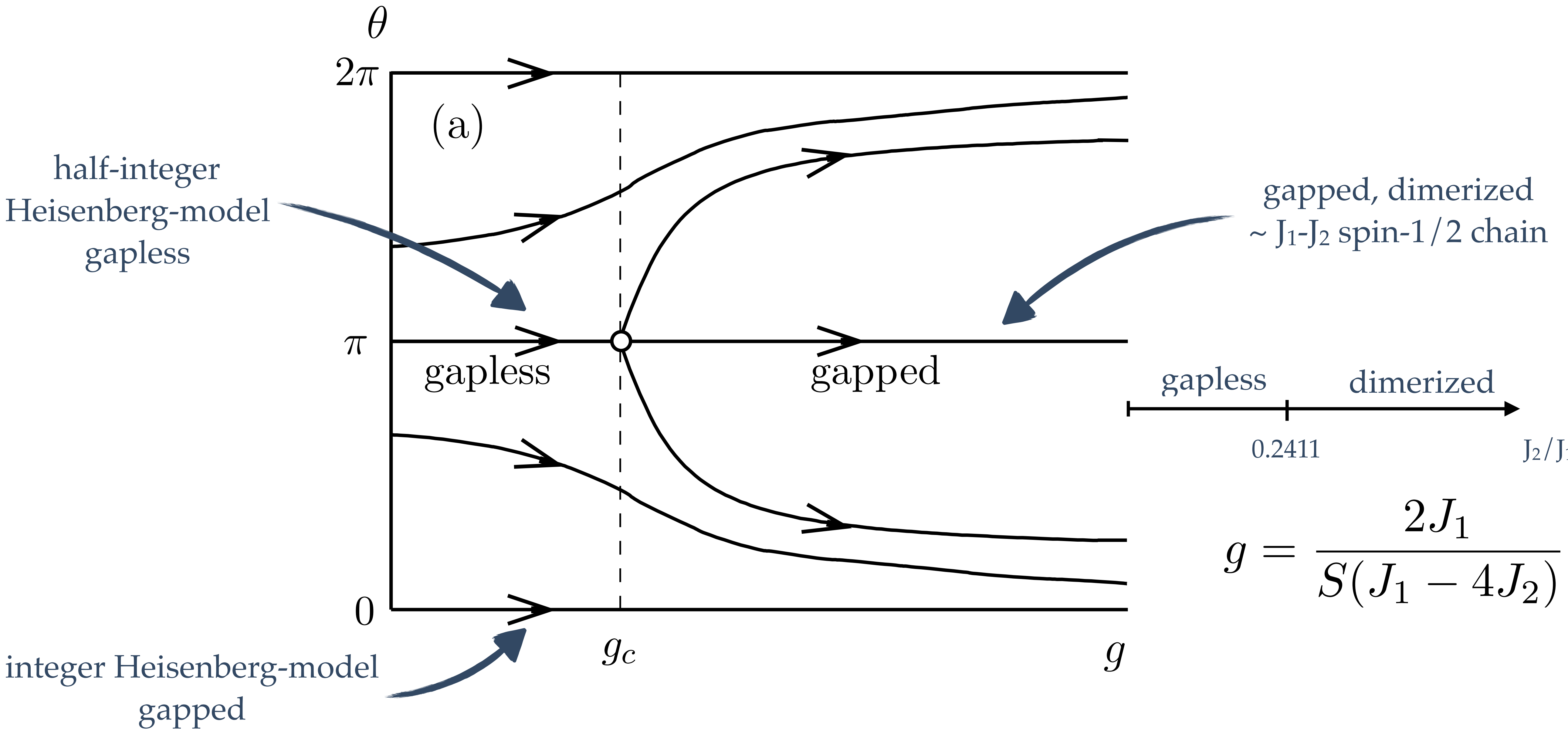
half-integer spins	O(3) NLSM with topological term
integer spins	O(3) NLSM without topological term



Affleck (1988)

$$S = \int dx d\tau \left(\frac{1}{2g} (\partial_\mu \mathbf{m})^2 + i2\pi S Q \right)$$

$$g = \frac{2}{S}$$



$$g = \frac{2J_1}{S(J_1 - 4J_2)}$$

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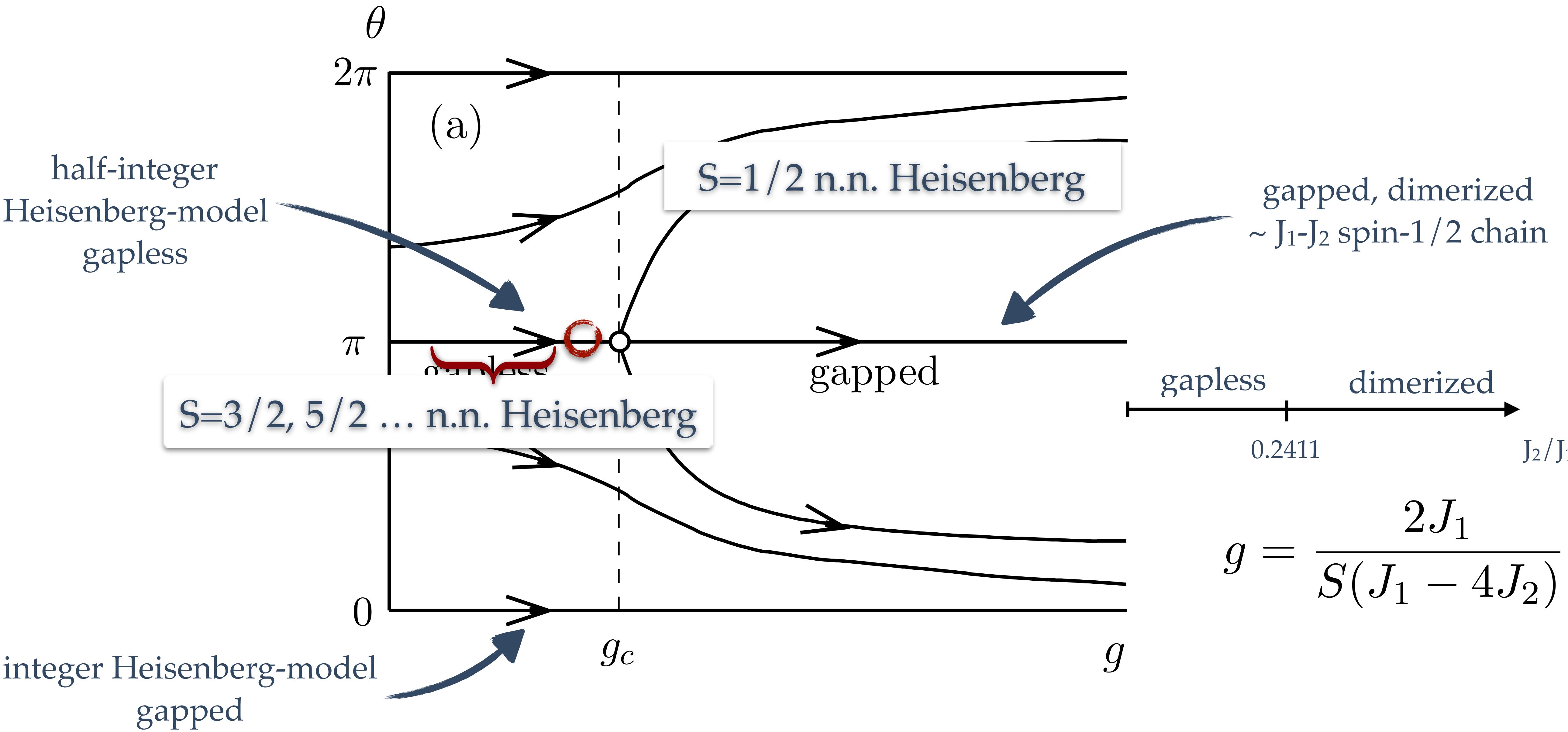
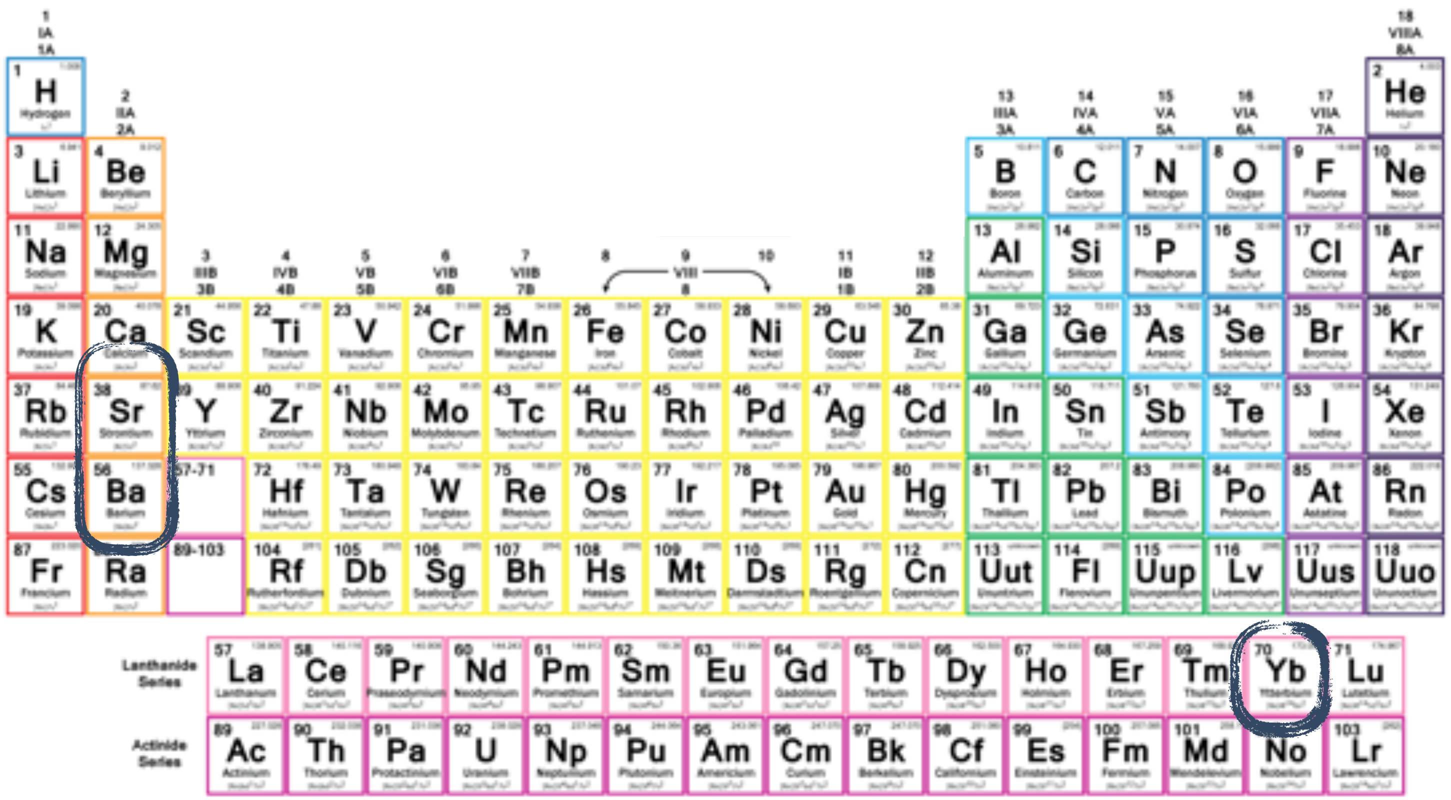
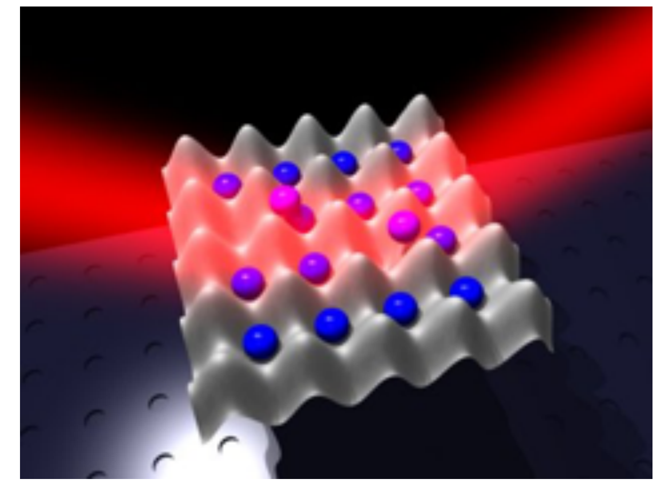


$$g = \frac{2J_1}{S(J_1 - 4J_2)}$$

SU(N) symmetry in cold atoms



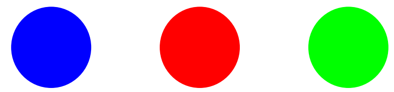


hopping process independent of nuclear spin
(N=2F+1 states)

SU(N) symmetric Heisenberg model in the Mott insulating phase

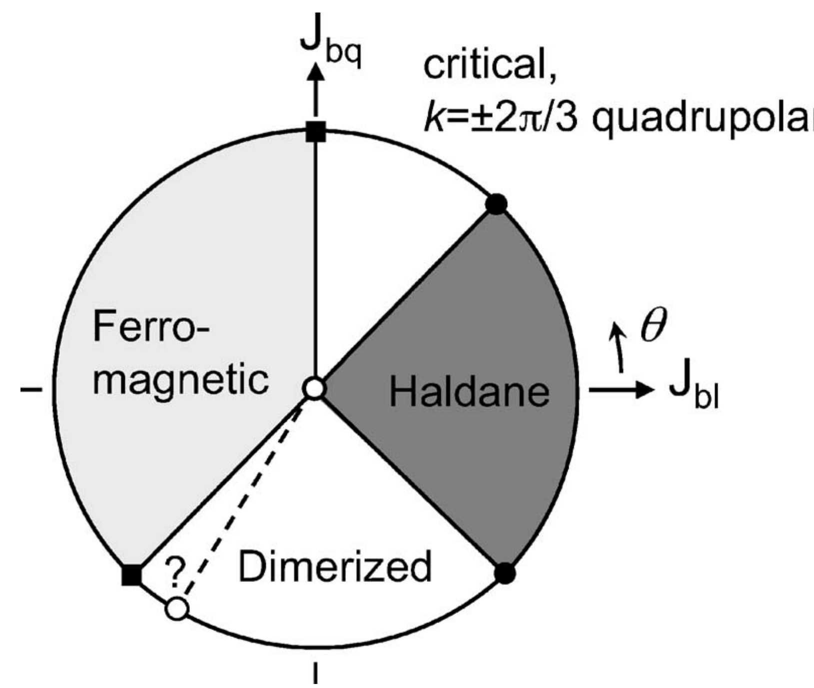
Gorshkov et al., Nature Physics, 6, 289 - 295 (2010)
Cazalilla et al., New J. Phys., 11, 103033 (2009)
Cazalilla, Rey, Rep. Prog. Phys. 77 124401 (2014)

SU(3) spin chains - fundamental spin

fundamental spin: 3 colors 

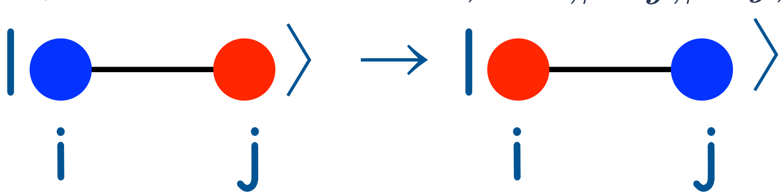
$$H = J_1 \sum_i \cos\theta(\mathbf{S_i}\cdot\mathbf{S_{i+1}}) + \sin\theta(\mathbf{S_i}\cdot\mathbf{S_{i+1}})^2$$

$$S^{\alpha}_{\beta} \sim c^{\dagger}_{\alpha} c_{\beta}$$



SU(3) symmetric Heisenberg interaction

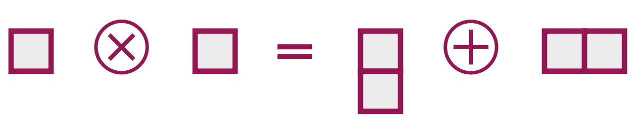
$$S^{\alpha}_{\beta}(i) S^{\beta}_{\alpha}(j) = c^{\dagger}_{i,\alpha} c_{i,\beta} c^{\dagger}_{j,\beta} c_{j,\alpha}$$



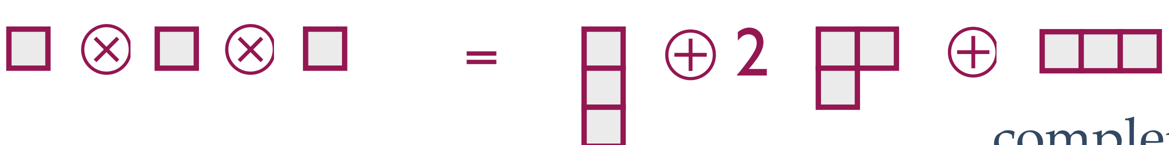
$$(\mathbf{S}_i \mathbf{S}_j) + (\mathbf{S}_i \mathbf{S}_j)^2 - 1 = \mathcal{P}^{(3)}_{ij}$$

Higher spins

3 × 3 = 3 + 6



3 × 3 × 3 = 1 + 2 × 8 + 10



singlet

completely symmetric

fully symmetric representation

p boxes:
p symmetrized fundamental spins

Nature of GS?

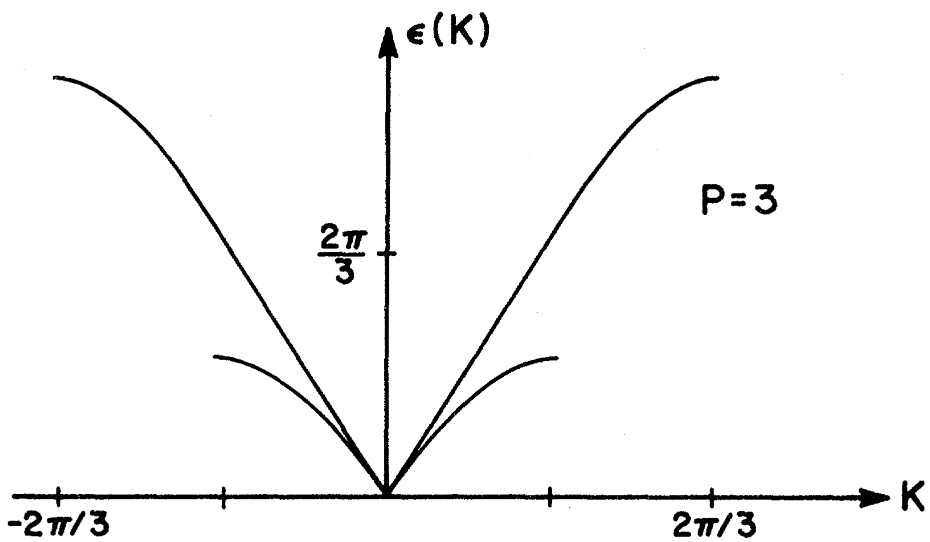
Lieb Schulz-Mattis-Affleck theorem:

p=3m±1 chains are either gapless or trimerized

Lieb, Schultz, Mattis (1961)
Affleck, Lieb (1986)

p=1

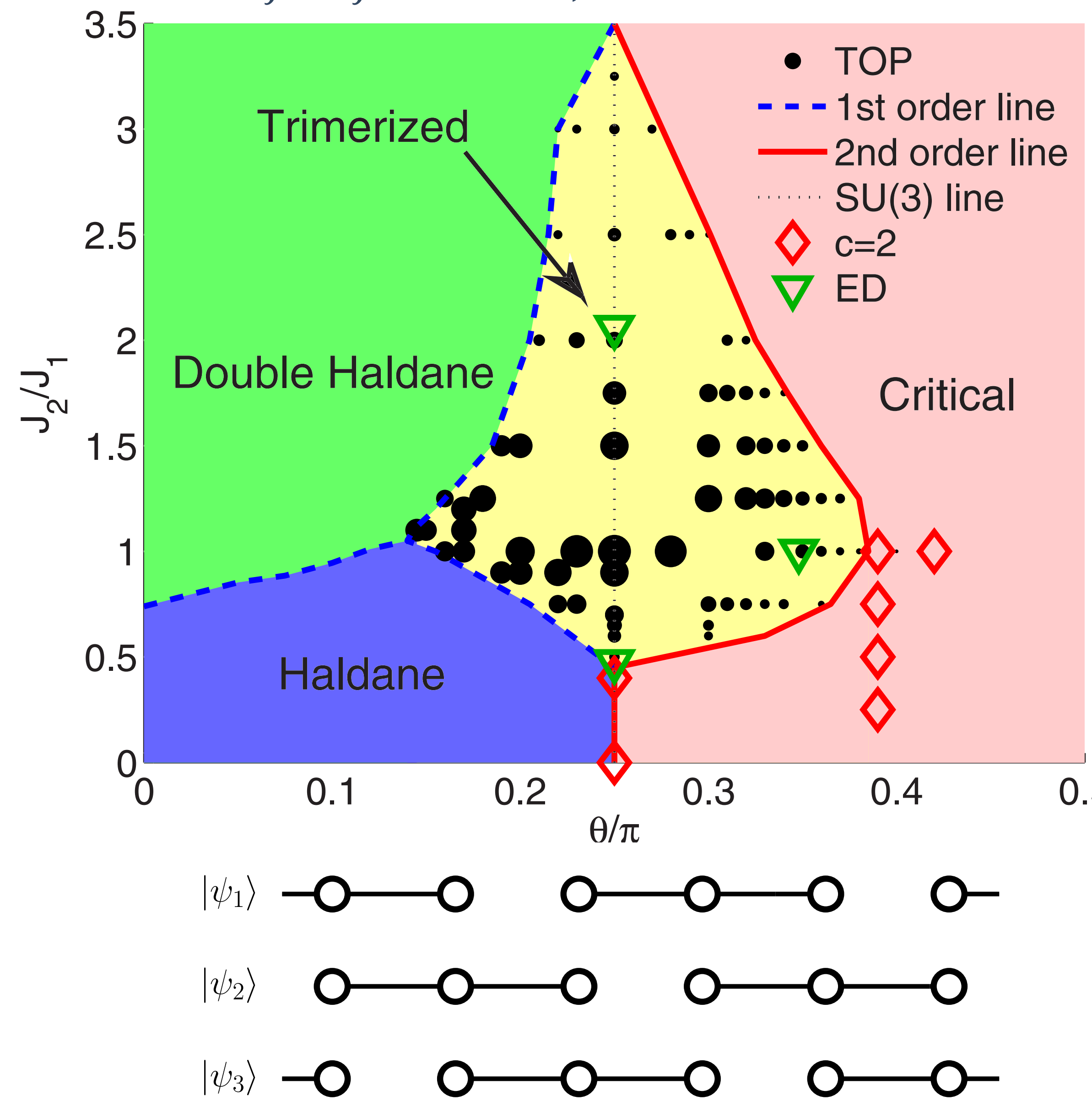
n.n Heisenberg point, Bethe solvable



$$H = J_1 \sum_i \cos\,\theta(\mathbf{S_i}\cdot\mathbf{S_{i+1}}) + \sin\,\theta(\mathbf{S_i}\cdot\mathbf{S_{i+1}})^2$$

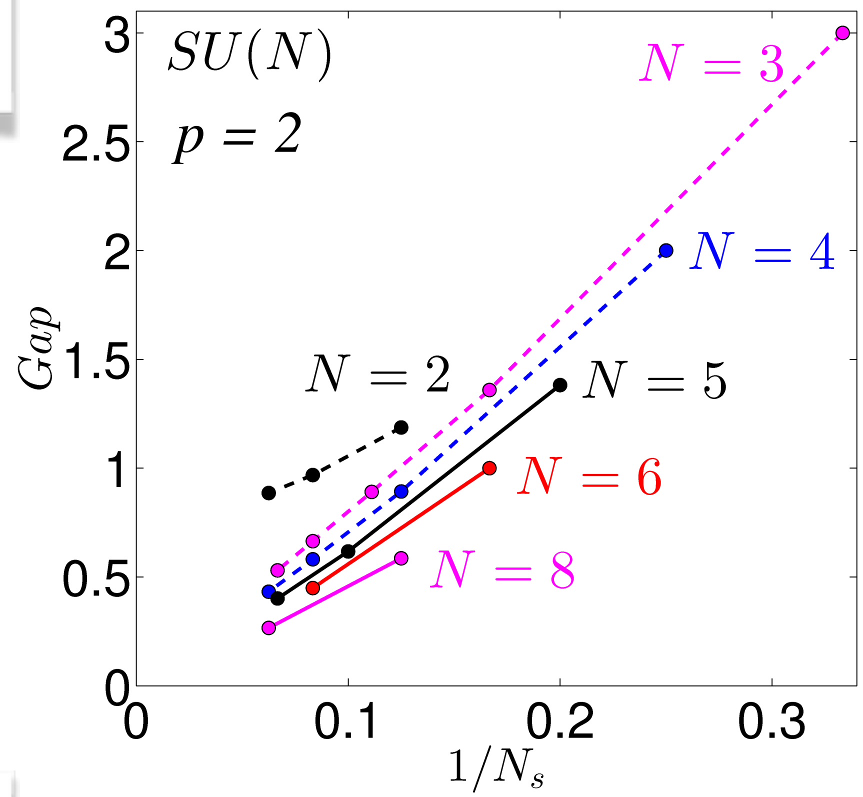
$$+ J_2 \sum_i \cos\,\theta(\mathbf{S_i}\cdot\mathbf{S_{i+2}}) + \sin\,\theta(\mathbf{S_i}\cdot\mathbf{S_{i+2}})^2.$$

*J*1 - *J*2 model, trimerized

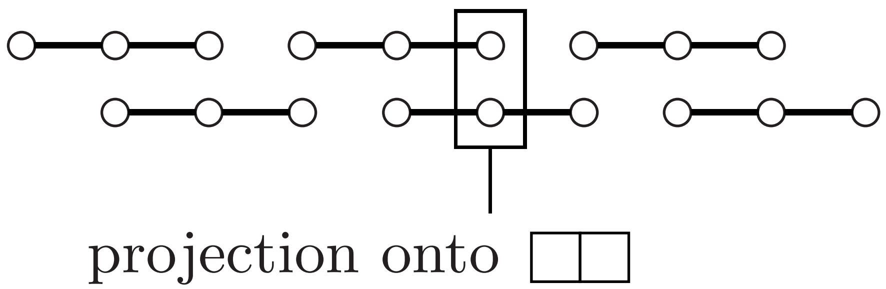


Nature of GS?

p=2

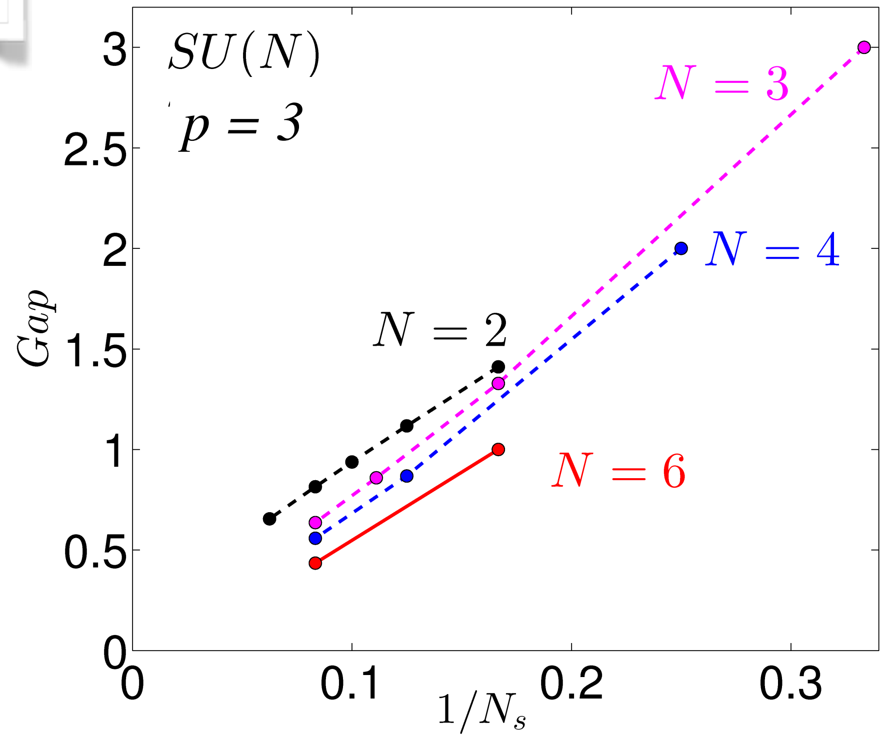


possible trimerized construction

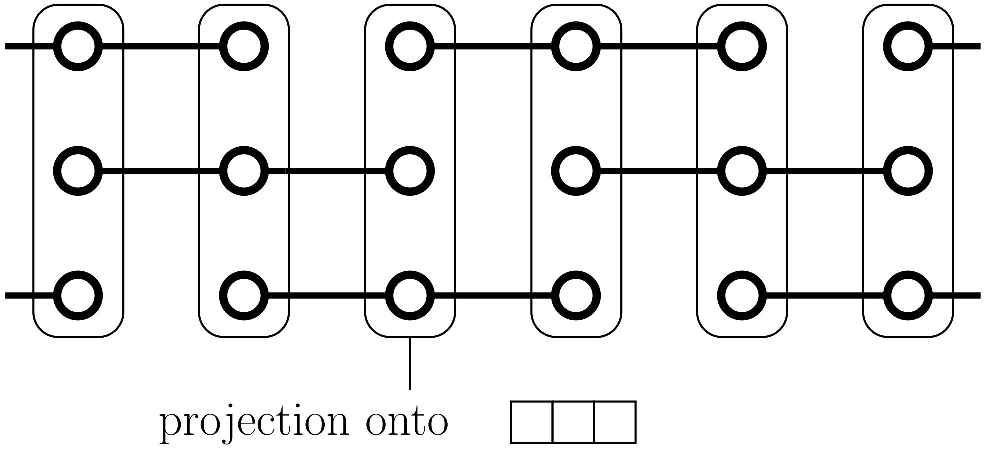


Lieb-Schultz-Mattis-Affleck ✓

p=3



p = 3: AKLT construction



Greiter, Rachel (2007)

$$H_{\mathbf{10}\mathrm{VBS}} = \sum_{i=1}^{N} \left((\boldsymbol{J}_i \boldsymbol{J}_{i+1})^2 + 5 \boldsymbol{J}_i \boldsymbol{J}_{i+1} + 6 \right),$$

is the p = 3 Heisenberg model gapped?

Nataf, Mila Phys. Rev. B 93, 155134 (2016)

SU(3) Heisenberg model & path integral

$$\mathcal{H} = \sum_{\langle i,j \rangle} S^{\alpha}_{\beta}(i) S^{\beta}_{\alpha}(j)$$

1	2	...	p

$\rightarrow$ p bosons per site

$$S^{\alpha}_{\beta}(i) = b^{\dagger}_{i,\alpha} b_{i,\beta} \qquad\qquad \mathcal{H} = \sum_{\langle i,j \rangle} b^{\dagger}_{i,\alpha} b_{i,\beta} b^{\dagger}_{j,\beta} b_{j,\alpha}$$

Path integral approach

$$\mathcal{Z} = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}\left(e^{-d\tau H} e^{-d\tau H} \ldots e^{-d\tau H}\right)$$

spin coherent state for the p-box case:

$$|\vec{\Phi}\rangle = \frac{1}{\sqrt{p!}} \left(\Phi^{(1)} b^{\dagger}_1 + \Phi^{(2)} b^{\dagger}_2 + \Phi^{(3)} b^{\dagger}_3\right)^p |0\rangle$$

with $|\vec{\Phi}|^2 = 1$ 3D complex fields

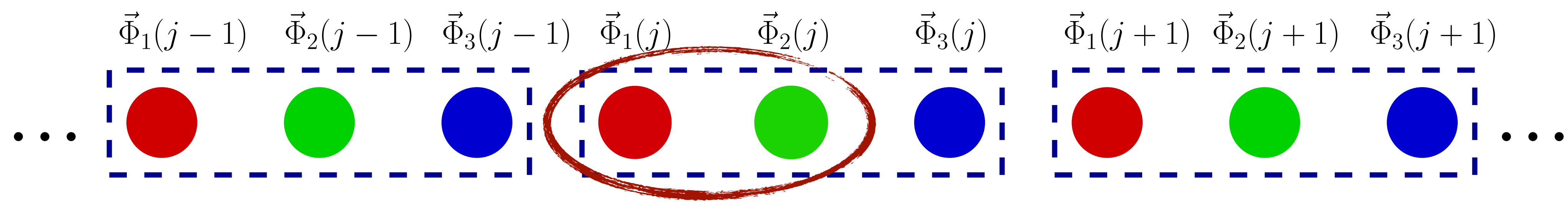
$$\mathcal{Z} = \int \mathcal{D}[\Phi] \exp\left(- \int d\tau \left[p^2 J \sum_i \left| \vec{\Phi}^*_{i,\tau} \vec{\Phi}_{i+1,\tau} \right|^2 + p \sum_i \vec{\Phi}^*_{i,\tau} \partial_\tau \vec{\Phi}_{i,\tau} \right] \right)$$

classical energy Berry phase (imaginary)

SU(3) case, coherent state path integral

$$\mathcal{Z} = \int \mathcal{D}[\Phi] \exp\left(- \int d\tau \left[p^2 J \sum_i \left| \vec{\Phi}^*_{i,\tau} \vec{\Phi}_{i+1,\tau} \right|^2 + p \sum_i \vec{\Phi}^*_{i,\tau} \partial_\tau \vec{\Phi}_{i,\tau} \right] \right)$$

classical ground state



local zero mode

short range **quantum fluctuations select 3-sublattice order** (order by disorder)
effective further neighbour (J_2, J_3) interactions

Low energy fluctutations in path integral:

non-orthogonality of the spin states in the unit cell

3 orthogonal states in the unit cell

$$\begin{pmatrix} \vec{\Phi}_1^T(3j,\tau) \\ \vec{\Phi}_2^T(3j+1,\tau) \\ \vec{\Phi}_3^T(3j+2,\tau) \end{pmatrix} = \begin{pmatrix} \sqrt{1-\frac{a^2}{p^2}(|L_{12}|^2+|L_{13}|^2)} & \frac{a}{p}L_{12} & \frac{a}{p}L_{13} \\ \frac{a}{p}L_{12}^* & \sqrt{1-\frac{a^2}{p^2}(|L_{12}|^2+|L_{23}|^2)} & \frac{a}{p}L_{23} \\ \frac{a}{p}L_{13}^* & \frac{a}{p}L_{23}^* & \sqrt{1-\frac{a^2}{p^2}(|L_{13}|^2+|L_{23}|^2)} \end{pmatrix} \begin{pmatrix} \vec{\phi}_1^T(j,\tau) \\ \vec{\phi}_2(j,\tau) \\ \vec{\phi}_3(j,\tau) \end{pmatrix}$$

L variables can be integrated out

$$S=\int dx d\tau \left(\sum_{n=1}^{3} \frac{1}{g} \left(\left|\vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2\right|^2 + \left|\vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3\right|^2 + \left|\vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1\right|^2 \right) + i \sum_{n=1}^{3} \frac{\theta_n}{2\pi i} \varepsilon_{\mu\nu} \left(\partial_\mu \vec{\phi}_n \cdot \partial_\nu \vec{\phi}_n^* \right) \right)$$

3 fields are orthogonal
from a unitary matrix

$$\mathrm{SU}(3)/\big(\mathrm{U}(1)\times \mathrm{U}(1)\big)$$

$$\begin{pmatrix} \vec{\phi}_1^T(j,\tau) \\ \vec{\phi}_2(j,\tau) \\ \vec{\phi}_3(j,\tau) \end{pmatrix}$$

$$g \sim \frac{1}{\sqrt{p}}$$

topological term

$$i\sum_{n=1}^{3} \theta_n Q_n$$

$$Q_1+Q_2+Q_3=0$$

$$\theta_1 = -\theta_3 = p\frac{2\pi}{3}$$

trivial for $p=3m$,
nontrivial otherwise

SU(3) spin chain with p=3m	NLSM without topological term
SU(3) spin chain with p=3m±1	NLSM with topological term

Monte Carlo simulation

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta \left(Q_1 - Q_3 \right) \quad \theta = p\frac{2\pi}{3}$$

imaginary term in action for real θ

we can make simulations for imaginary $\theta = i\vartheta$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) - \vartheta \left(Q_1 - Q_3 \right)$$

Monte Carlo simulation

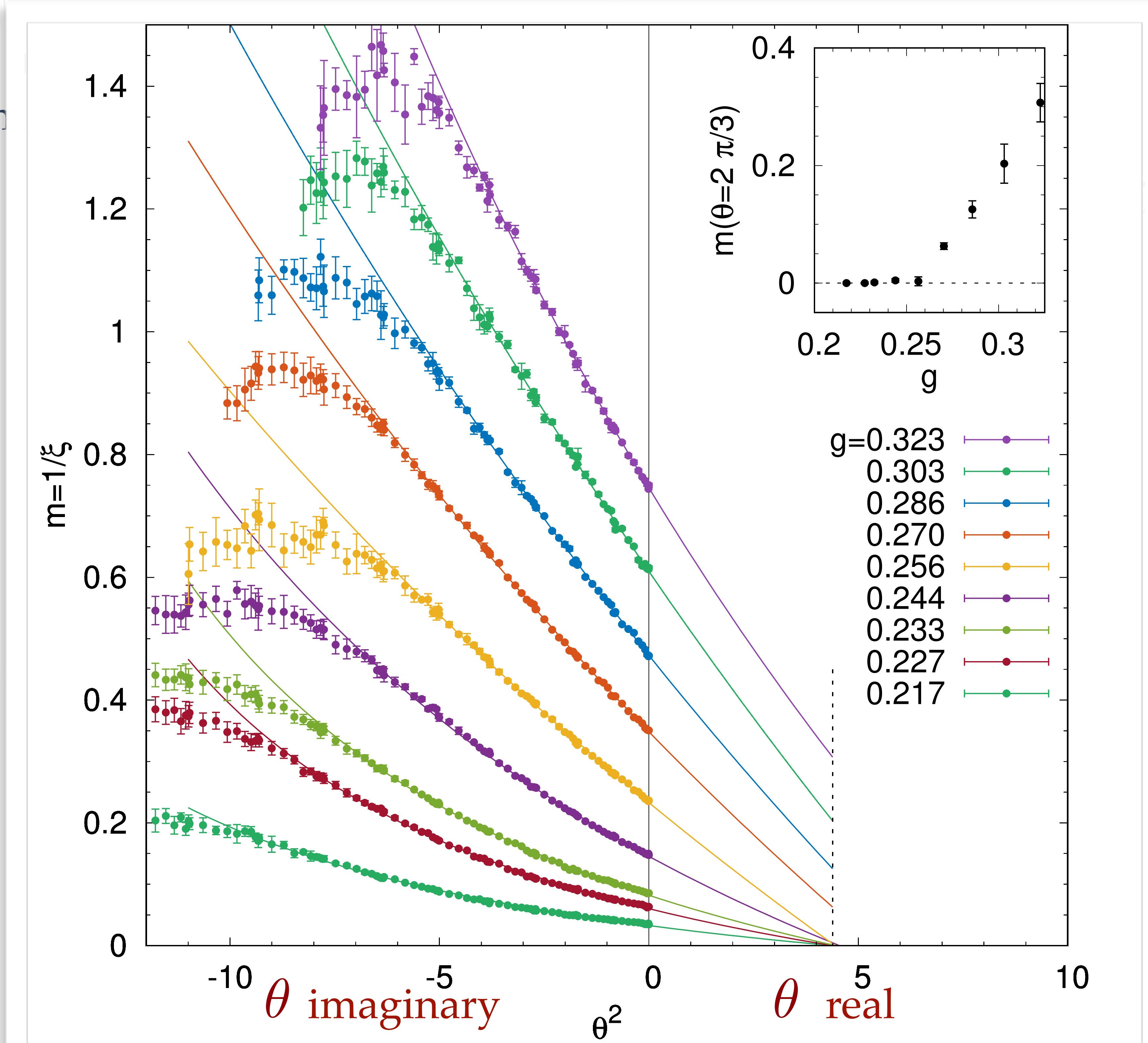
$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta \left(Q_1 - Q_3 \right) \qquad \theta = p\frac{2\pi}{3}$$

imaginary term

we can make sin

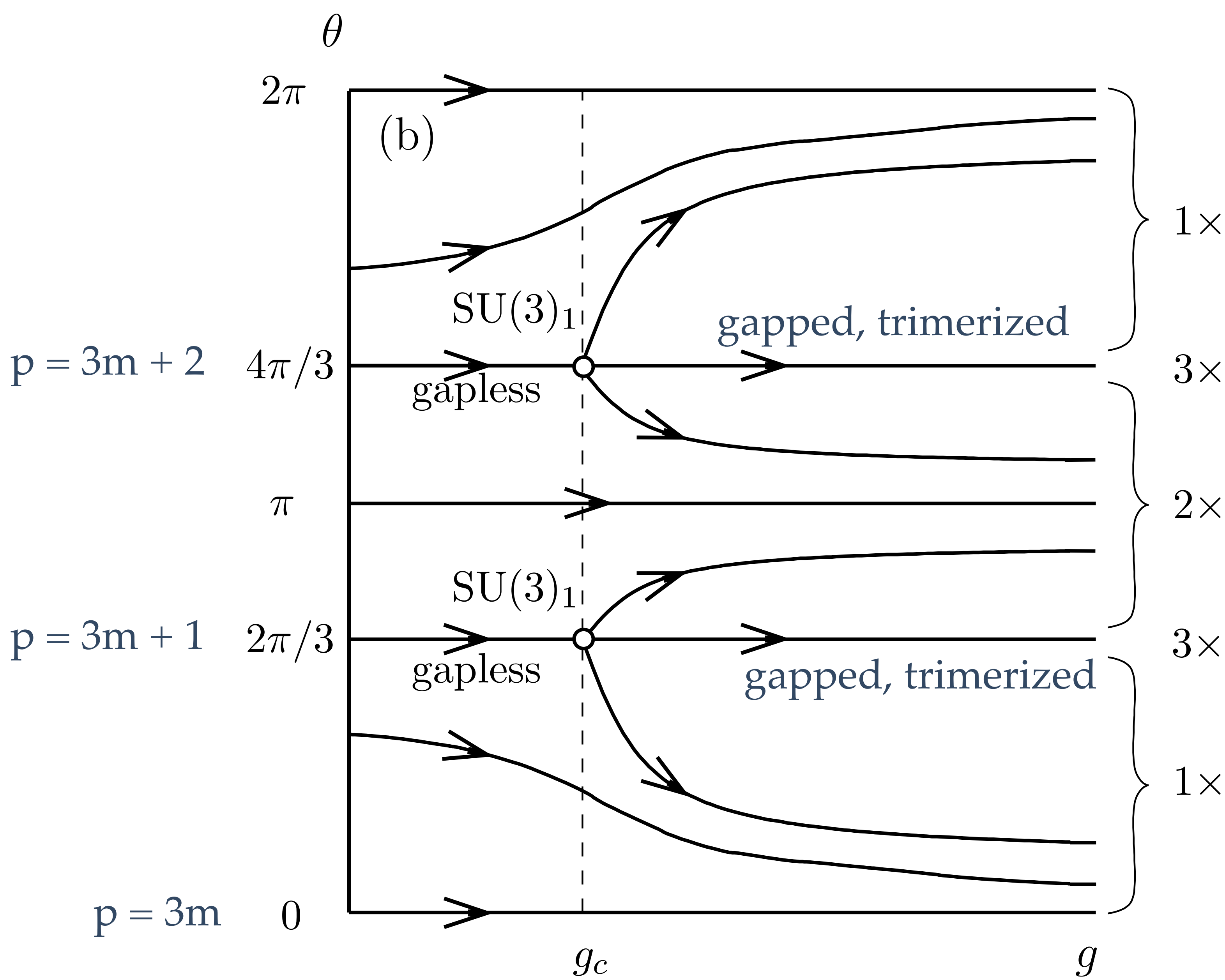
$$S = \int dx d\tau \frac{1}{g}$$

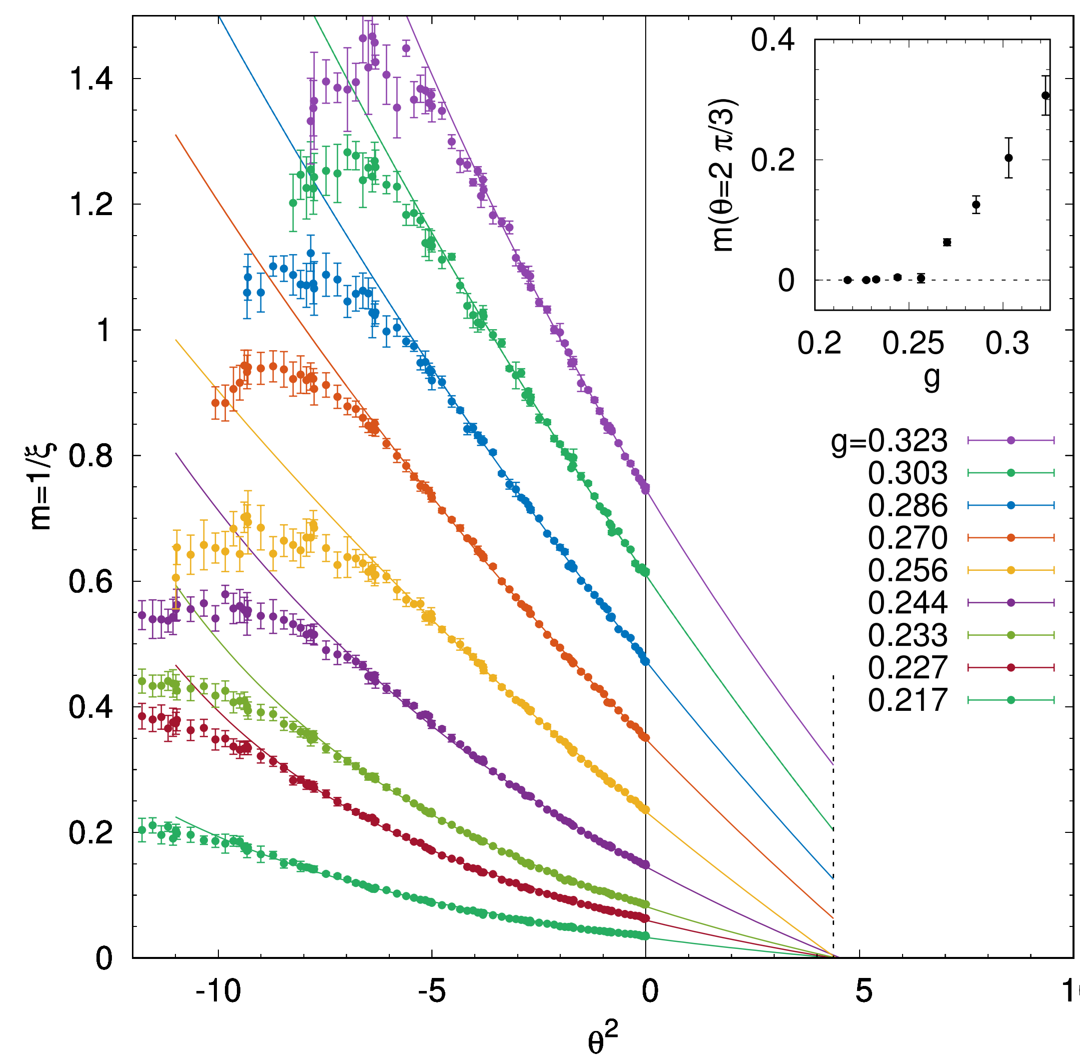
$_3)$



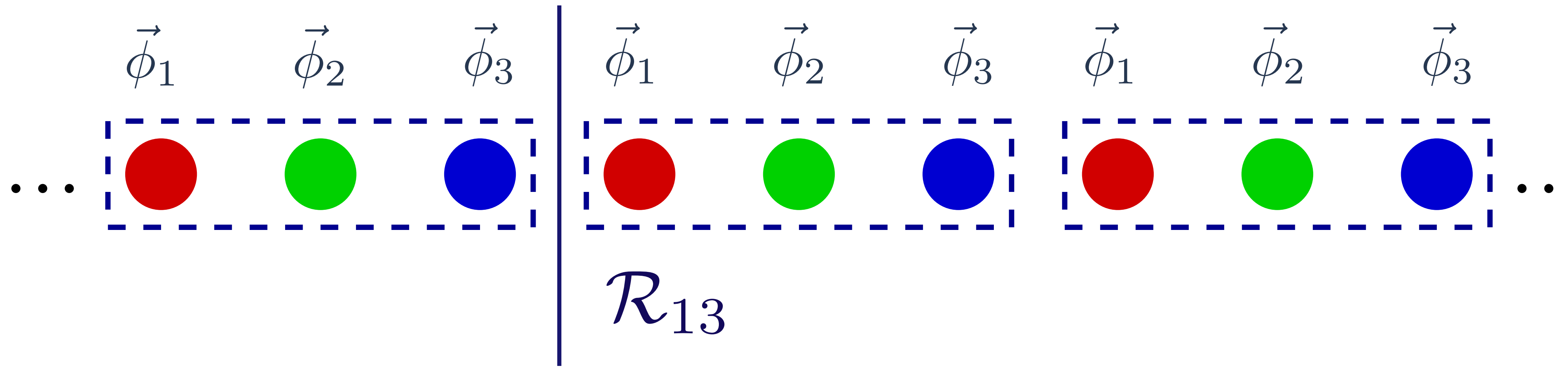
Phase diagram

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta \left(Q_1 - Q_3 \right) \qquad \theta = p\frac{2\pi}{3}$$





Symmetries of the field theory



$$S=\int dx d\tau \frac{1}{g}\left(\left|\vec{\phi}_1^*\cdot\partial_\mu\vec{\phi}_2\right|^2+\left|\vec{\phi}_2^*\cdot\partial_\mu\vec{\phi}_3\right|^2+\left|\vec{\phi}_3^*\cdot\partial_\mu\vec{\phi}_1\right|^2\right)+i\theta\left(Q_1-Q_3\right) \qquad \theta=p\frac{2\pi}{3}$$

Translation by one site: $\vec{\phi}_1\rightarrow\vec{\phi}_2\rightarrow\vec{\phi}_3\rightarrow\vec{\phi}_1$ $\qquad \mathbb{Z}_3$ symmetry in NLSM

$$ip\frac{2\pi}{3}(Q_1-Q_3)\rightarrow ip\frac{2\pi}{3}(Q_2-Q_1)=ip\frac{2\pi}{3}(Q_1-Q_3)+ip\frac{2\pi}{3}(Q_1+Q_2+Q_3)-i3p\frac{2\pi}{3}Q_1.$$

Mirror symmetry: $\vec{\phi}_{1(3)}(x,\tau)\leftrightarrow\vec{\phi}_{3(1)}(-x,\tau)$ $\qquad \vec{\phi}_2(x,\tau)\leftrightarrow\vec{\phi}_2(-x,\tau)$

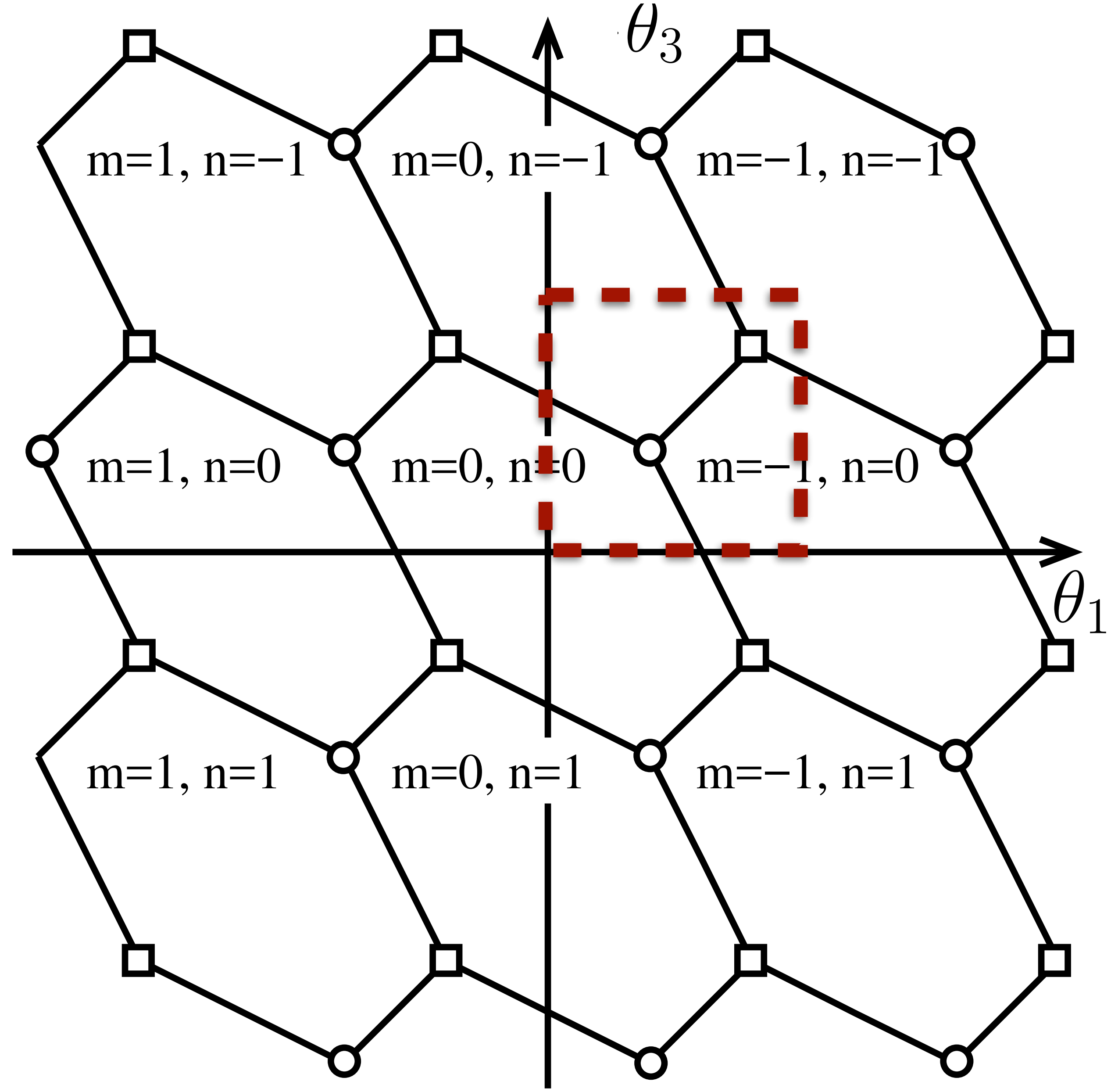
$$\theta(Q_1-Q_3)\rightarrow-\theta(Q_3-Q_1),$$

$\mathbb{Z}_2$ symmetry in NLSM

General phase diagram: $\theta_1 \neq -\theta_3$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i \left(\theta_1 Q_1 - \theta_3 Q_3 \right)$$

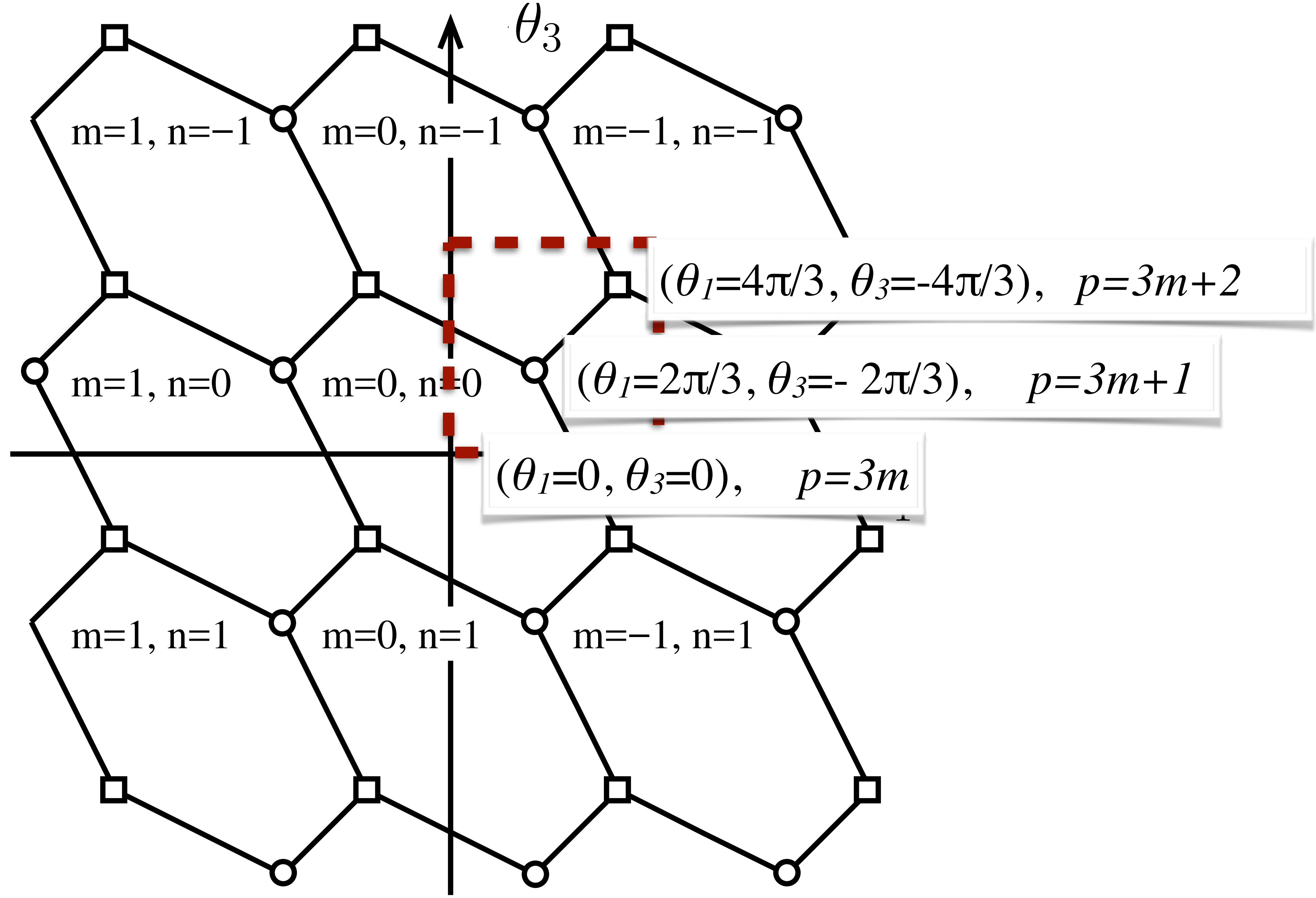
$g \to \infty$ is integrable, $S = i \left(\theta_1 Q_1 - \theta_3 Q_3 \right)$



General phase diagram: $\theta_1 \neq -\theta_3$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i \left(\theta_1 Q_1 - \theta_3 Q_3 \right)$$

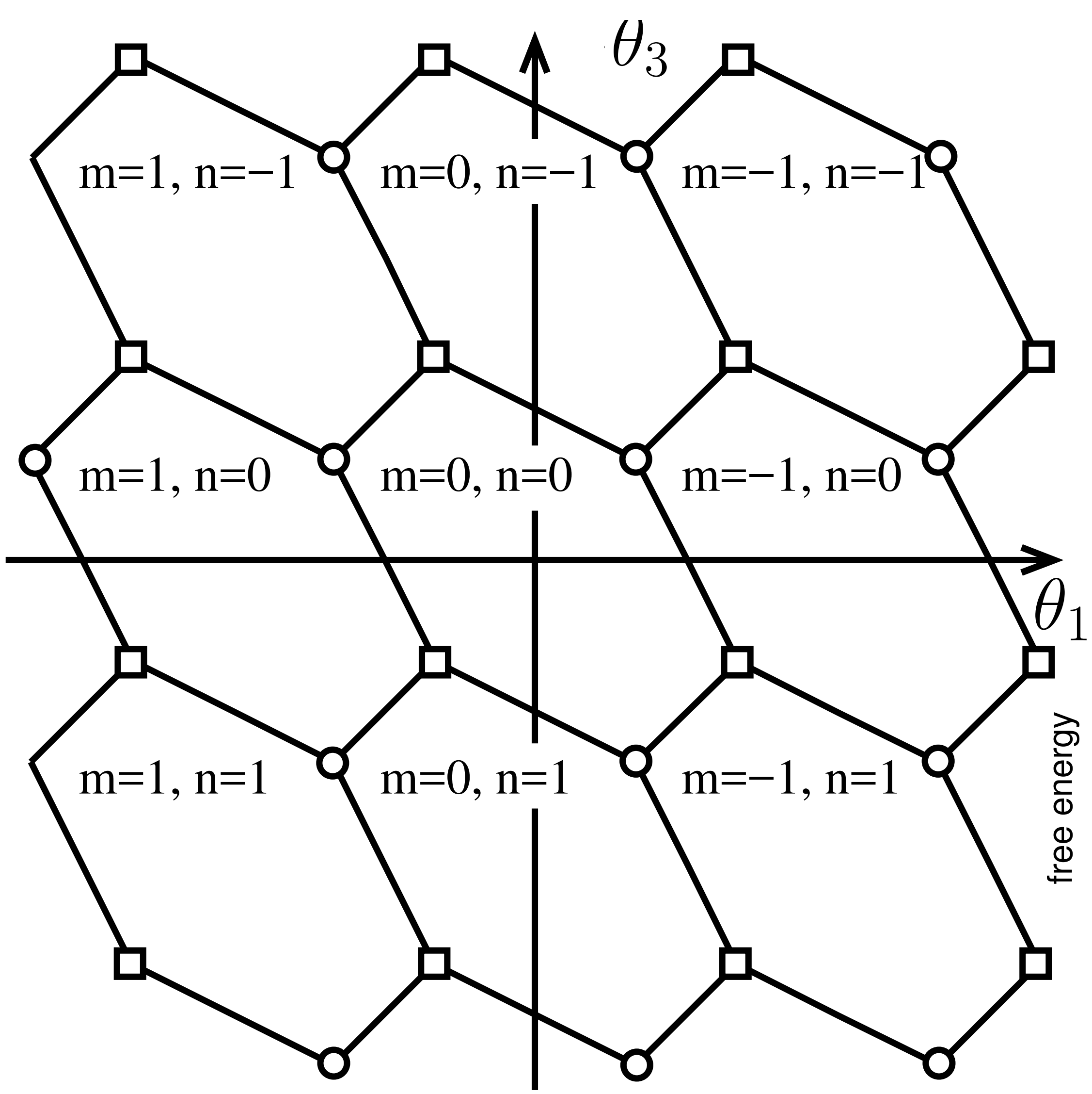
$g \to \infty$ is integrable, $S = i\left(\theta_1 Q_1 - \theta_3 Q_3\right)$



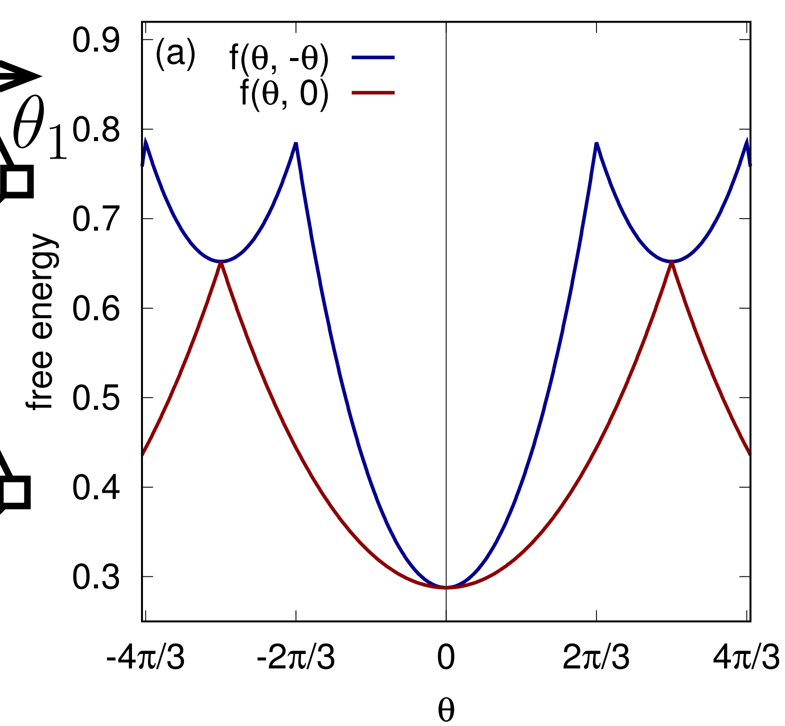
General phase diagram: $\theta_1 \neq -\theta_3$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i \left(\theta_1 Q_1 - \theta_3 Q_3 \right)$$

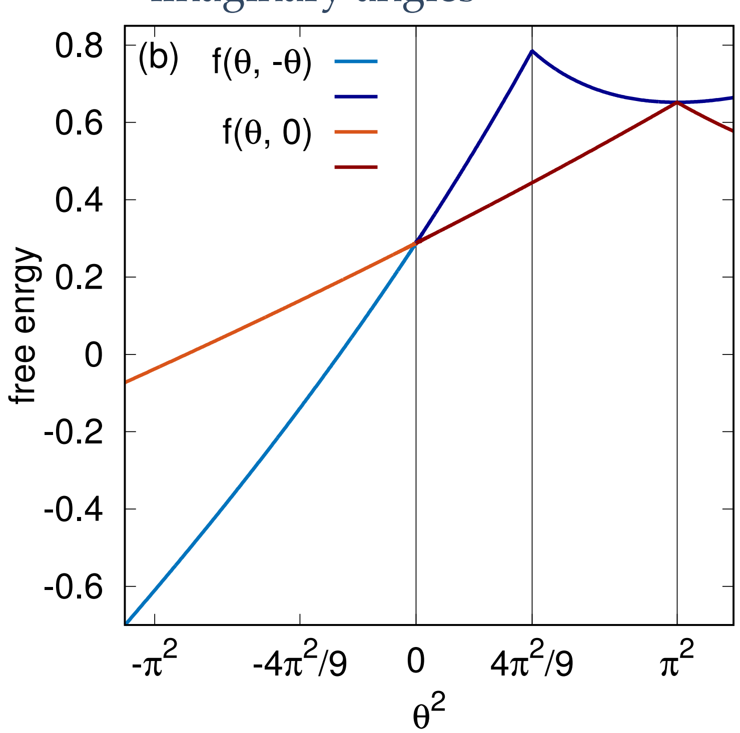
$g \to \infty$ is integrable, $S = i \left(\theta_1 Q_1 - \theta_3 Q_3 \right)$



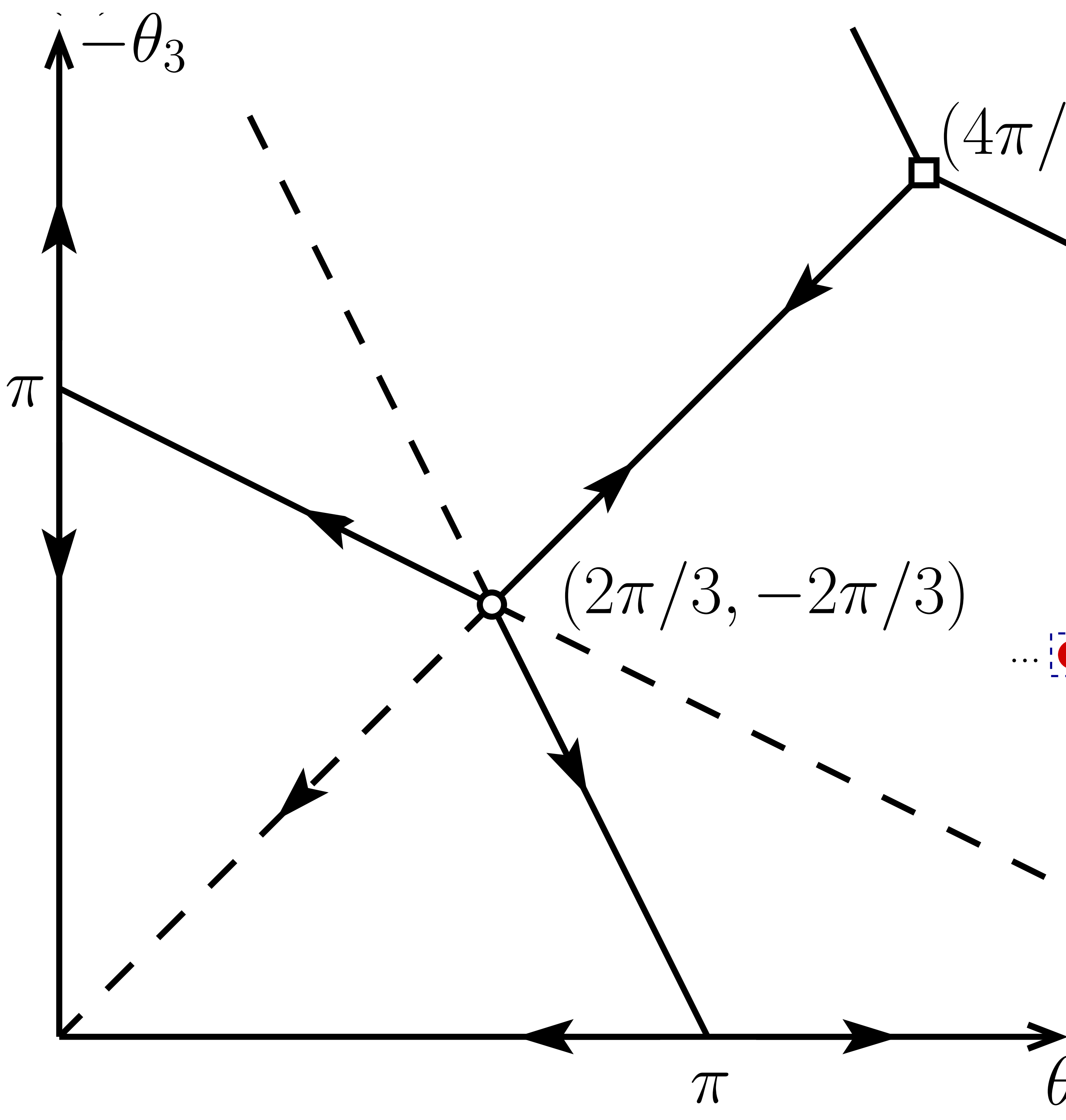
free energy



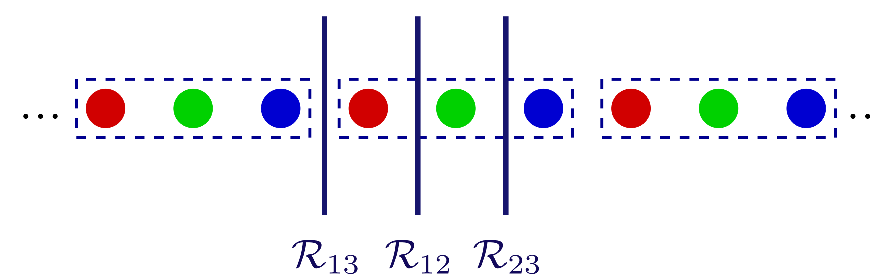
extrapolating from imaginary angles



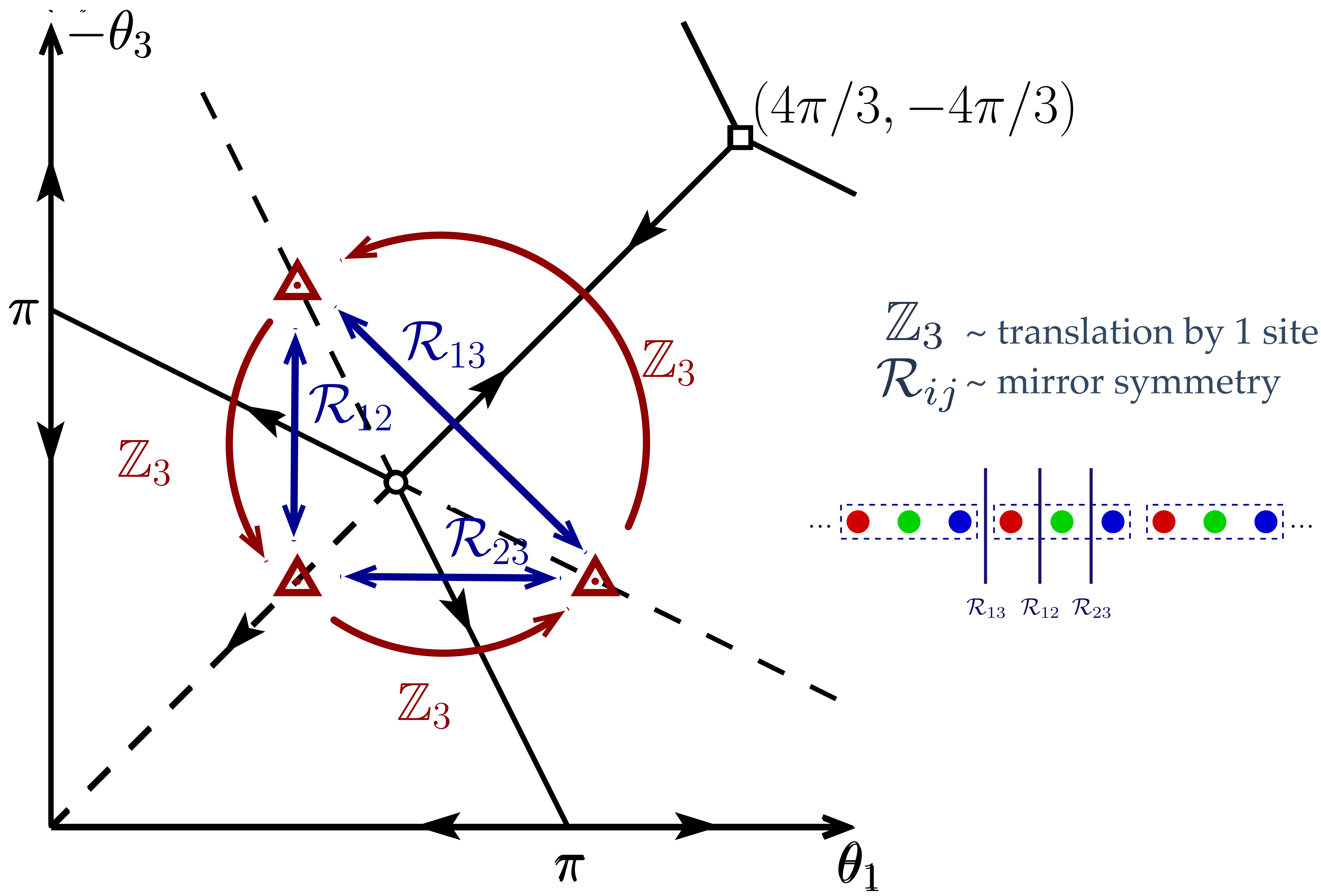
General phase diagram: $\theta_1 \neq -\theta_3$



$\mathbb{Z}_3$ ~ translation by 1 site
$\mathcal{R}_{ij}$ ~ mirror symmetry



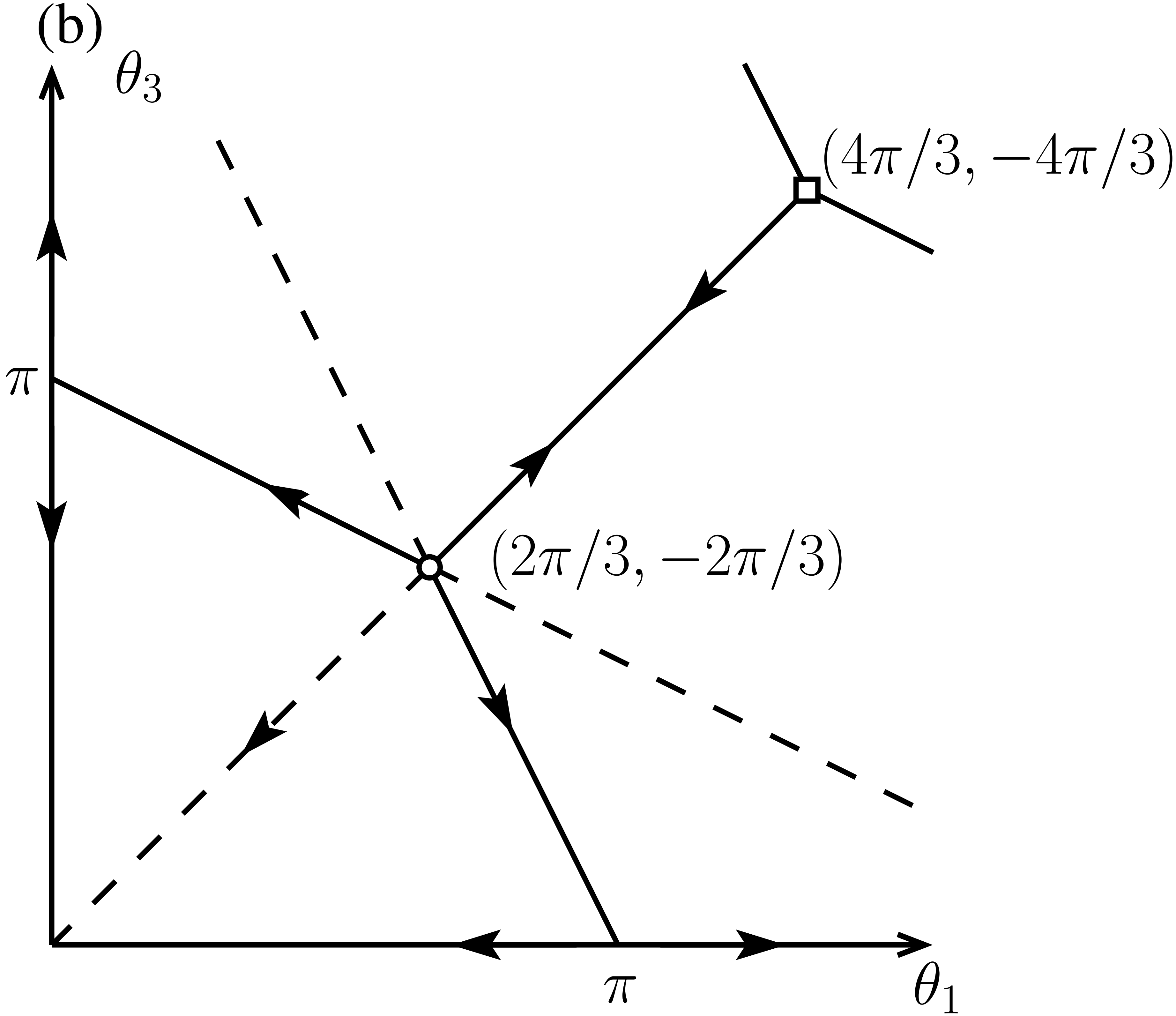
General phase diagram: $\theta_1 \neq -\theta_3$

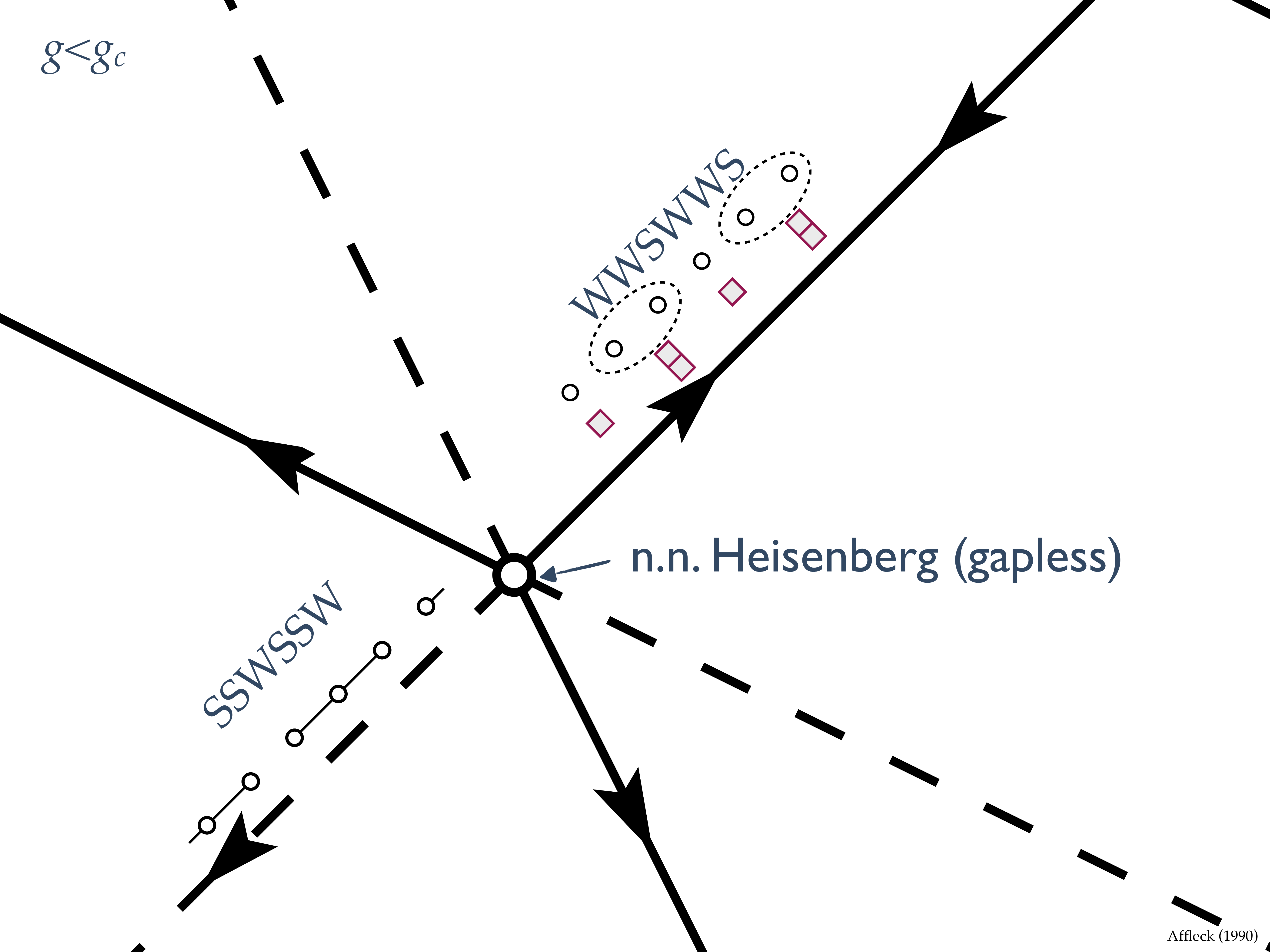


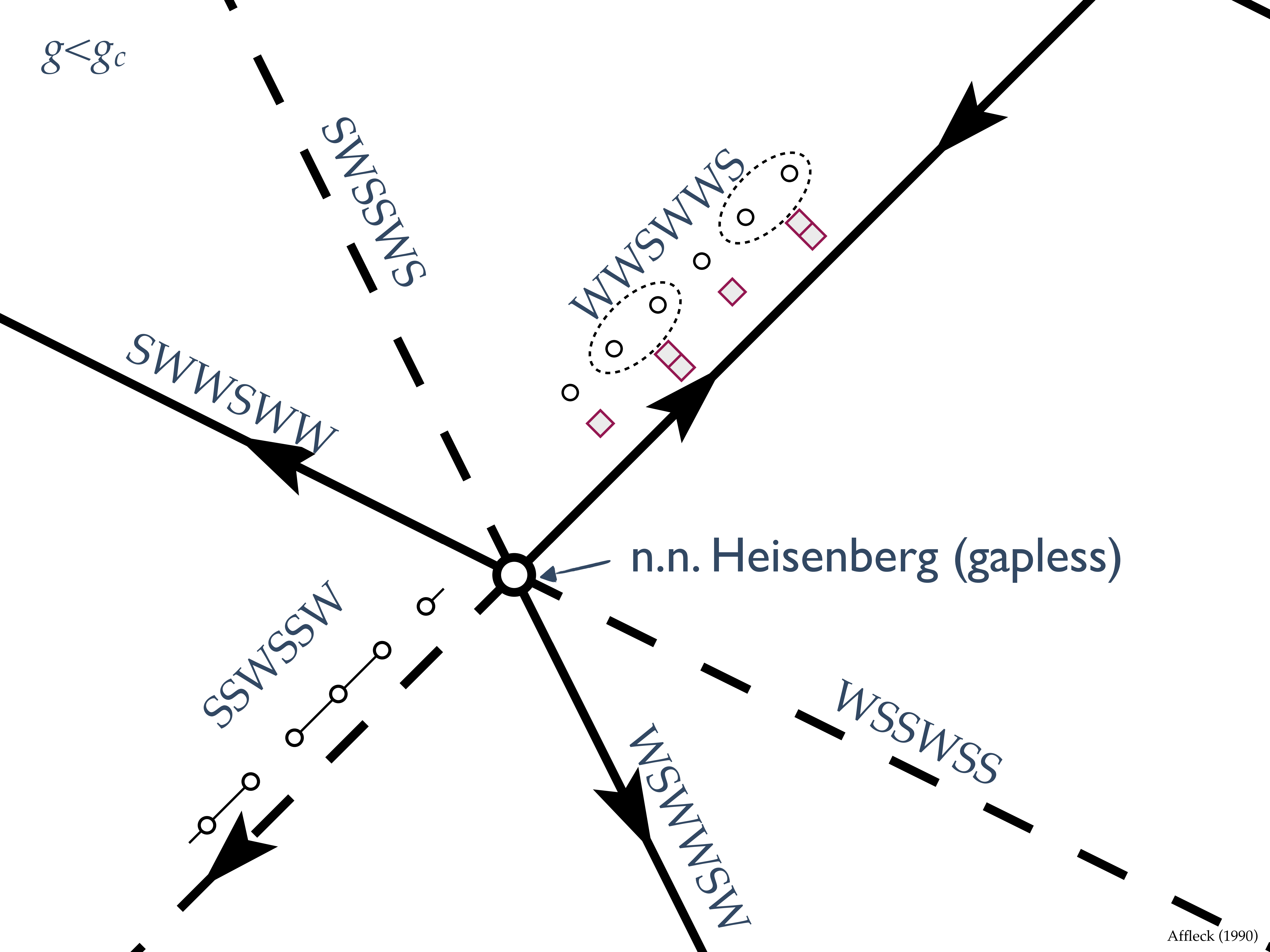
$\mathbb{Z}_3$ ~ translation by 1 site

$\mathcal{R}_{ij}$ ~ mirror symmetry

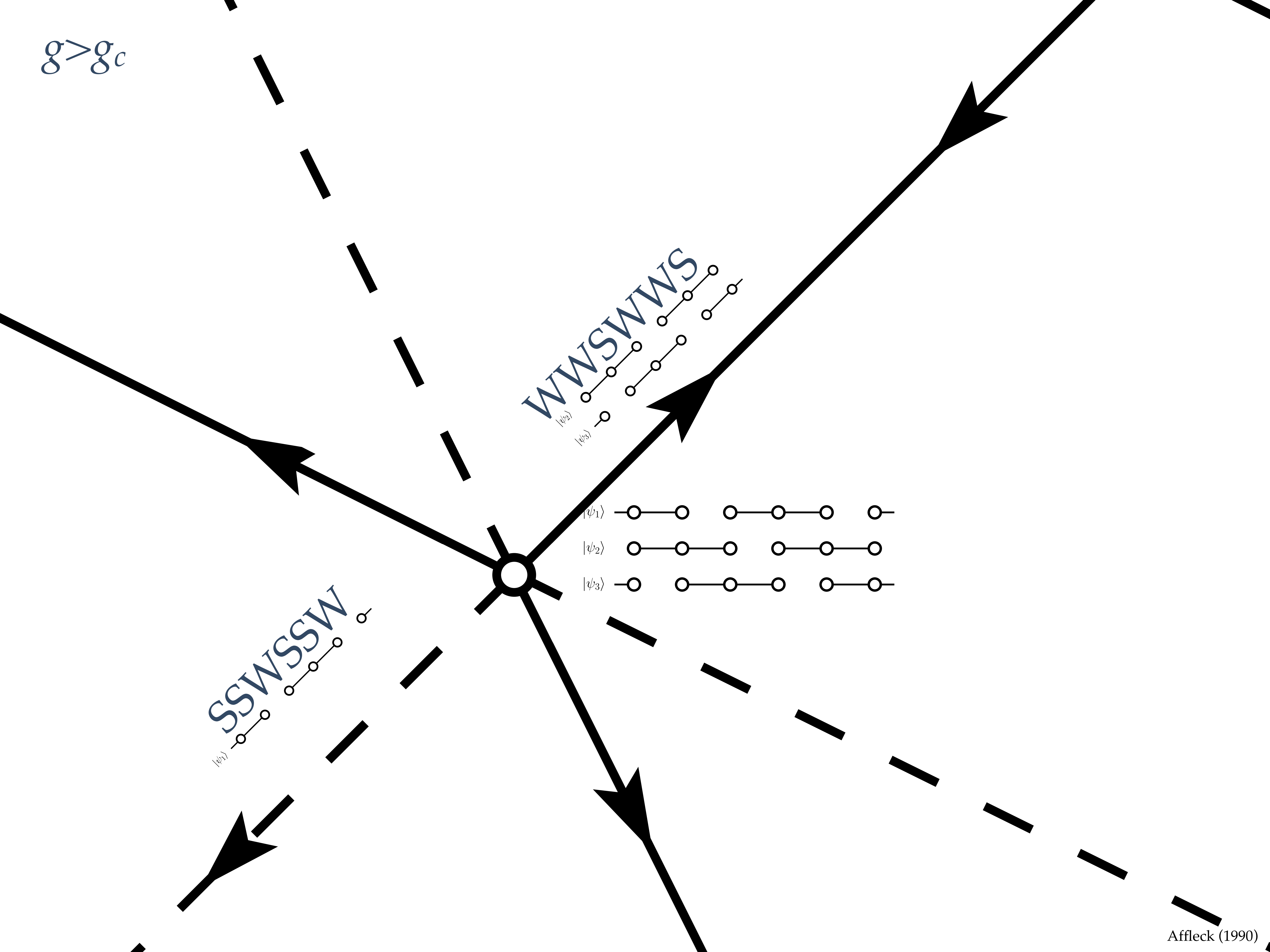
$g < g_c$



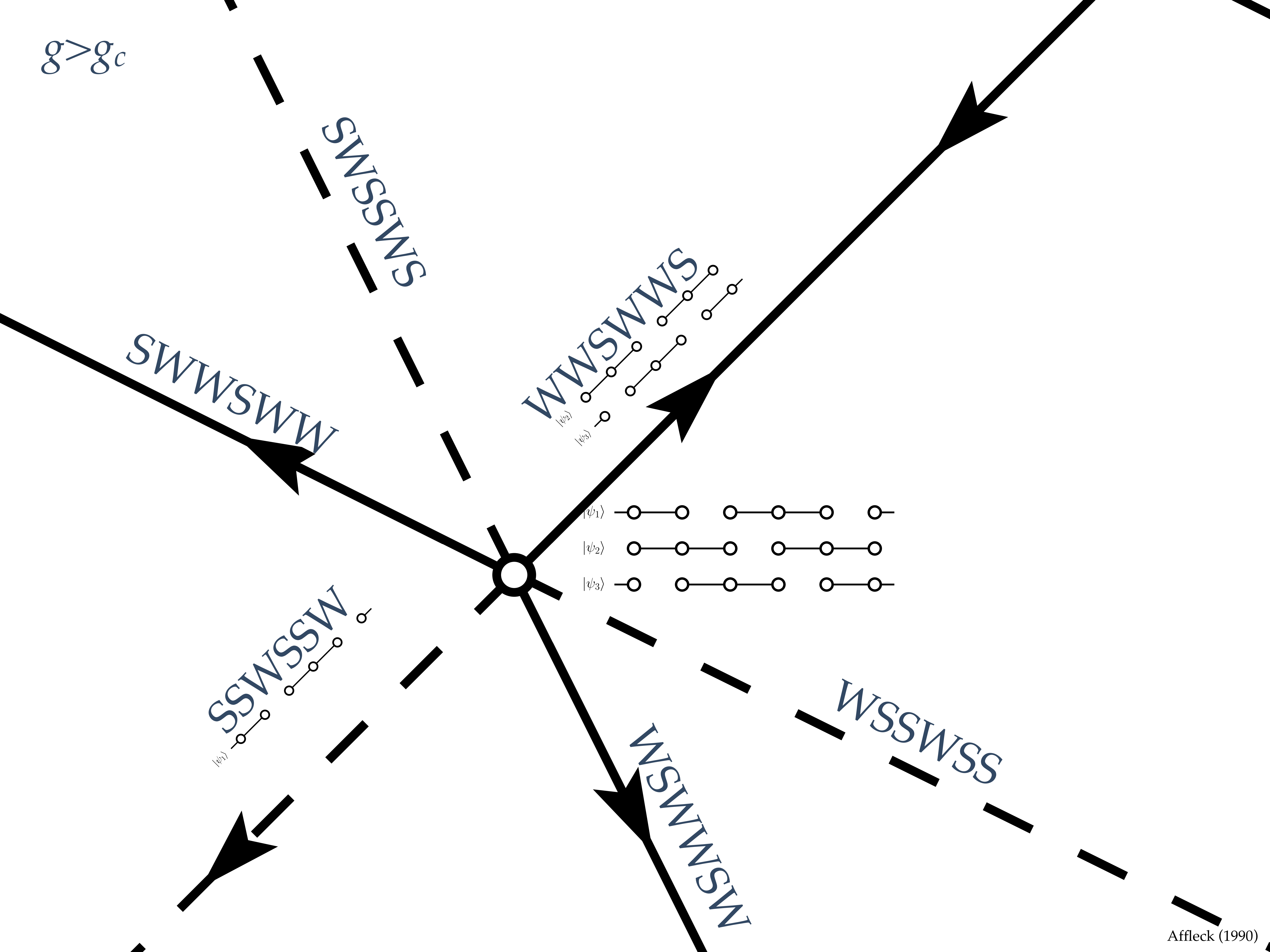

g<gc
WWSWWS
n.n. Heisenberg (gapless)
SSWSSW
Affleck (1990)



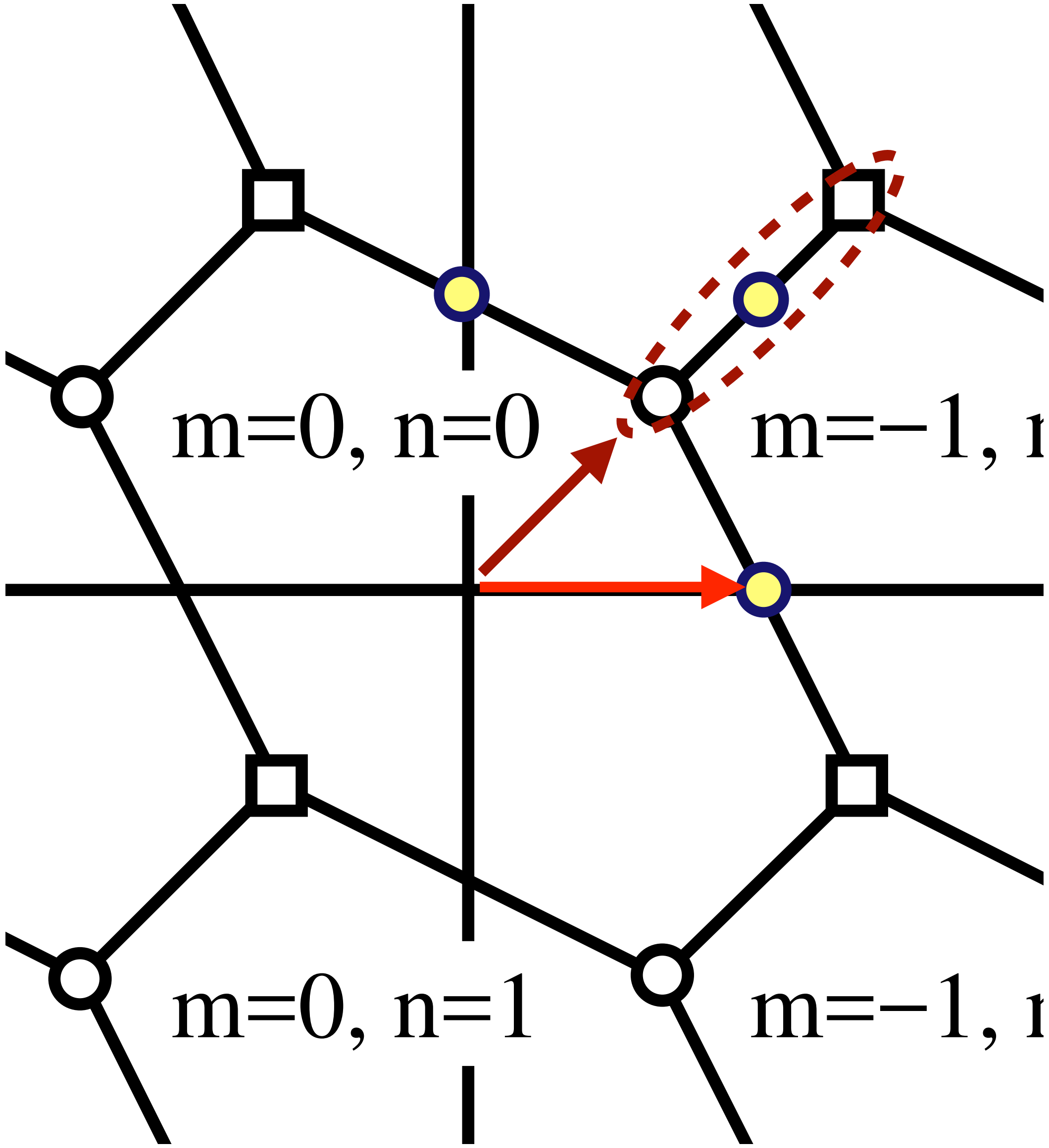
$g<g_c$
SWSSWS
WWSWWS
SWWSWW
n.n. Heisenberg (gapless)
SSWSSW
WSSWSS
WSWWSW
Affleck (1990)

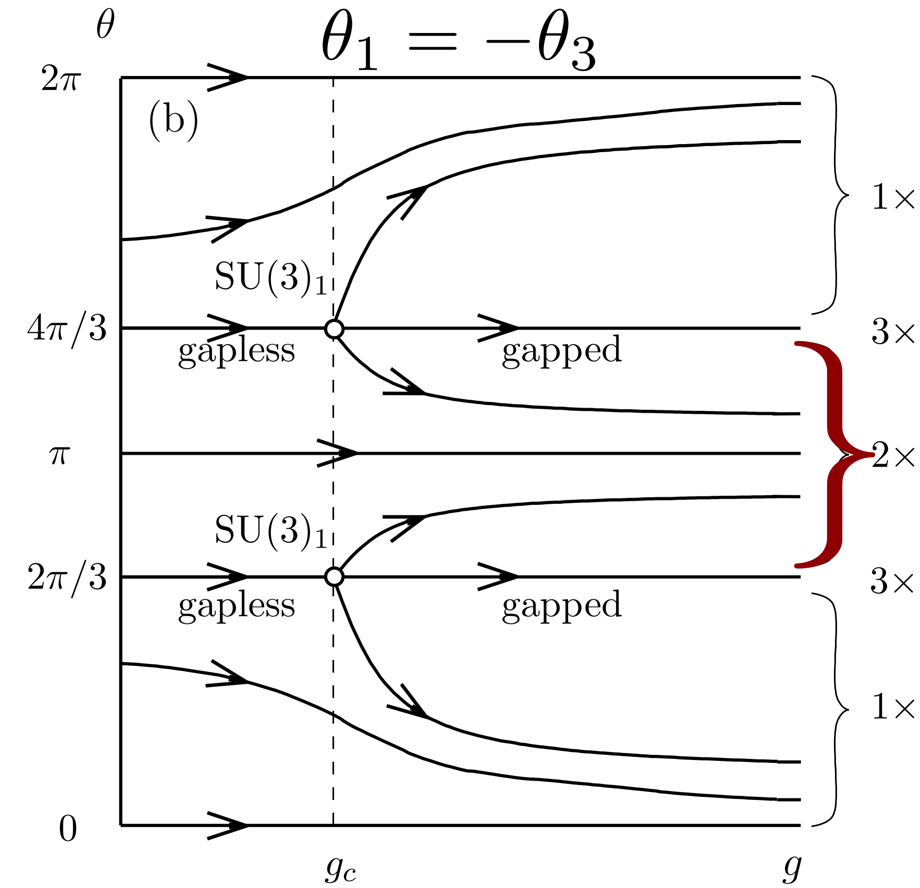


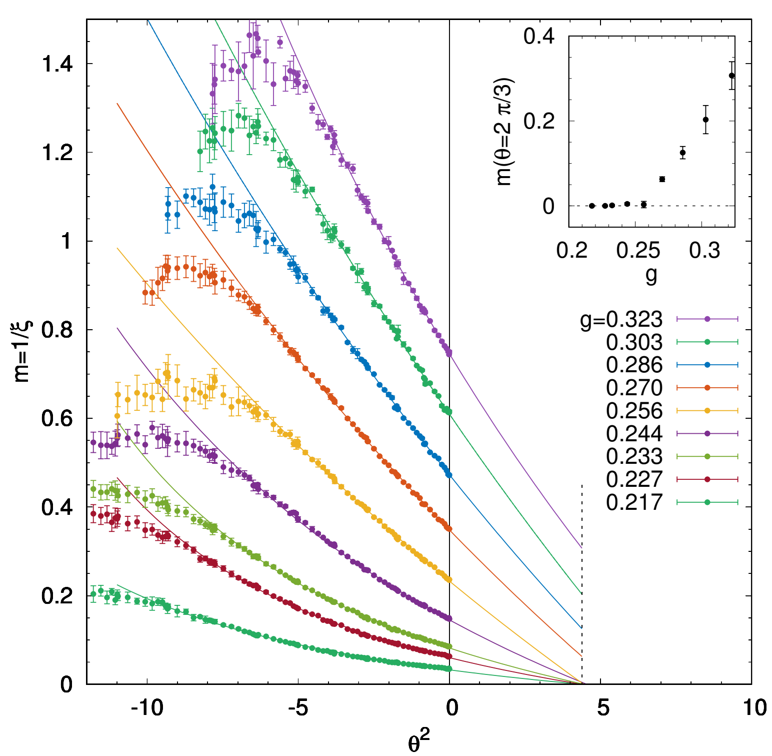
$g > g_c$
WWSWWS
SSWSSW
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$
Affleck (1990)

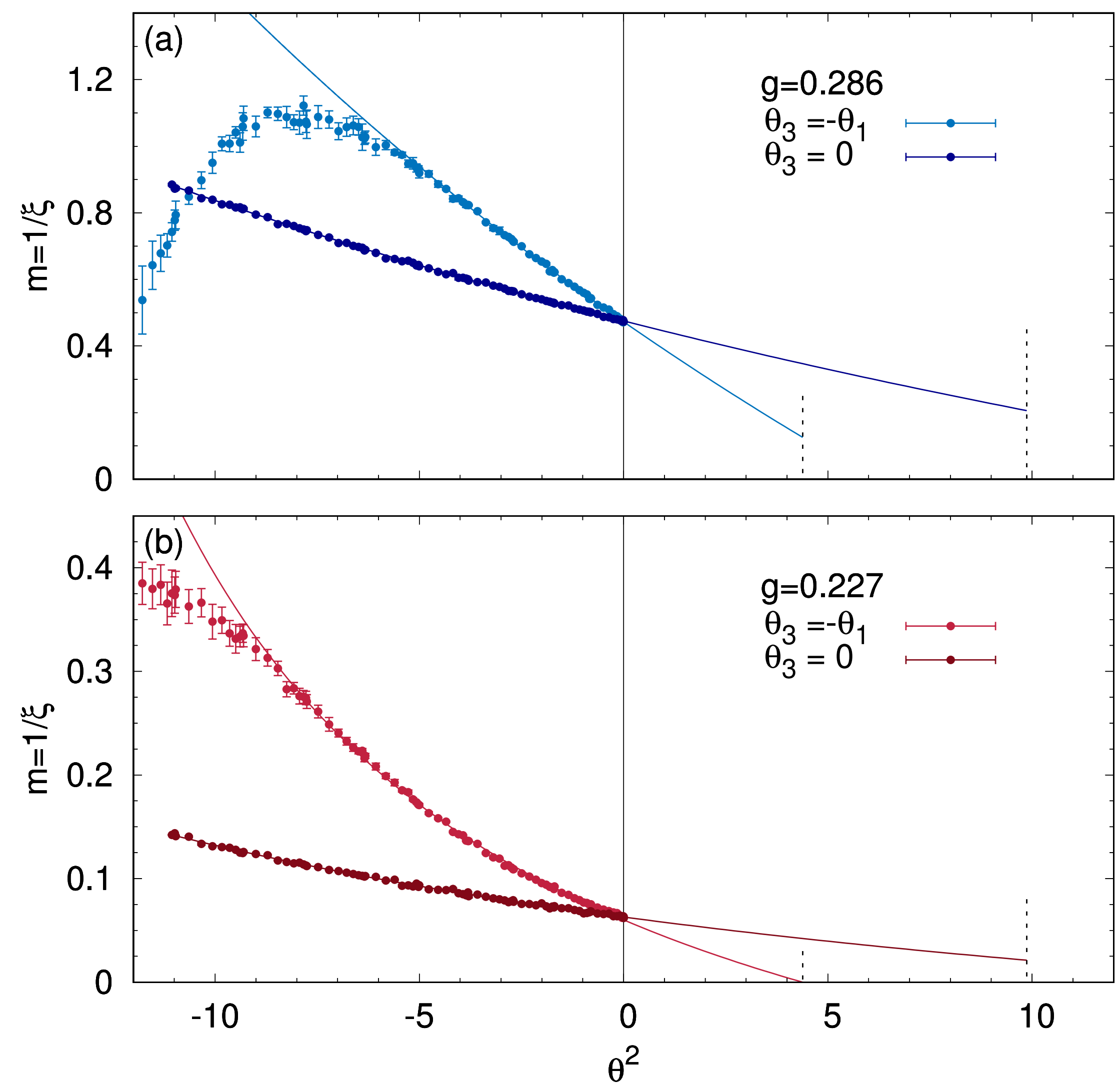


$g > g_c$
SWSSWS
WWSWWS
SWWSWW
SSWSSW
WSSWSS
WSWWSW
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$
Affleck (1990)

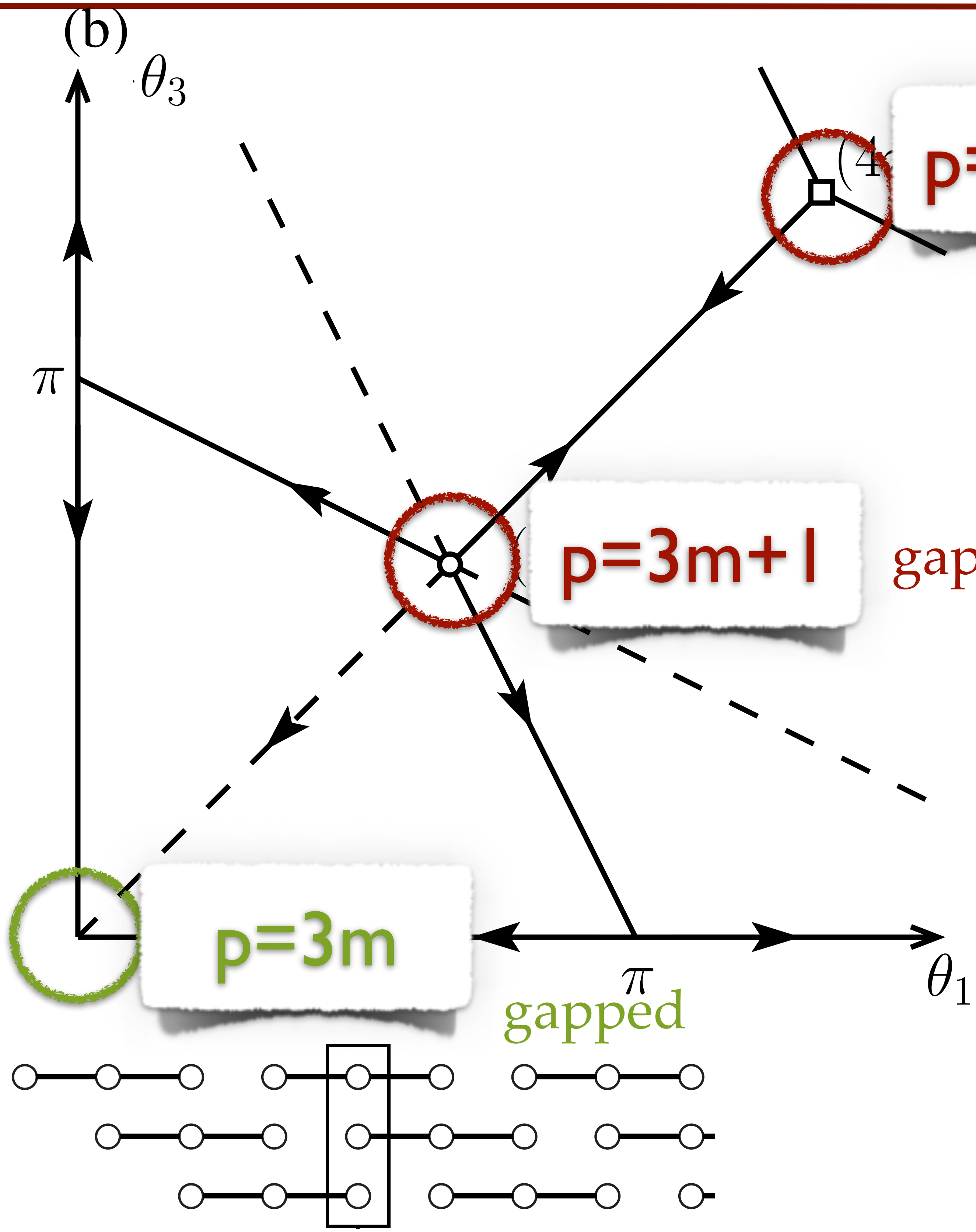

m=0, n=0
m=−1, n
m=0, n=1
m=−1, n


$\theta_1 = -\theta_3$
(b)
θ
2π
$4\pi/3$
π
$2\pi/3$
0
SU(3)$_1$
SU(3)$_1$
gapless
gapped
gapless
gapped
g_c
g
1×
3×
2×
3×
1×

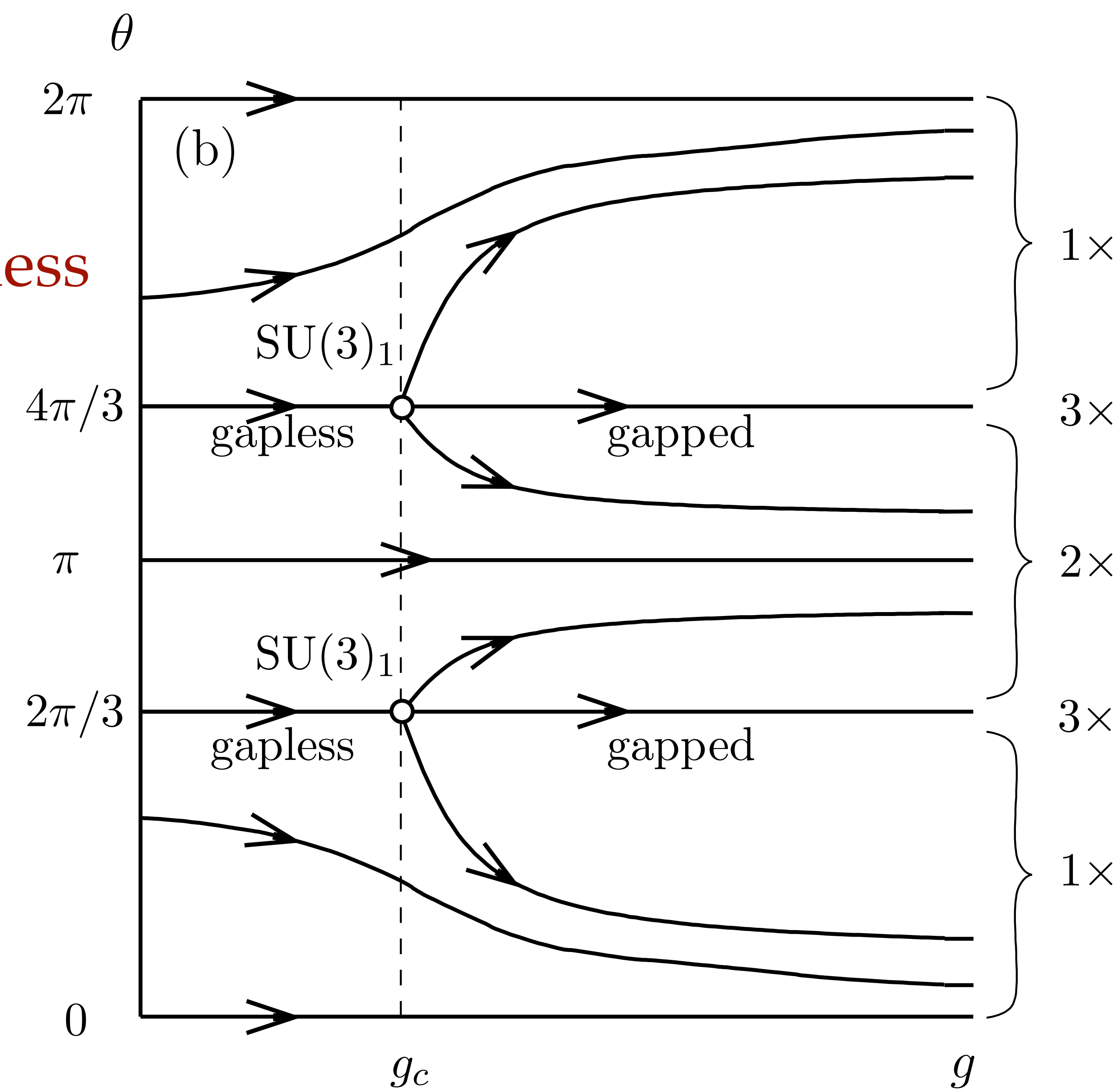

m=1/ξ
θ²
m(θ=2 π/3)
g
g=0.323
0.303
0.286
0.270
0.256
0.244
0.233
0.227
0.217


(a)
g=0.286
θ3 =-θ1
θ3 = 0
m=1/ξ
(b)
g=0.227
θ3 =-θ1
θ3 = 0
m=1/ξ
θ²

What we learned



Greiter, Rachel (2007)



Thank you for your attention