

Generalization of the Haldane conjecture to $SU(3)$ spin chains



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26 February, 2018



Haldane:

The $SU(2)$ symmetric Heisenberg spin chain behaves differently for integer and half-integer spins

based on a mapping of the low energy degrees of freedom into a
1+1 dimensional $O(3)$ nonlinear sigma model

The $SU(3)$ symmetric Heisenberg spin chain behaves differently for $p=3m$ and $p=3m\pm 1$ spins in the fully symmetric representation

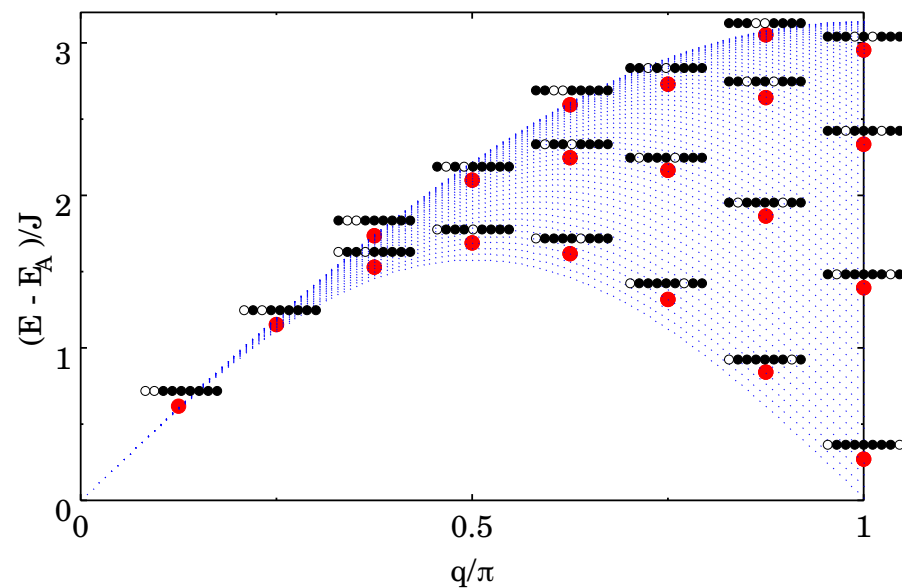
based on a mapping of the low energy degrees of freedom into a
1+1 dimensional $SU(3)/(U(1) \times U(1))$ nonlinear sigma model

Half-integer spins

$S=1/2$

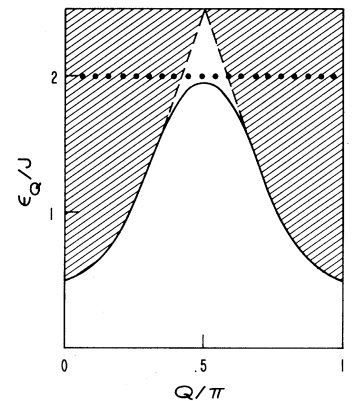
$$H = \sum_j J_1 \mathbf{S}_j \mathbf{S}_{j+1} + J_2 \mathbf{S}_j \mathbf{S}_{j+2}$$

n.n Heisenberg chain
Bethe ansatz solvable,
gapless



J_1 - J_2 Heisenberg chain dimerized,
gapped

$$H_{MG} = \sum_i \mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{2} \sum_i \mathbf{S}_i \mathbf{S}_{i+2}$$



gapless

0.2411

gapped and dimerized

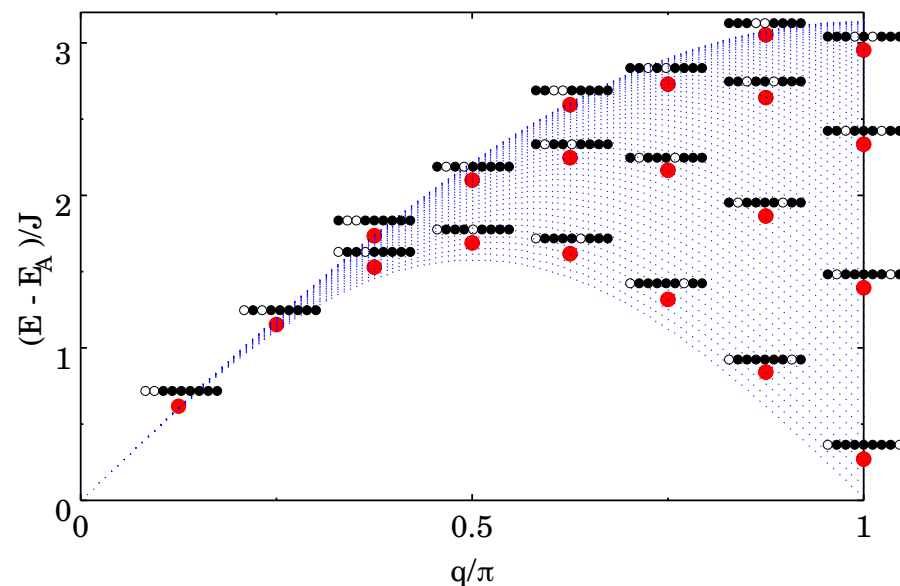
J_2/J_1

Half-integer spins

$S=1/2$

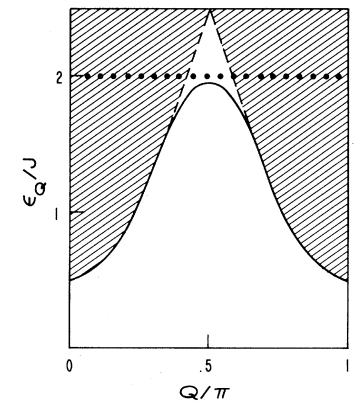
$$H = \sum_j J_1 \mathbf{S}_j \mathbf{S}_{j+1} + J_2 \mathbf{S}_j \mathbf{S}_{j+2}$$

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$$H_{MG} = \sum_i \mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{2} \sum_i \mathbf{S}_i \mathbf{S}_{i+2}$$



gapless

Lieb Schulz-Mattis-Affleck theorem:

half integer spin chain is either gapless or dimerized

Lieb, Schultz, Mattis (1961)
Affleck, Lieb (1986)

dimerized

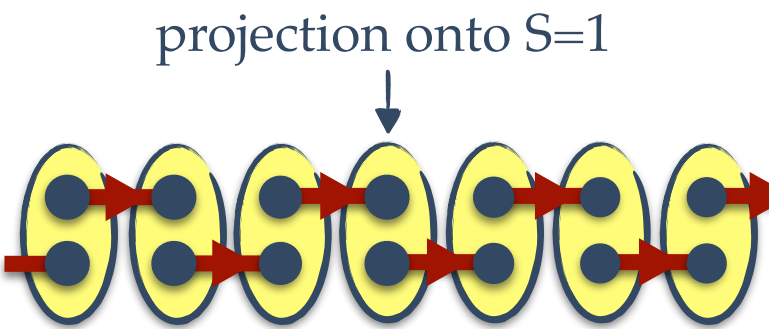
J_2/J_1

Integer spins

S=1

AKLT model

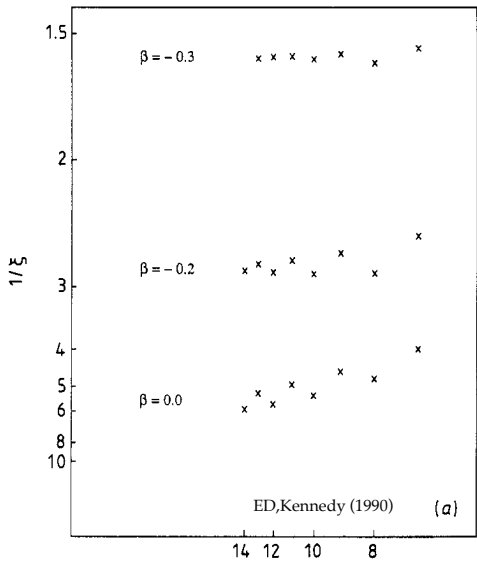
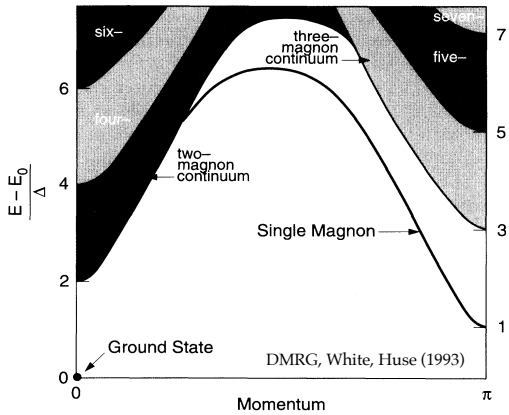
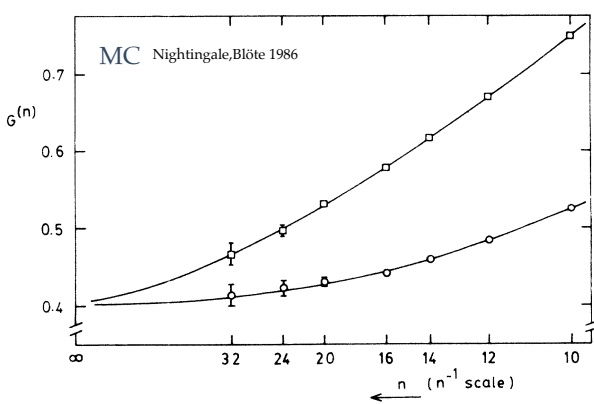
$$H_{\text{AKLT}} = \sum_i \text{Proj} (|\mathbf{S}_i + \mathbf{S}_{i+1}| = 2)$$
$$= \sum_i \frac{1}{3} + \frac{1}{2} \mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{6} (\mathbf{S}_i \mathbf{S}_{i+1})^2$$



Unique gapped ground state

Affleck, Kennedy, Lieb, Tasaki (1988).

nearest neighbour Heisenberg model



Compound	Chemical formula	
	CsNiCl ₃	Johnson (1979)
NENP	Ni(C ₂ H ₈ N ₂) ₂ NO ₂ ClO ₄	Meyer (1982)
NENF	Ni(C ₂ H ₈ N ₂) ₂ NO ₂ PF ₆	
NINO	Ni(C ₃ H ₁₀ N ₂) ₂ NO ₂ ClO ₄	Renald(1988)
NINAZ	Ni(C ₃ H ₁₀ N ₂) ₂ N ₃ ClO ₄	Renald(1990)
NDMAZ	Ni(C ₅ H ₁₄ N ₂) ₂ N ₃ ClO ₄	Yamashita(1995)
NDMAP	Ni(C ₅ H ₁₄ N ₂) ₂ N ₃ PF ₆	Momfort(1996)
TMNIN	(CH ₃) ₄ NNi(NO ₂) ₃	Gadet(1991)
YBANO	Y ₂ BaNiO ₅	Cheong (1992)
	AgVP ₂ S ₆	Lee (1986)

Path integral for SU(2) Heisenberg model

Haldane (1983)

$$\mathcal{Z} = \text{Tr} (e^{-\beta H}) = \text{Tr} (e^{-d\tau H} e^{-d\tau H} \dots e^{-d\tau H})$$

inserting a complete set of states at every time step



1D quantum model $\longrightarrow$ 1+1D classical fields

$$\int \mathcal{D}[\mathbf{n}] \exp \left(- \int d\tau \left[JS^2 \sum_i \mathbf{n}_{i,\tau} \mathbf{n}_{i+1,\tau} + 2S \sum_i \mathbf{n}_{i,\tau} \partial_\tau \mathbf{n}_{i,\tau} \right] \right)$$

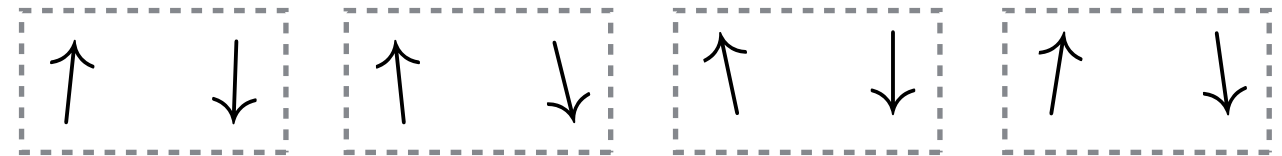
$$\int \mathcal{D}[\mathbf{n}] \exp \left(- \int d\tau \left[JS^2 \sum_i \mathbf{n}_{i,\tau} \mathbf{n}_{i+1,\tau} + 2S \sum_i \mathbf{n}_{i,\tau} \partial_\tau \mathbf{n}_{i,\tau} \right] \right)$$

classical energy

Berry phase (imaginary)

$$\mathbf{n}_{2j} = \frac{S}{\sqrt{1+\mathbf{l}^2}} (\mathbf{m}_j + \mathbf{l}_j)$$

$$\mathbf{n}_{2j+1} = \frac{S}{\sqrt{1+\mathbf{l}^2}} (-\mathbf{m}_j + \mathbf{l}_j)$$



$\mathbf{l}_i$ terms can be integrated out ($\sim$ Legendre transformation)

$$S[\mathbf{m}] = \int dx dt \frac{1}{2g} \left(v(\partial_x \mathbf{m})^2 + \frac{1}{v} (\partial_t \mathbf{m})^2 + \frac{i\theta}{8\pi} \epsilon_{ij} \mathbf{m} \cdot (\partial_i \mathbf{m} \times \partial_j \mathbf{m}) \right)$$

$$g = \frac{2}{S}$$

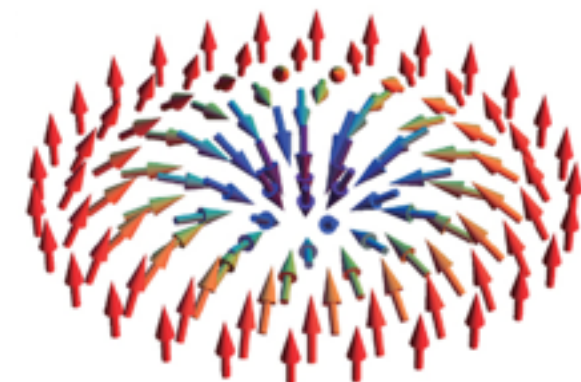
$\theta = 2\pi S$ the rest is an integer topological charge

half-integer spins

O(3) NLSM with topological term

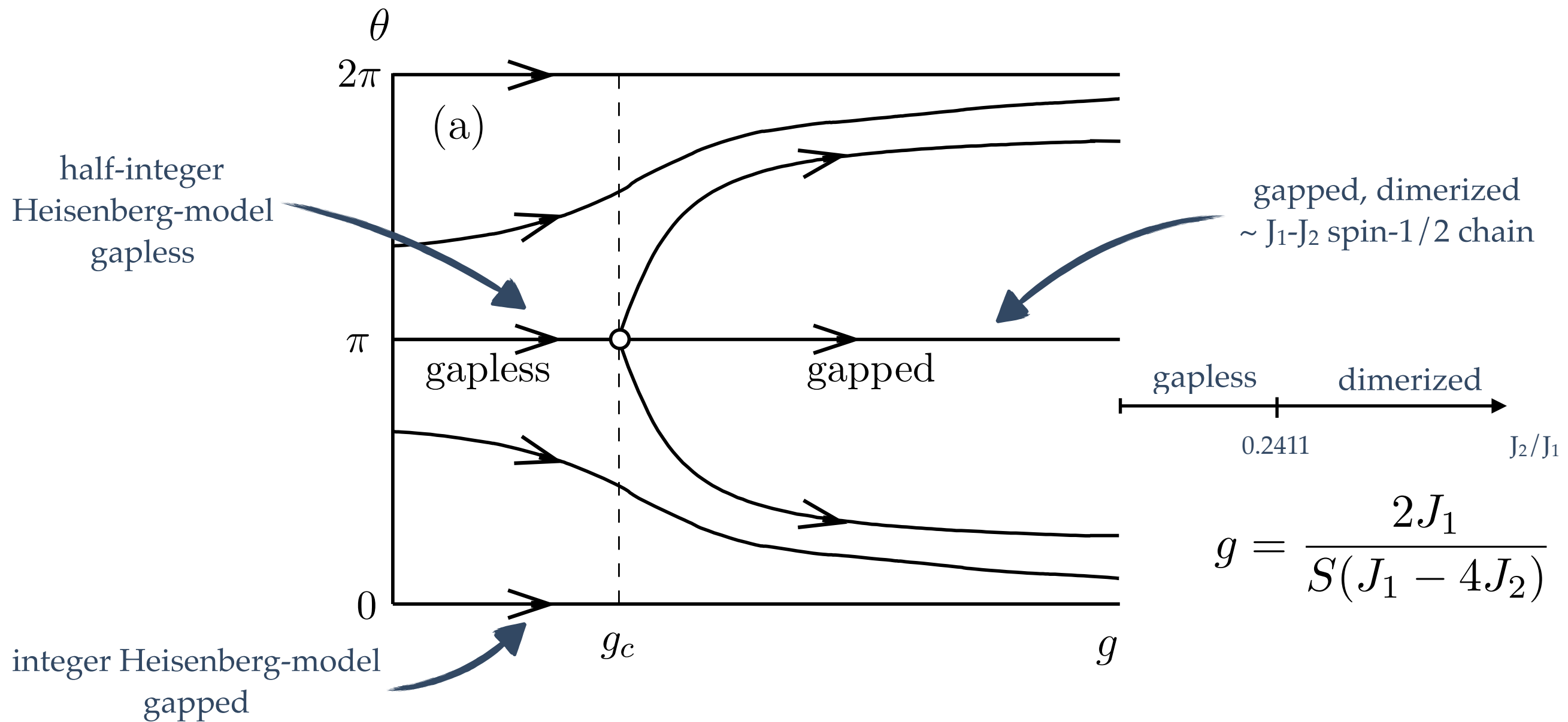
integer spins

O(3) NLSM without topological term



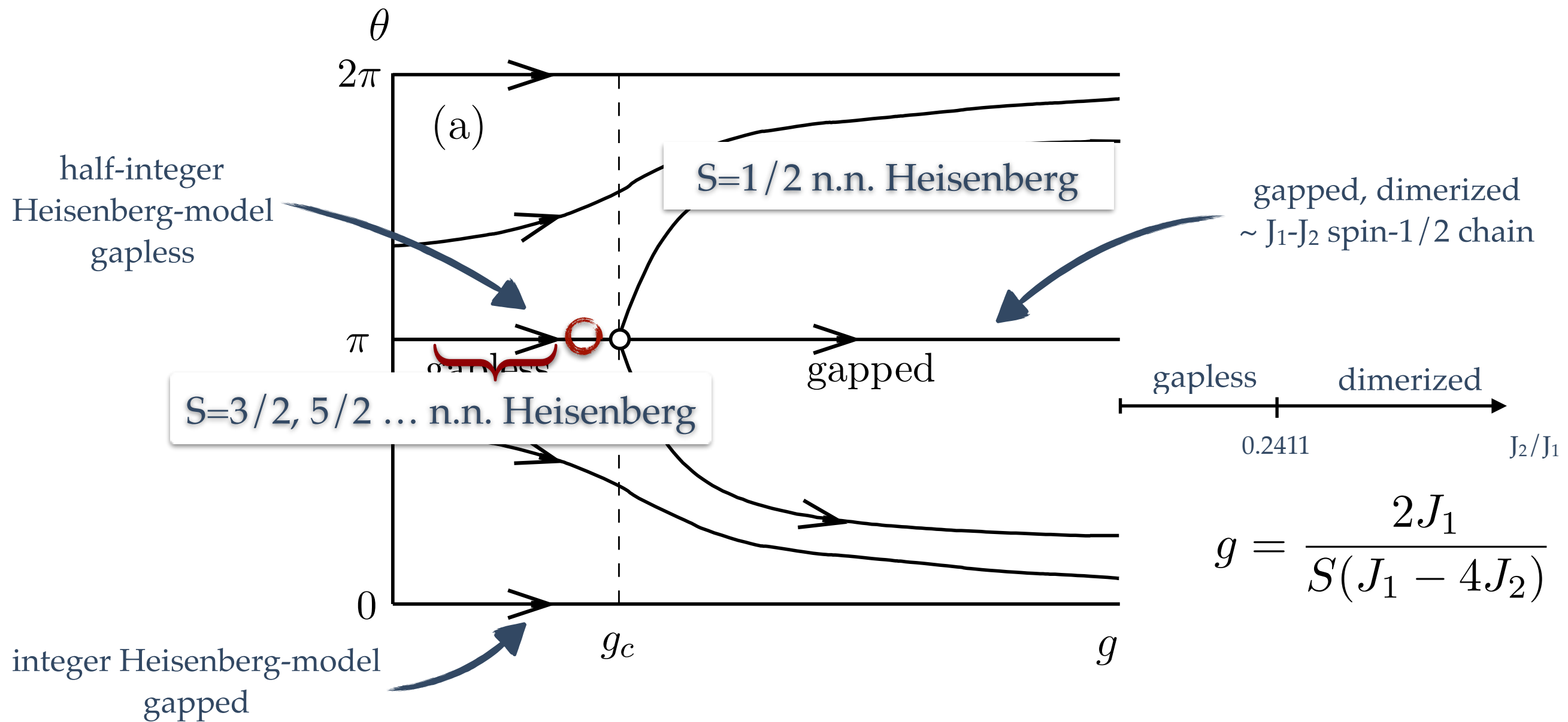
$$S = \int dx d\tau \left(\frac{1}{2g} (\partial_\mu \mathbf{m})^2 + i2\pi S Q \right)$$

$$g = \frac{2}{S}$$



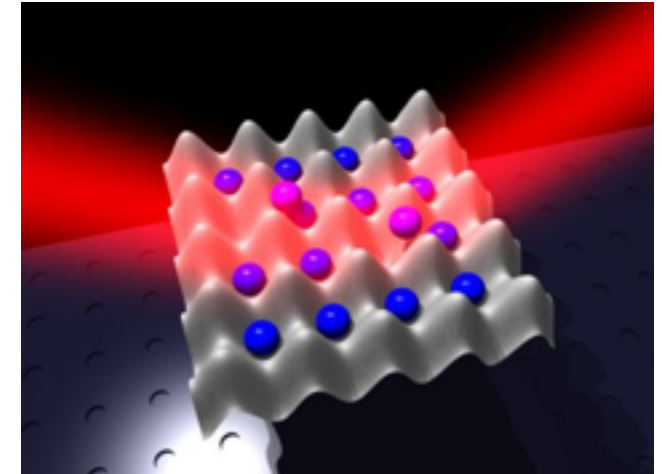
$$S = \int dx d\tau \left(\frac{1}{2g} (\partial_\mu \mathbf{m})^2 + i2\pi S Q \right)$$

$$g = \frac{2}{S}$$



SU(N) symmetry in cold atoms

1 1A 1A 1 H Hydrogen 1.008																		2 2A 2A 4 He Helium 4.003																																																							
3 1A 1A 3 Li Lithium 6.941				4 2A 2A 9 Be Beryllium 9.012														5 3A 3A 10.811 B Boron 10.811				6 4A 4A 12.011 C Carbon 12.011				7 5A 5A 14.007 N Nitrogen 14.007				8 6A 6A 15.999 O Oxygen 15.999				9 7A 7A 18.998 F Fluorine 18.998				10 8A 8A 20.180 Ne Neon 20.180																																			
11 1A 1A 22.990 Na Sodium 22.990				12 2A 2A 24.305 Mg Magnesium 24.305				13 3B 3B 26.982 Al Aluminum 26.982				14 4B 4B 28.086 Si Silicon 28.086				15 5B 5B 30.974 P Phosphorus 30.974				16 6B 6B 32.06 S Sulfur 32.06				17 7B 7B 35.45 Cl Chlorine 35.45				18 8B 8B 39.948 Ar Argon 39.948																																													
19 1A 1A 39.098 K Potassium 39.098				20 2A 2A 40.078 Ca Calcium 40.078				21 3 3 44.956 Sc Scandium 44.956				22 4 4 47.88 Ti Titanium 47.88				23 5 5 50.942 V Vanadium 50.942				24 6 6 51.996 Cr Chromium 51.996				25 7 7 54.938 Mn Manganese 54.938				26 8 8 55.845 Fe Iron 55.845				27 9 9 58.933 Co Cobalt 58.933				28 10 10 58.693 Ni Nickel 58.693				29 11 11 63.546 Cu Copper 63.546				30 12 12 65.38 Zn Zinc 65.38																													
37 1A 1A 85.468 Rb Rubidium 85.468				38 2A 2A 87.62 Sr Strontium 87.62				39 3 3 88.906 Y Yttrium 88.906				40 4 4 91.224 Zr Zirconium 91.224				41 5 5 92.906 Nb Niobium 92.906				42 6 6 95.94 Mo Molybdenum 95.94				43 7 7 98.906 Tc Technetium 98.906				44 8 8 101.07 Ru Ruthenium 101.07				45 9 9 102.905 Rh Rhodium 102.905				46 10 10 106.42 Pd Palladium 106.42				47 11 11 107.868 Ag Silver 107.868				48 12 12 112.414 Cd Cadmium 112.414																													
55 1A 1A 132.905 Cs Cesium 132.905				56 2A 2A 137.327 Ba Barium 137.327				57-71 3 3 138.905 La Lanthanum 138.905				72 4 4 175.4 Hf Hafnium 175.4				73 5 5 180.948 Ta Tantalum 180.948				74 6 6 183.84 W Tungsten 183.84				75 7 7 186.207 Re Rhenium 186.207				76 8 8 190.23 Os Osmium 190.23				77 9 9 192.22 Ir Iridium 192.22				78 10 10 195.084 Pt Platinum 195.084				79 11 11 196.967 Au Gold 196.967				80 12 12 200.59 Hg Mercury 200.59																													
87 1A 1A 223 Fr Francium 223				88 2A 2A 226 Ra Radium 226				89-103 3 3 227 Ac Actinium 227				104 4 4 261.101 Rf Rutherfordium 261.101				105 5 5 262.101 Db Dubnium 262.101				106 6 6 263.101 Sg Seaborgium 263.101				107 7 7 264.101 Bh Bohrium 264.101				108 8 8 265.101 Hs Hassium 265.101				109 9 9 266.101 Mt Meitnerium 266.101				110 10 10 267.101 Ds Darmstadtium 267.101				111 11 11 268.101 Rg Roentgenium 268.101				112 12 12 269.101 Cn Copernicium 269.101																													
57 3 3 138.905 La Lanthanum 138.905																		58 4 4 140.12 Ce Cerium 140.12				59 5 5 140.908 Pr Praseodymium 140.908				60 6 6 144.24 Nd Neodymium 144.24				61 7 7 144.913 Pm Promethium 144.913				62 8 8 150.36 Sm Samarium 150.36				63 9 9 151.964 Eu Europium 151.964				64 10 10 157.25 Gd Gadolinium 157.25				65 11 11 158.925 Tb Terbium 158.925				66 12 12 162.50 Dy Dysprosium 162.50				67 13 13 164.930 Ho Holmium 164.930				68 14 14 167.259 Er Erbium 167.259				69 15 15 168.934 Tm Thulium 168.934				70 16 16 173.054 Yb Ytterbium 173.054				71 17 17 174.967 Lu Lutetium 174.967			
89 3 3 227.028 Ac Actinium 227.028																		90 4 4 232.038 Th Thorium 232.038				91 5 5 231.036 Pa Protactinium 231.036				92 6 6 238.029 U Uranium 238.029				93 7 7 237.048 Np Neptunium 237.048				94 8 8 244.064 Pu Plutonium 244.064				95 9 9 243.061 Am Americium 243.061				96 10 10 247.07 Cm Curium 247.07				97 11 11 247.07 Bk Berkelium 247.07				98 12 12 251.08 Cf Californium 251.08				99 13 13 252.083 Es Einsteinium 252.083				100 14 14 257.10 Fm Fermium 257.10				101 15 15 258.10 Md Mendelevium 258.10				102 16 16 259.10 No Nobelium 259.10				103 17 17 262 Lr Lawrencium 262			



hopping process independent of nuclear spin
($N=2F+1$ states)

SU(N) symmetric Heisenberg model in the Mott insulating phase

Gorshkov et al., Nature Physics, 6, 289 - 295 (2010)

Cazalilla et al., New J. Phys., 11, 103033 (2009)

Cazalilla, Rey, Rep. Prog. Phys. 77 124401 (2014)

SU(3) spin chains - fundamental spin

fundamental spin: 3 colors

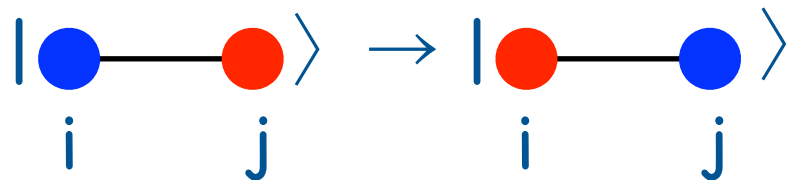


$$H = J_1 \sum_i \cos \theta (\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + \sin \theta (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2$$

$$S_\beta^\alpha \sim c_\alpha^\dagger c_\beta$$

SU(3) symmetric Heisenberg interaction

$$S_\beta^\alpha(i) S_\alpha^\beta(j) = c_{i,\alpha}^\dagger c_{i,\beta} c_{j,\beta}^\dagger c_{j,\alpha}$$



Higher spins

$$3 \times 3 = 3 + 6$$

$$\square \otimes \square = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \oplus \square\square$$

$$3 \times 3 \times 3 = 1 + 2 \times 8 + 10$$

$$\square \otimes \square \otimes \square = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \oplus 2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \square\square\square$$

singlet

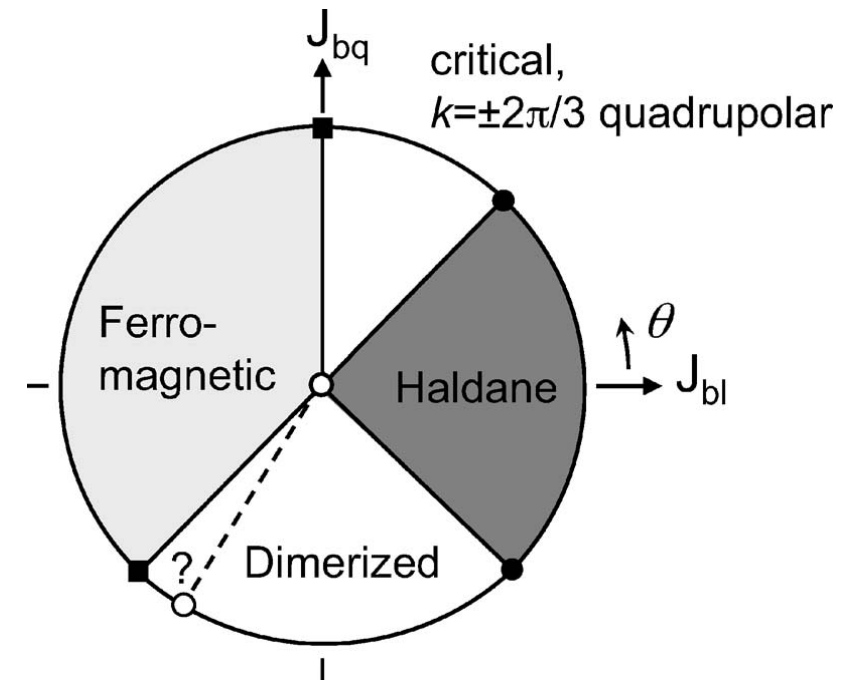
completely symmetric



fully symmetric representation

p boxes:

p symmetrized fundamental spins



$$(\mathbf{S}_i \mathbf{S}_j) + (\mathbf{S}_i \mathbf{S}_j)^2 - 1 = \mathcal{P}_{ij}^{(3)}$$

Nature of GS?

Lieb Schulz-Mattis-Affleck theorem:

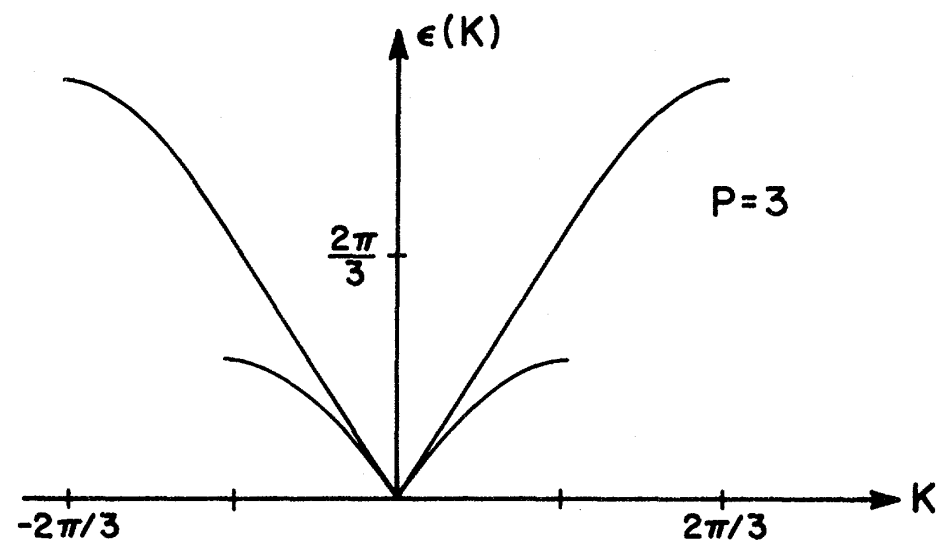
$p=3m\pm 1$ chains are either gapless or trimerized

Lieb, Schultz, Mattis (1961)

Affleck, Lieb (1986)

$p=1$

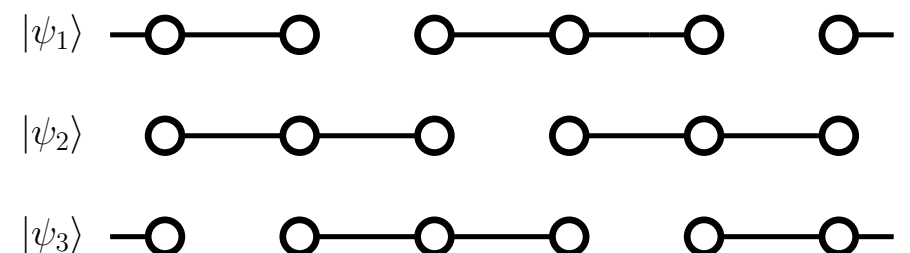
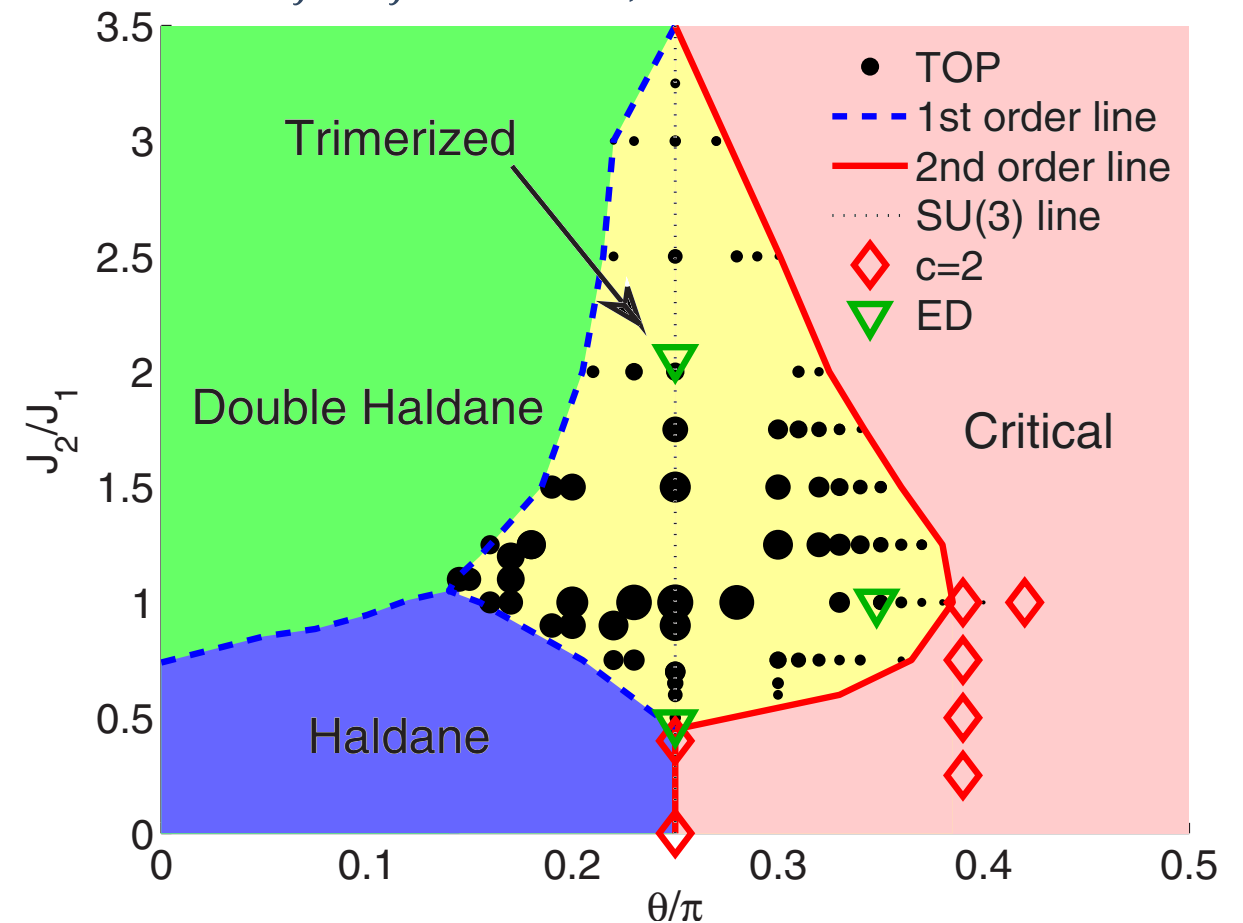
n.n Heisenberg point, Bethe solvable



$$H = J_1 \sum_i \cos \theta(\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + \sin \theta(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2$$

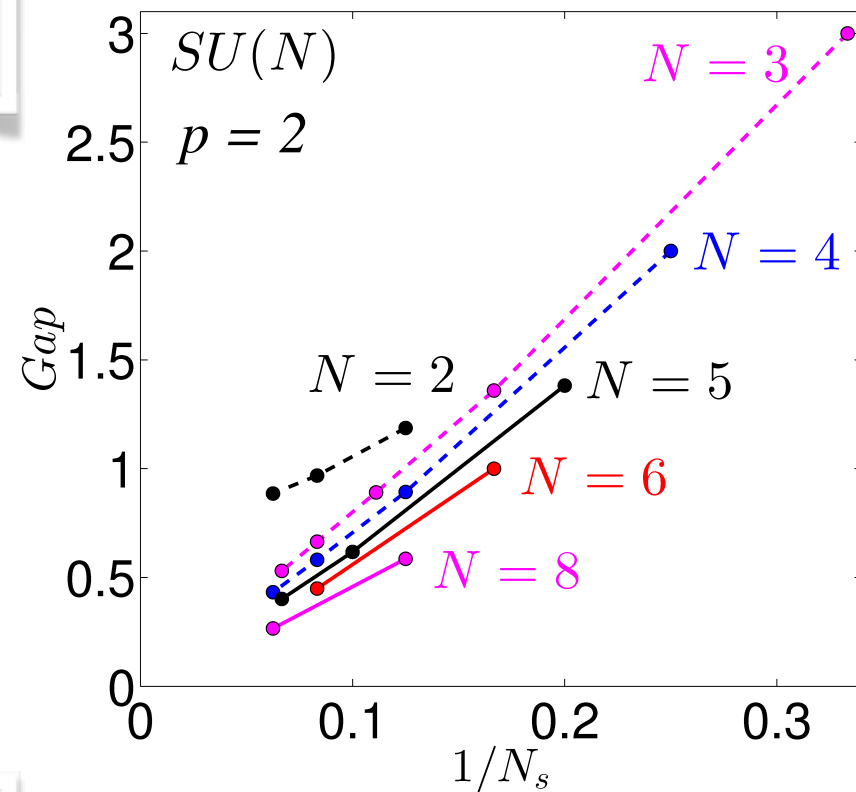
$$+ J_2 \sum_i \cos \theta(\mathbf{S}_i \cdot \mathbf{S}_{i+2}) + \sin \theta(\mathbf{S}_i \cdot \mathbf{S}_{i+2})^2.$$

$J_1 - J_2$ model, trimerized

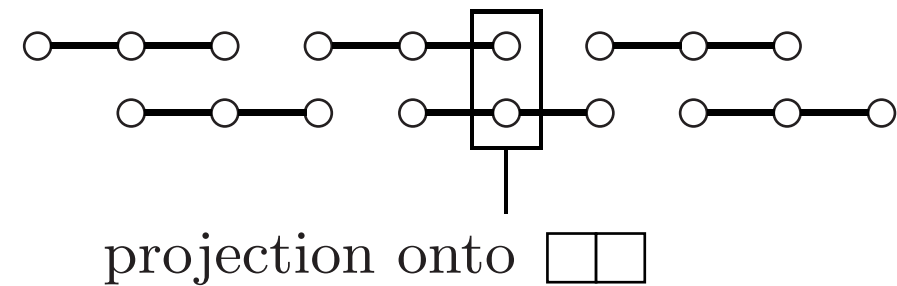


Nature of GS?

$p=2$

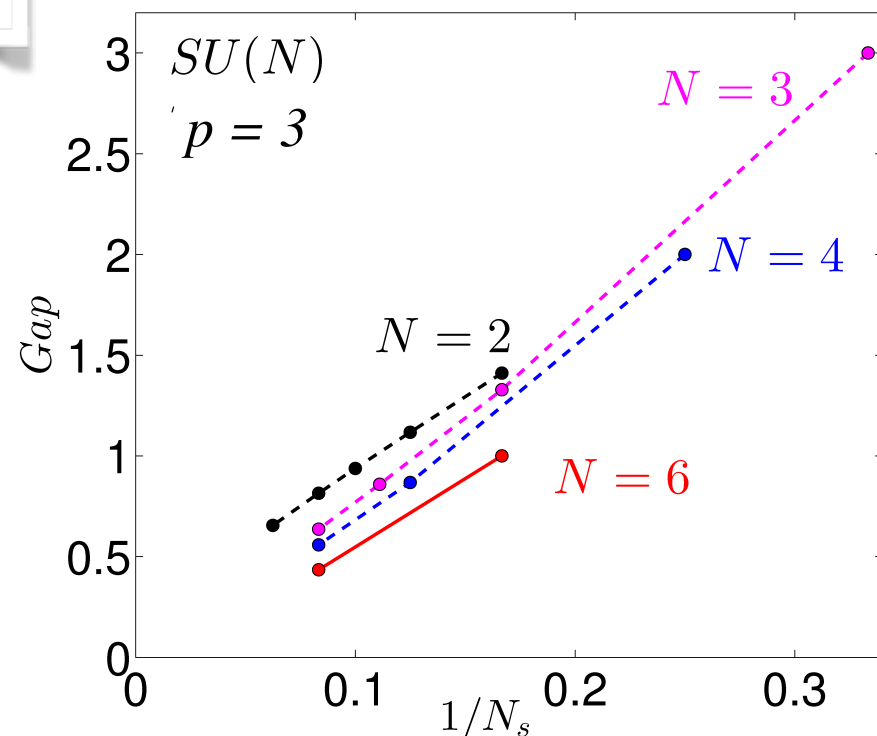


possible trimerized construction

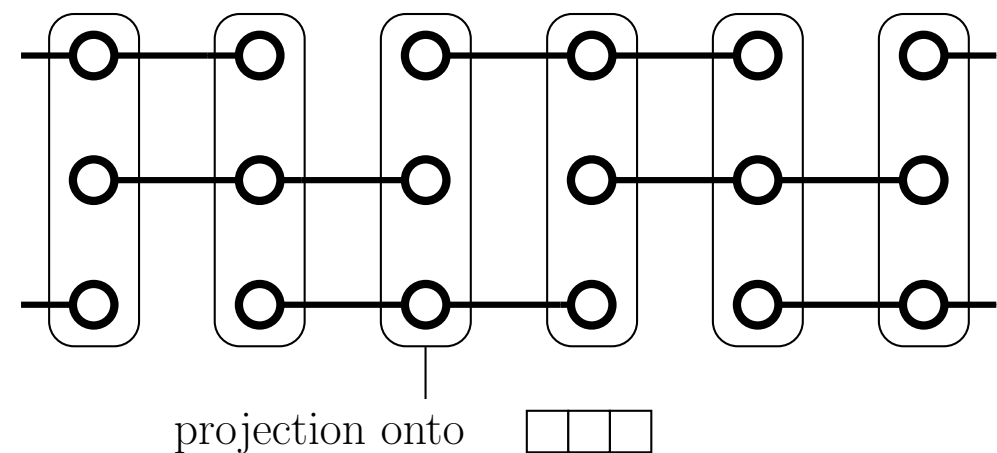


Lieb-Schultz-Mattis-Affleck ✓

$p=3$



$p=3$: AKLT construction



Greiter, Rachel (2007)

$$H_{10\text{VBS}} = \sum_{i=1}^N ((J_i J_{i+1})^2 + 5J_i J_{i+1} + 6),$$

is the $p=3$ Heisenberg model gapped?

SU(3) Heisenberg model & path integral

$$\mathcal{H} = \sum_{\langle i,j \rangle} S_{\beta}^{\alpha}(i) S_{\alpha}^{\beta}(j)$$

1	2	...	p
---	---	-----	---

 $\rightarrow$ p bosons per site

$$S_{\beta}^{\alpha}(i) = b_{i,\alpha}^{\dagger} b_{i,\beta}$$

$$\mathcal{H} = \sum_{\langle i,j \rangle} b_{i,\alpha}^{\dagger} b_{i,\beta} b_{j,\beta}^{\dagger} b_{j,\alpha}$$

Path integral approach

$$\mathcal{Z} = \text{Tr} (e^{-\beta H}) = \text{Tr} (e^{-d\tau H} e^{-d\tau H} \dots e^{-d\tau H})$$

spin coherent state for the p-box case:

$$|\vec{\Phi}\rangle = \frac{1}{\sqrt{p!}} (\Phi^{(1)} b_1^{\dagger} + \Phi^{(2)} b_2^{\dagger} + \Phi^{(3)} b_3^{\dagger})^p |0\rangle \quad \text{with} \quad |\vec{\Phi}|^2 = 1 \quad \text{3D complex fields}$$

$$\mathcal{Z} = \int \mathcal{D}[\Phi] \exp \left(- \int d\tau \left[p^2 J \sum_i \left| \vec{\Phi}_{i,\tau}^* \vec{\Phi}_{i+1,\tau} \right|^2 + p \sum_i \vec{\Phi}_{i,\tau}^* \partial_{\tau} \vec{\Phi}_{i,\tau} \right] \right)$$

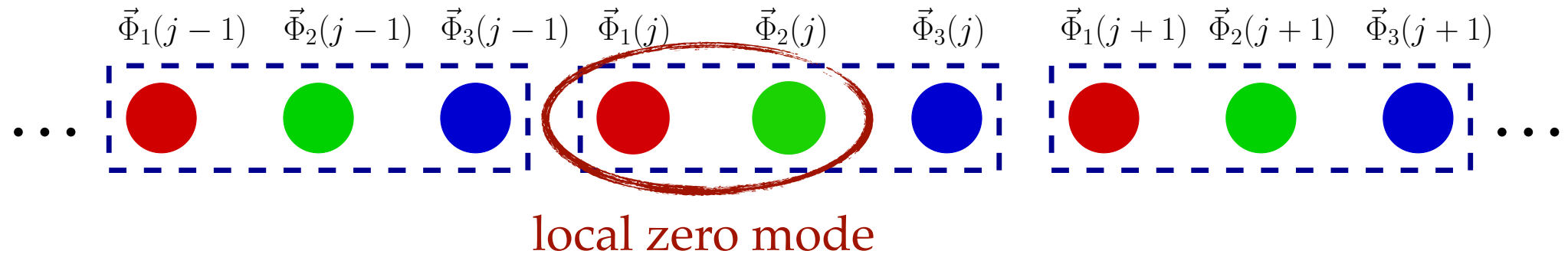
classical energy

Berry phase (imaginary)

SU(3) case, coherent state path integral

$$\mathcal{Z} = \int \mathcal{D}[\Phi] \exp \left(- \int d\tau \left[p^2 J \sum_i \left| \vec{\Phi}_{i,\tau}^* \vec{\Phi}_{i+1,\tau} \right|^2 + p \sum_i \vec{\Phi}_{i,\tau}^* \partial_\tau \vec{\Phi}_{i,\tau} \right] \right)$$

classical ground state



short range **quantum fluctuations select 3-sublattice order** (order by disorder)
effective further neighbour (J_2, J_3) interactions

Low energy fluctuations in path integral:

non-orthogonality of the spin states in the unit cell

3 orthogonal states
in the unit cell

$$\begin{pmatrix} \vec{\Phi}_1^T(3j, \tau) \\ \vec{\Phi}_2^T(3j+1, \tau) \\ \vec{\Phi}_3^T(3j+2, \tau) \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \frac{a^2}{p^2}(|L_{12}|^2 + |L_{13}|^2)} & \frac{a}{p}L_{12} & \frac{a}{p}L_{13} \\ \frac{a}{p}L_{12}^* & \sqrt{1 - \frac{a^2}{p^2}(|L_{12}|^2 + |L_{23}|^2)} & \frac{a}{p}L_{23} \\ \frac{a}{p}L_{13}^* & \frac{a}{p}L_{23}^* & \sqrt{1 - \frac{a^2}{p^2}(|L_{13}|^2 + |L_{23}|^2)} \end{pmatrix} \begin{pmatrix} \vec{\phi}_1^T(j, \tau) \\ \vec{\phi}_2^T(j, \tau) \\ \vec{\phi}_3^T(j, \tau) \end{pmatrix}$$

L variables can be integrated out

$$S = \int dx d\tau \left(\sum_{n=1}^3 \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i \sum_{n=1}^3 \frac{\theta_n}{2\pi i} \varepsilon_{\mu\nu} \left(\partial_\mu \vec{\phi}_n \cdot \partial_\nu \vec{\phi}_n^* \right) \right)$$

3 fields are orthogonal
from a unitary matrix

$$\text{SU}(3)/(\text{U}(1) \times \text{U}(1))$$

$$\begin{pmatrix} \vec{\phi}_1^T(j, \tau) \\ \vec{\phi}_2(j, \tau) \\ \vec{\phi}_3(j, \tau) \end{pmatrix}$$

$$g \sim \frac{1}{\sqrt{p}}$$

topological term

$$i \sum_{n=1}^3 \theta_n Q_n$$

$$Q_1 + Q_2 + Q_3 = 0$$

$$\theta_1 = -\theta_3 = p \frac{2\pi}{3}$$

trivial for $p = 3m$,
nontrivial otherwise

SU(3) spin chain with $p=3m$

NLSM without topological term

SU(3) spin chain with $p=3m \pm 1$

NLSM with topological term

Monte Carlo simulation

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta (Q_1 - Q_3) \quad \theta = p \frac{2\pi}{3}$$

imaginary term in action for real θ

we can make simulations for imaginary $\theta = i\vartheta$

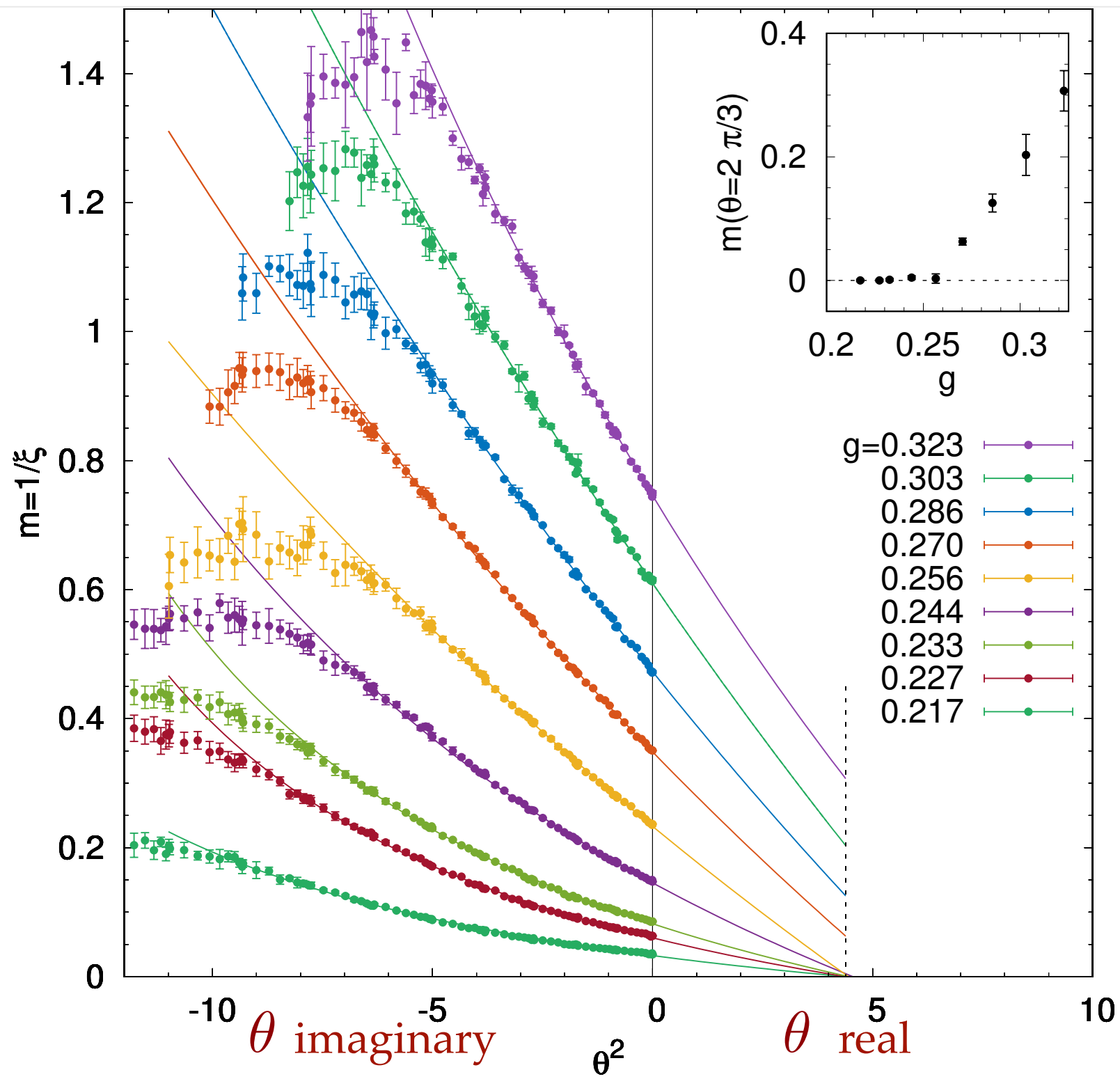
$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) - \vartheta (Q_1 - Q_3)$$

Monte Carlo simulation

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta (Q_1 - Q_3) \quad \theta = p \frac{2\pi}{3}$$

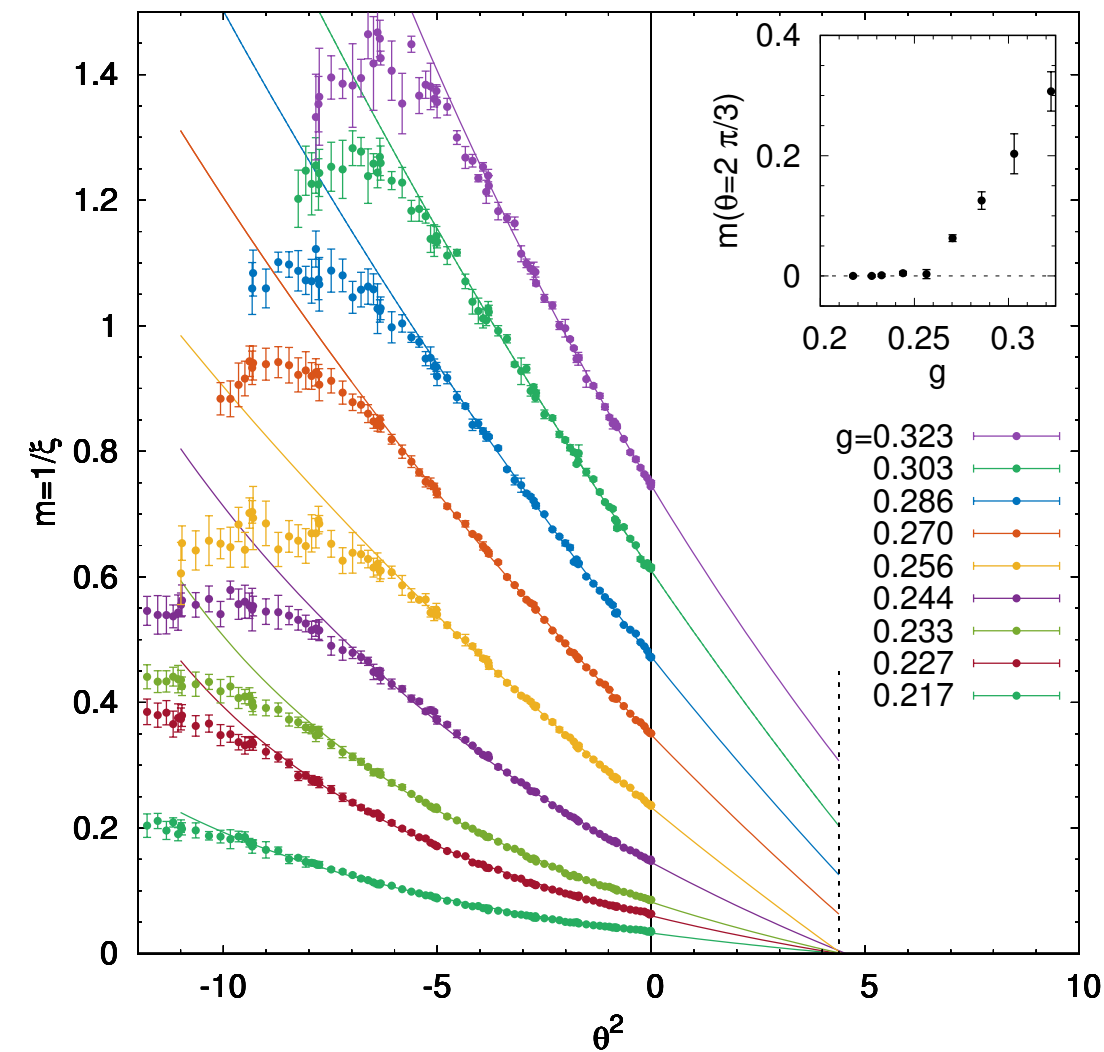
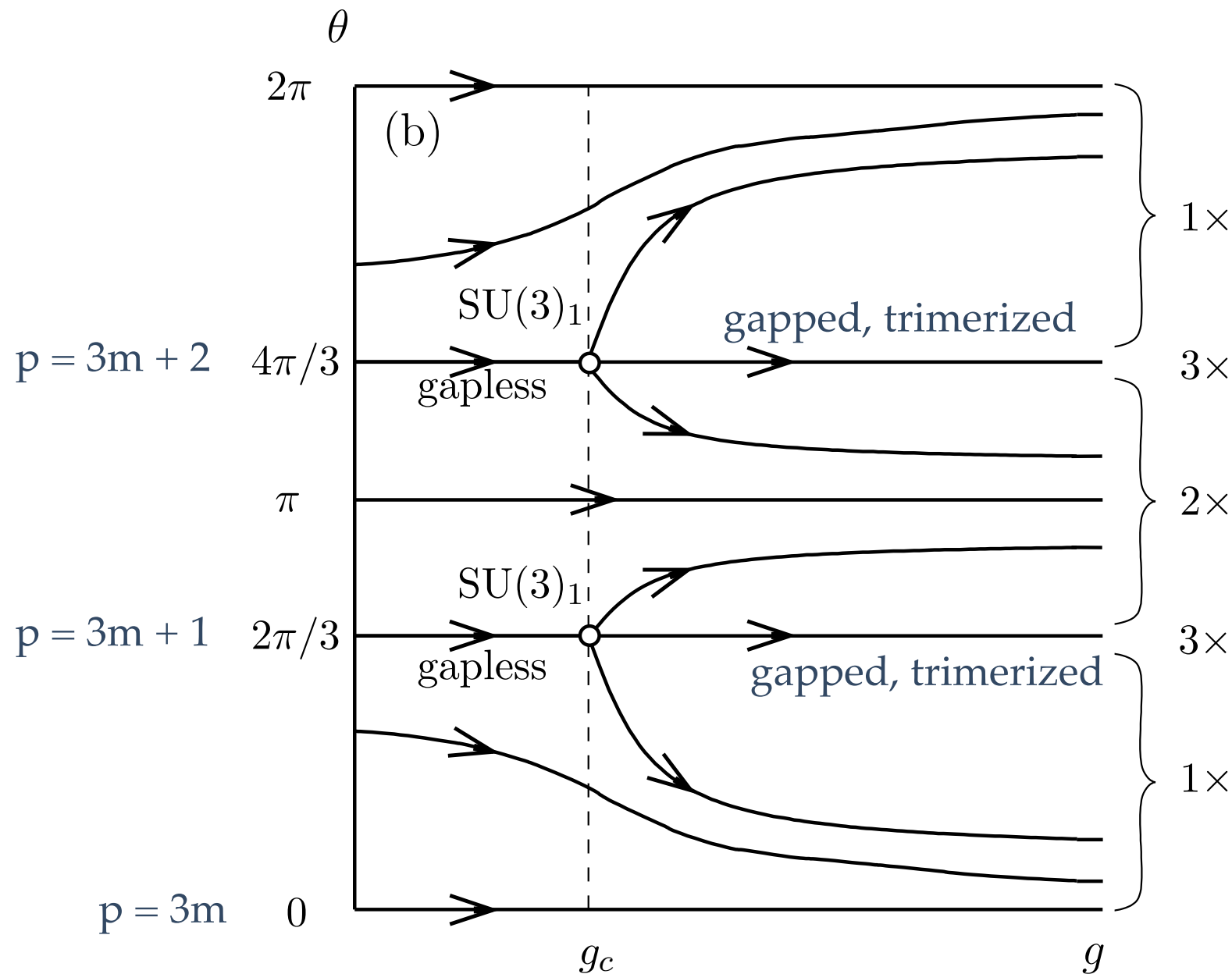
imaginary term
we can make sin

$$S = \int dx d\tau \frac{1}{g}$$

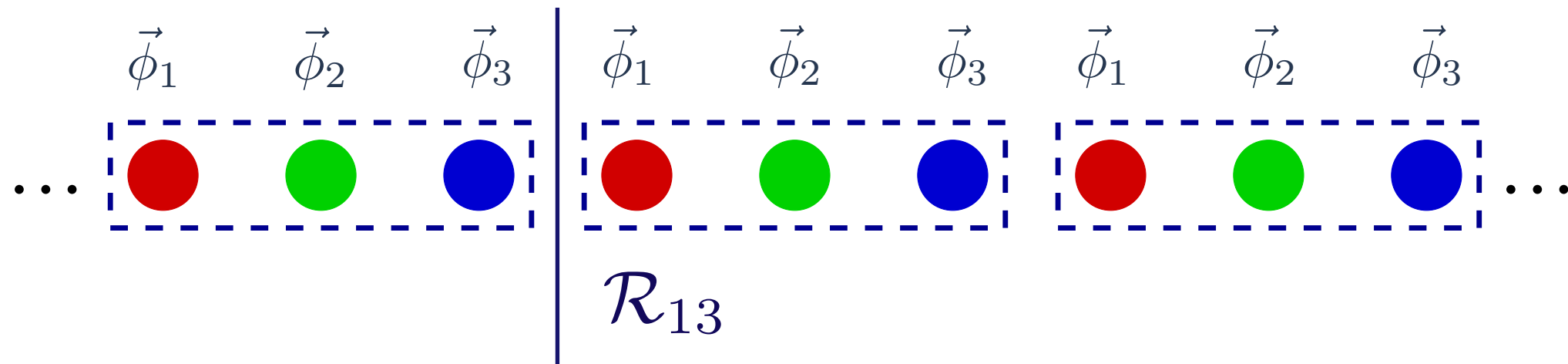


Phase diagram

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta (Q_1 - Q_3) \quad \theta = p \frac{2\pi}{3}$$



Symmetries of the field theory



$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i\theta (Q_1 - Q_3) \quad \theta = p \frac{2\pi}{3}$$

Translation by one site: $\vec{\phi}_1 \rightarrow \vec{\phi}_2 \rightarrow \vec{\phi}_3 \rightarrow \vec{\phi}_1$ $\mathbb{Z}_3$ symmetry in NLSM

$$ip \frac{2\pi}{3} (Q_1 - Q_3) \rightarrow ip \frac{2\pi}{3} (Q_2 - Q_1) = ip \frac{2\pi}{3} (Q_1 - Q_3) + ip \frac{2\pi}{3} (Q_1 + Q_2 + Q_3) - i3p \frac{2\pi}{3} Q_1.$$

Mirror symmetry: $\vec{\phi}_{1(3)}(x, \tau) \leftrightarrow \vec{\phi}_{3(1)}(-x, \tau)$ $\vec{\phi}_2(x, \tau) \leftrightarrow \vec{\phi}_2(-x, \tau)$

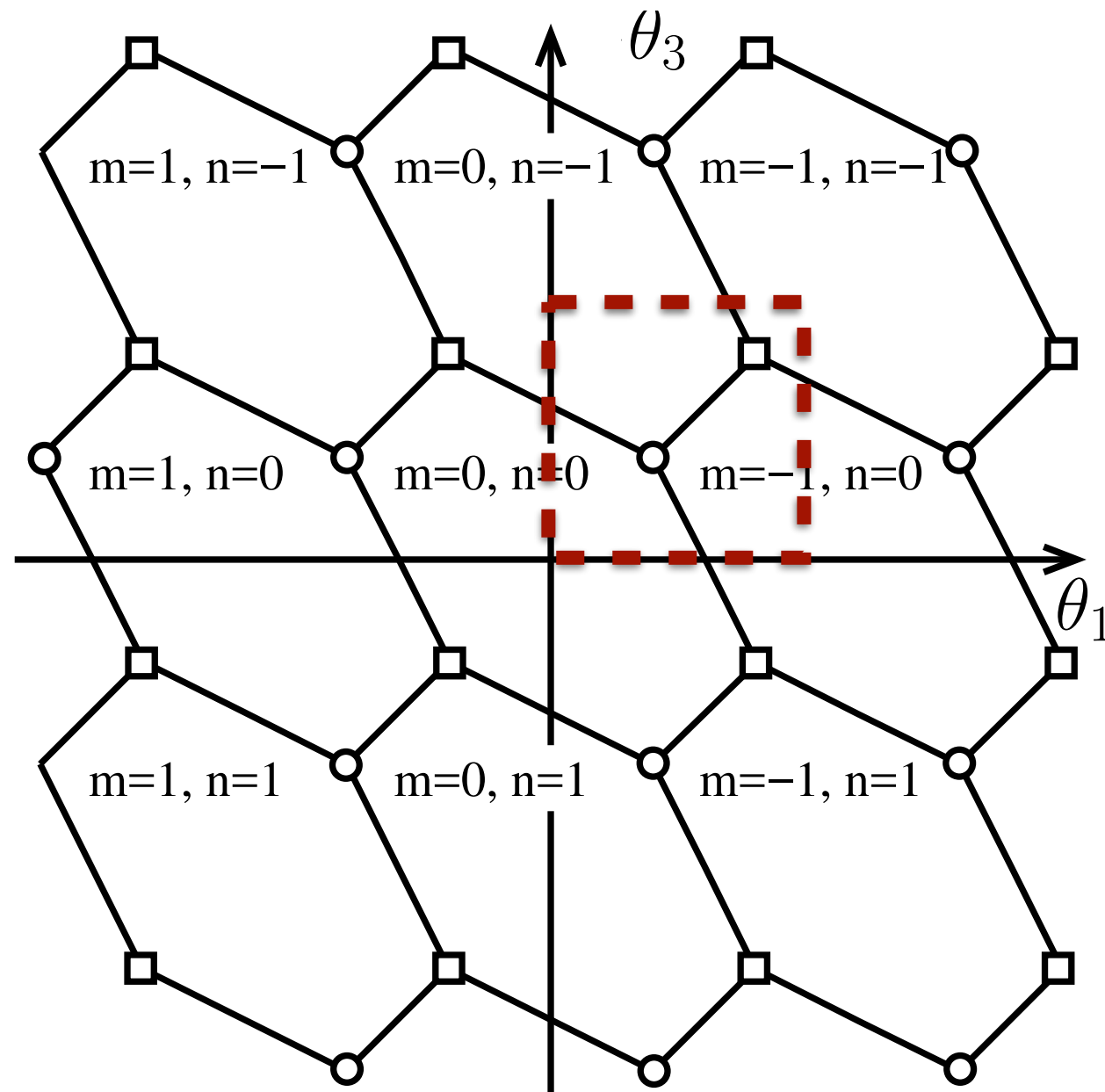
$$\theta(Q_1 - Q_3) \rightarrow -\theta(Q_3 - Q_1),$$

$\mathbb{Z}_2$ symmetry in NLSM

General phase diagram: $\theta_1 \neq -\theta_3$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i (\theta_1 Q_1 - \theta_3 Q_3)$$

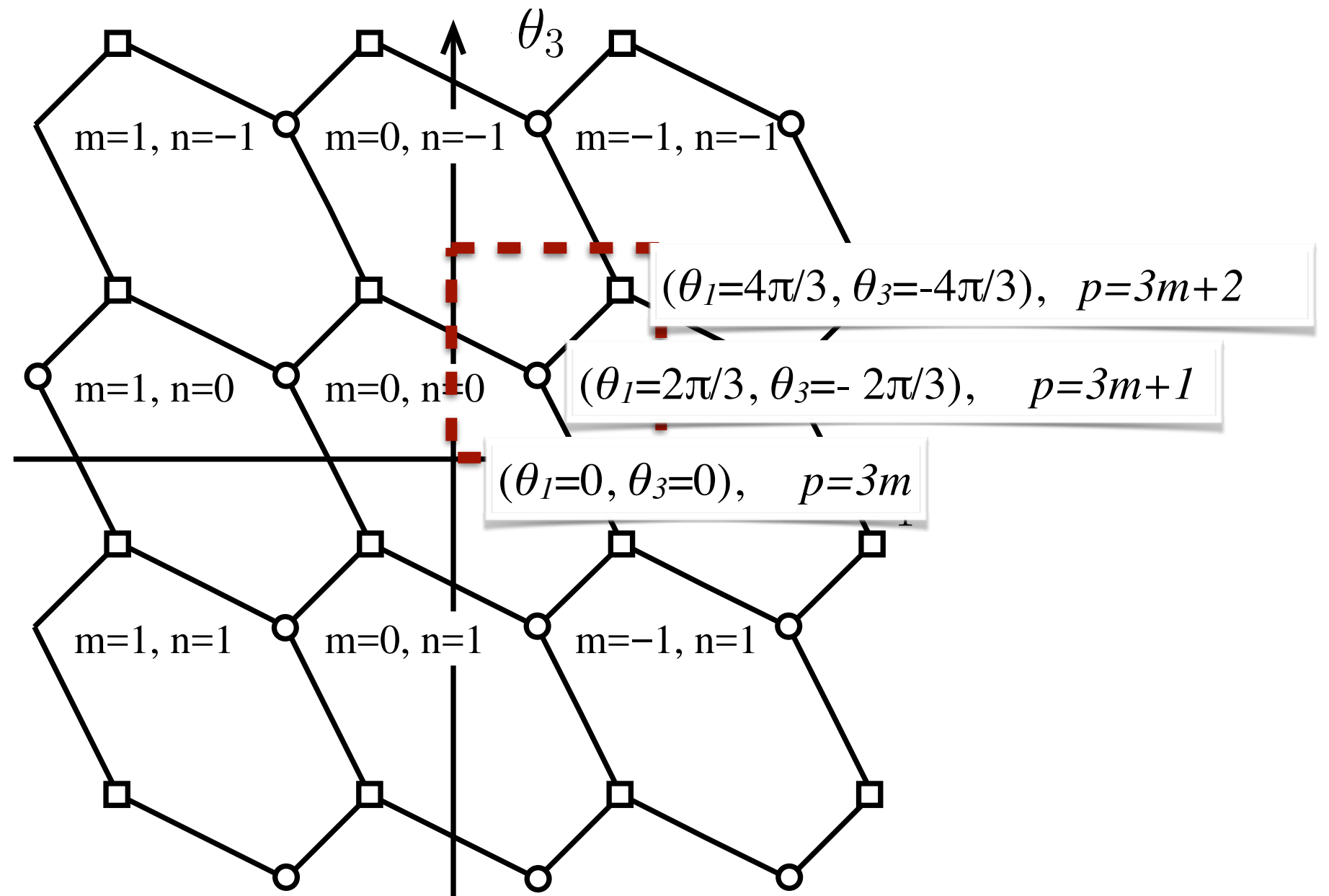
$g \rightarrow \infty$ is integrable, $S = i (\theta_1 Q_1 - \theta_3 Q_3)$



General phase diagram: $\theta_1 \neq -\theta_3$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i (\theta_1 Q_1 - \theta_3 Q_3)$$

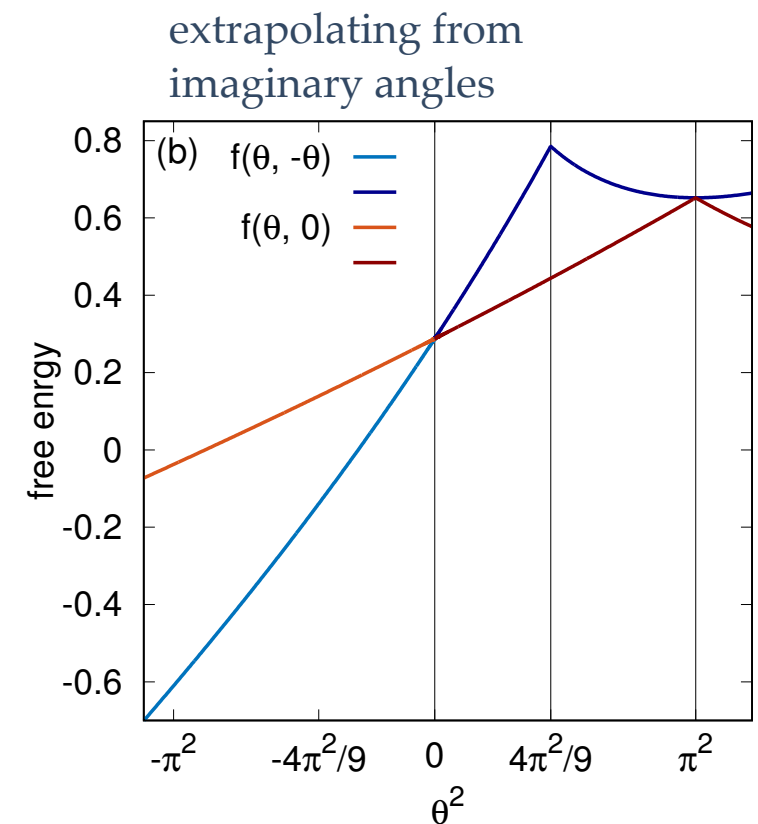
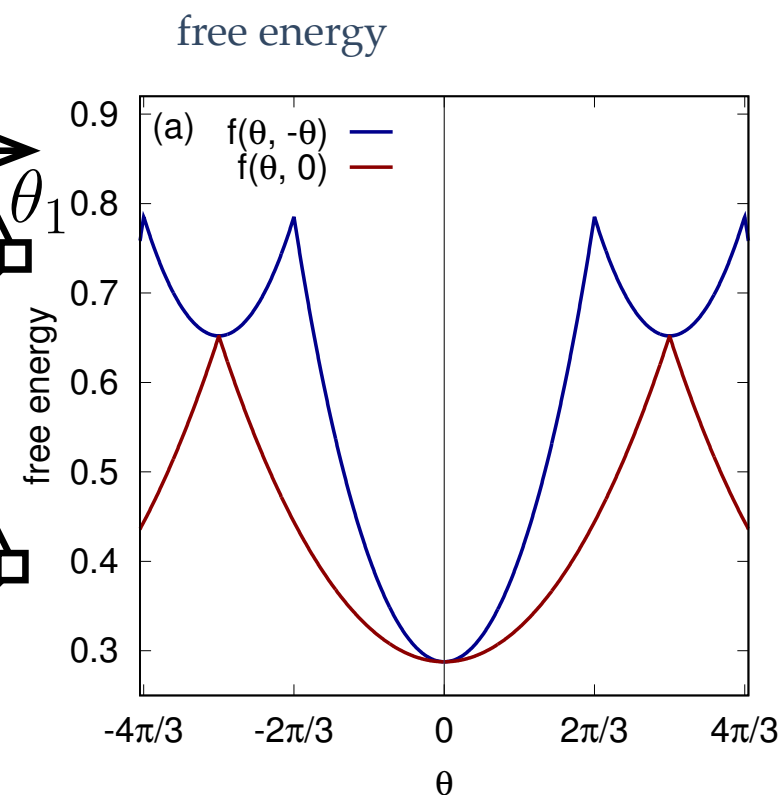
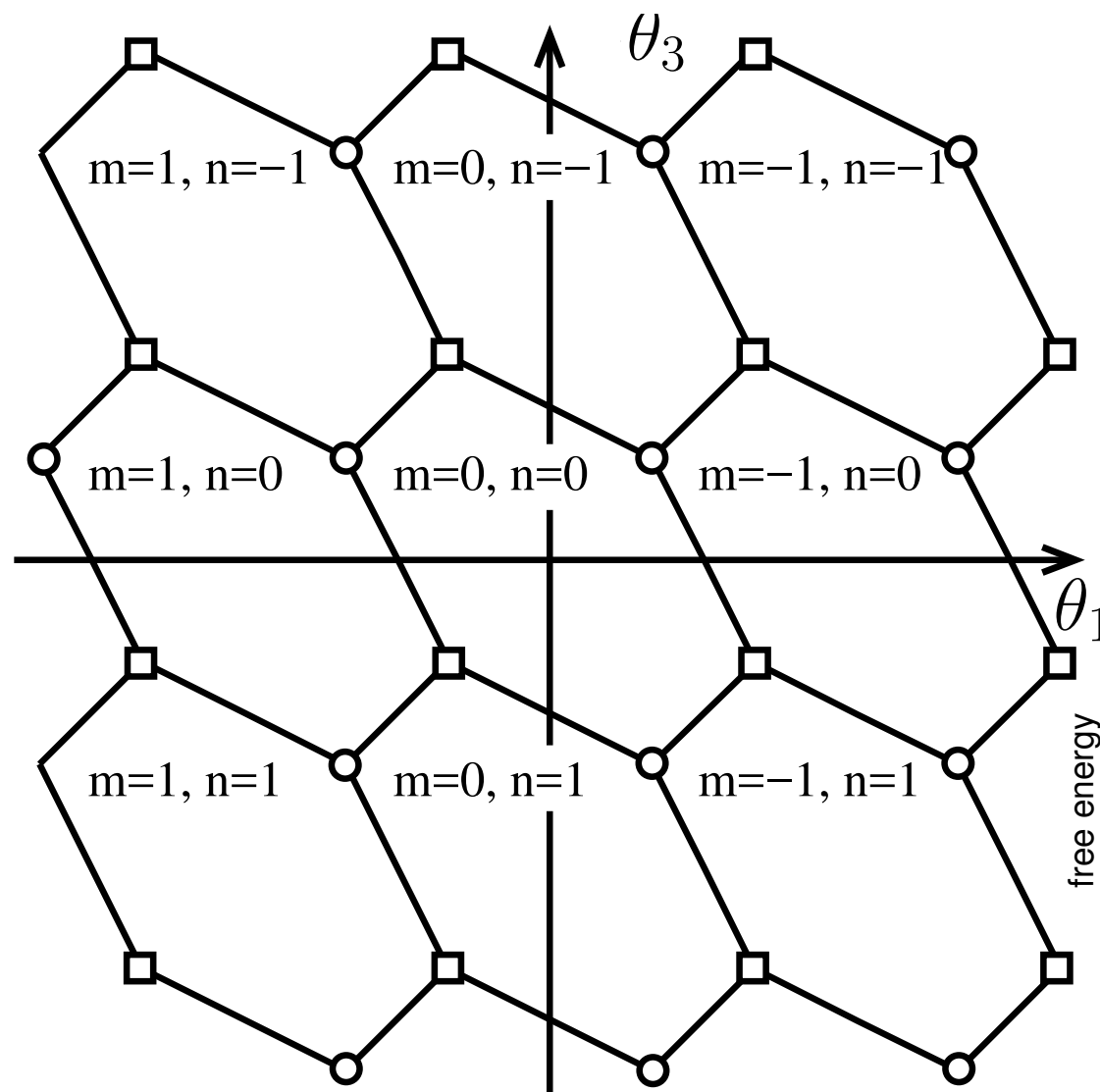
$g \rightarrow \infty$ is integrable, $S = i (\theta_1 Q_1 - \theta_3 Q_3)$



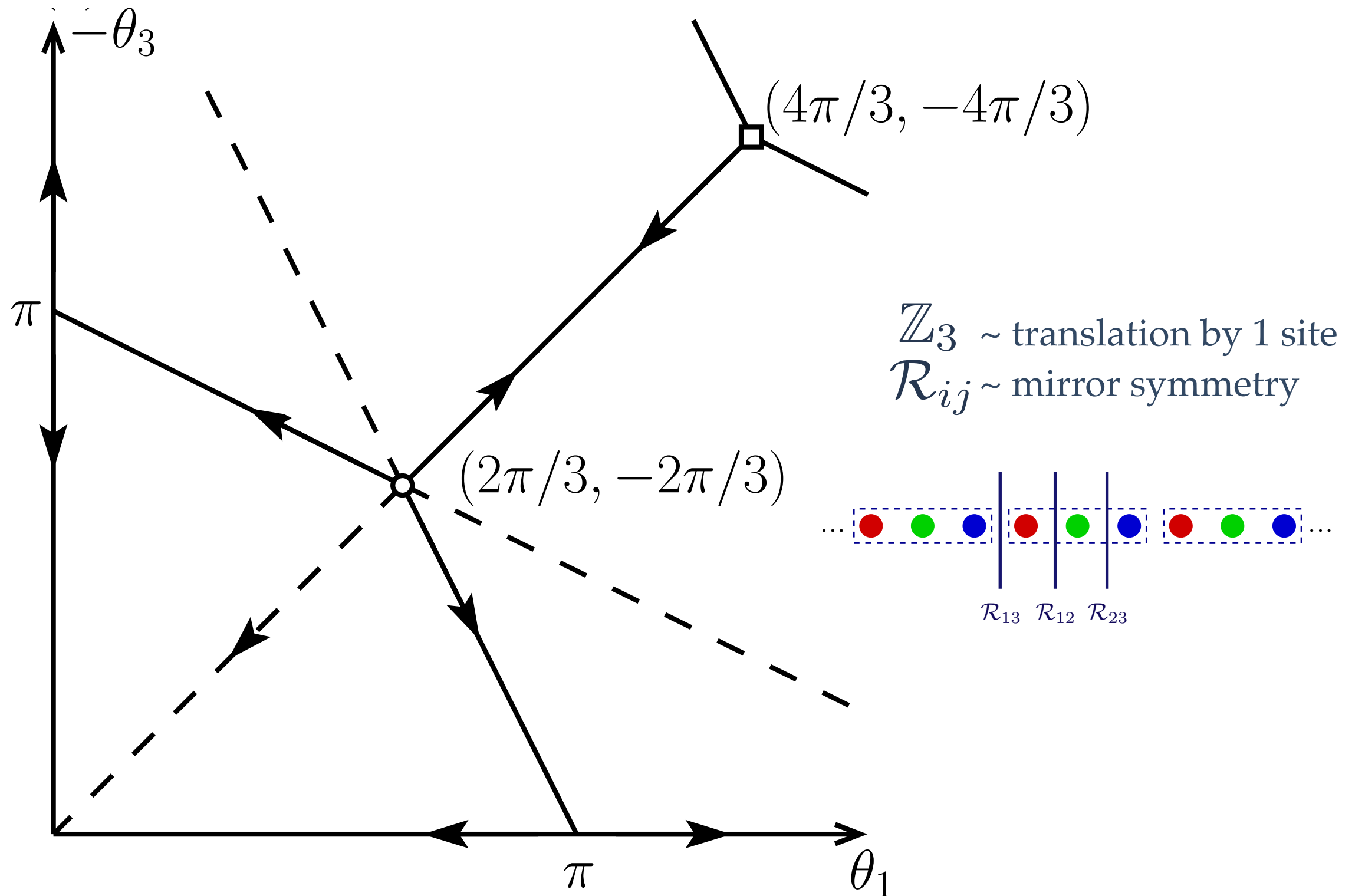
General phase diagram: $\theta_1 \neq -\theta_3$

$$S = \int dx d\tau \frac{1}{g} \left(\left| \vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_2 \right|^2 + \left| \vec{\phi}_2^* \cdot \partial_\mu \vec{\phi}_3 \right|^2 + \left| \vec{\phi}_3^* \cdot \partial_\mu \vec{\phi}_1 \right|^2 \right) + i (\theta_1 Q_1 - \theta_3 Q_3)$$

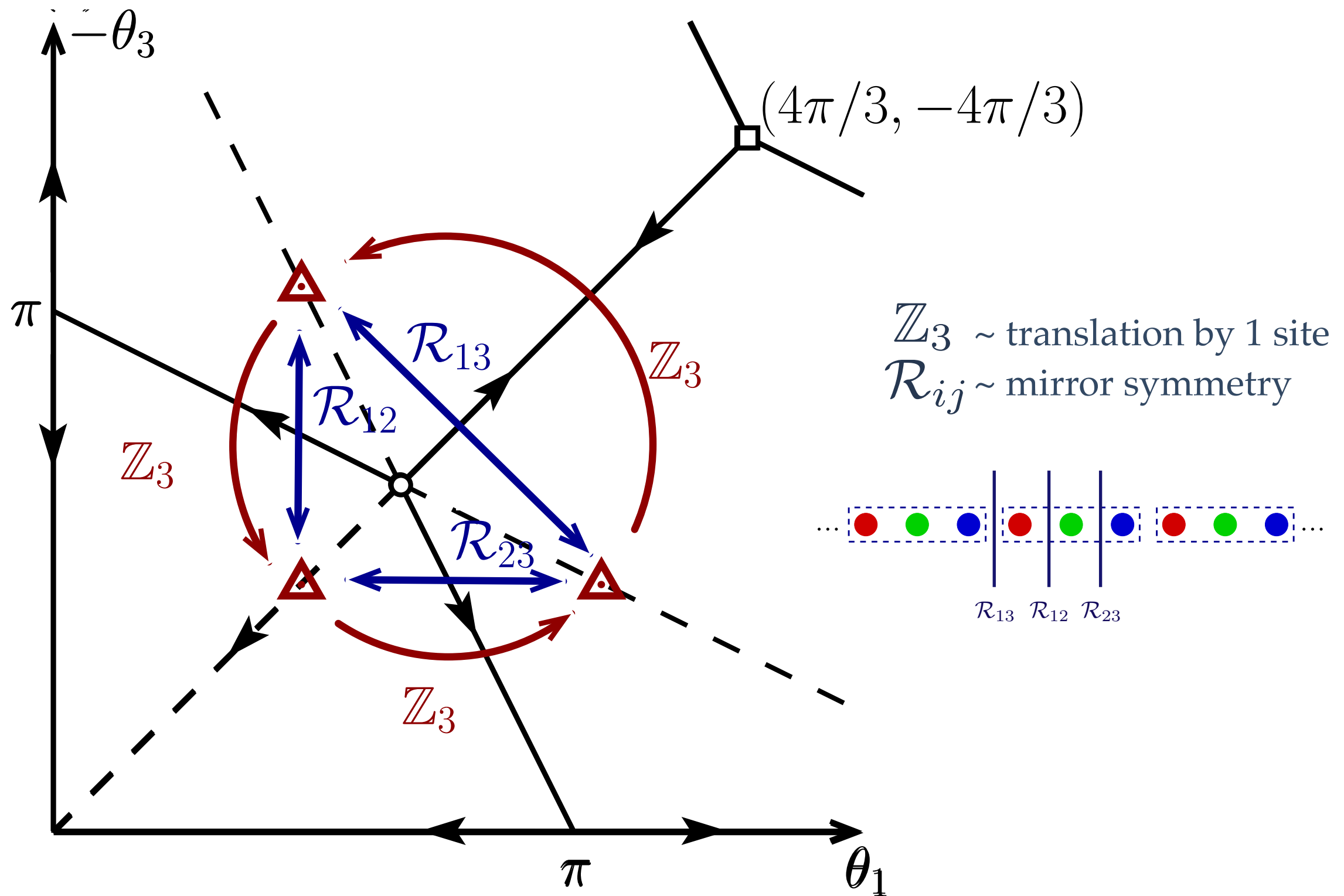
$g \rightarrow \infty$ is integrable, $S = i (\theta_1 Q_1 - \theta_3 Q_3)$



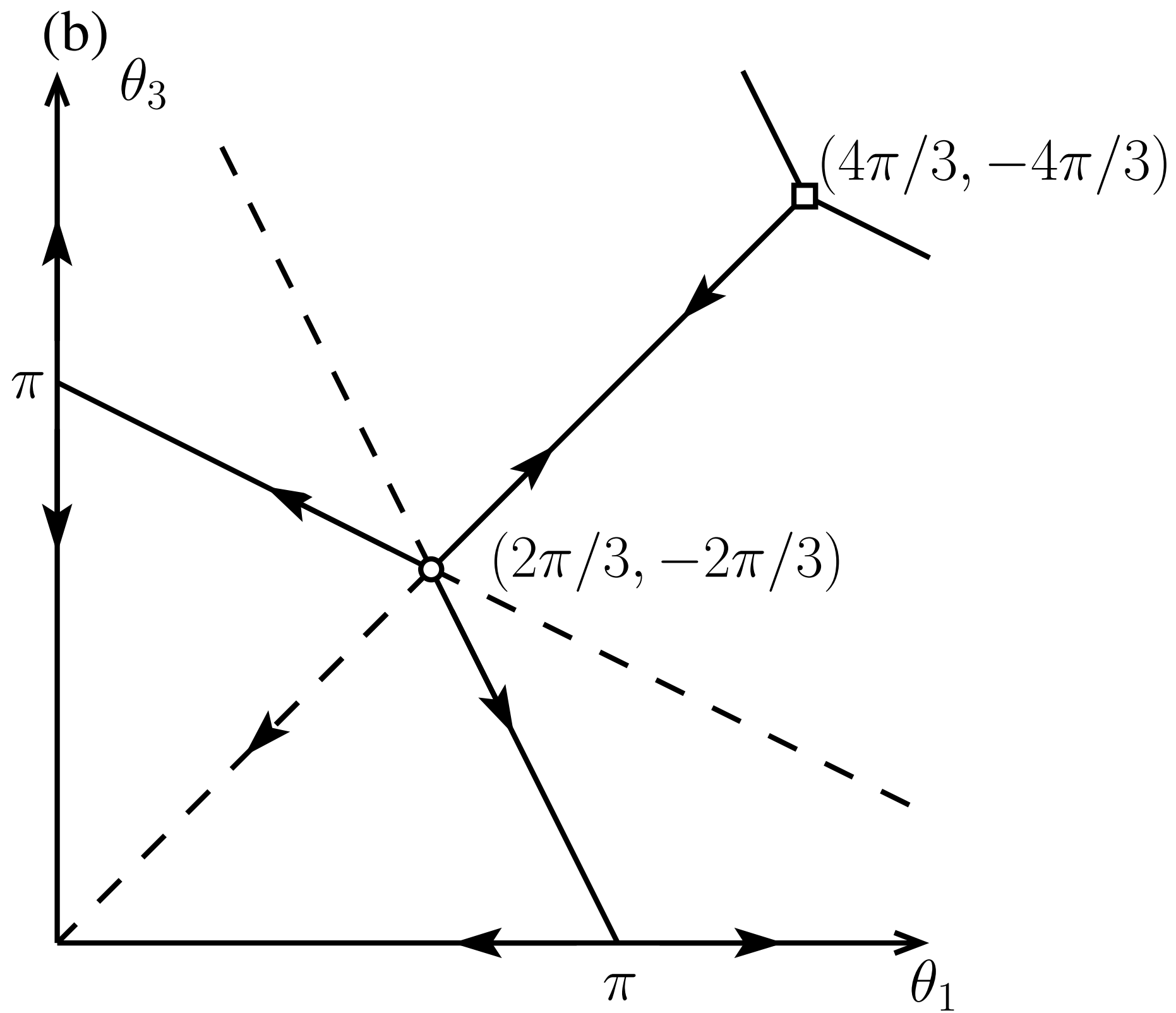
General phase diagram: $\theta_1 \neq -\theta_3$



General phase diagram: $\theta_1 \neq -\theta_3$



$$g < g_c$$

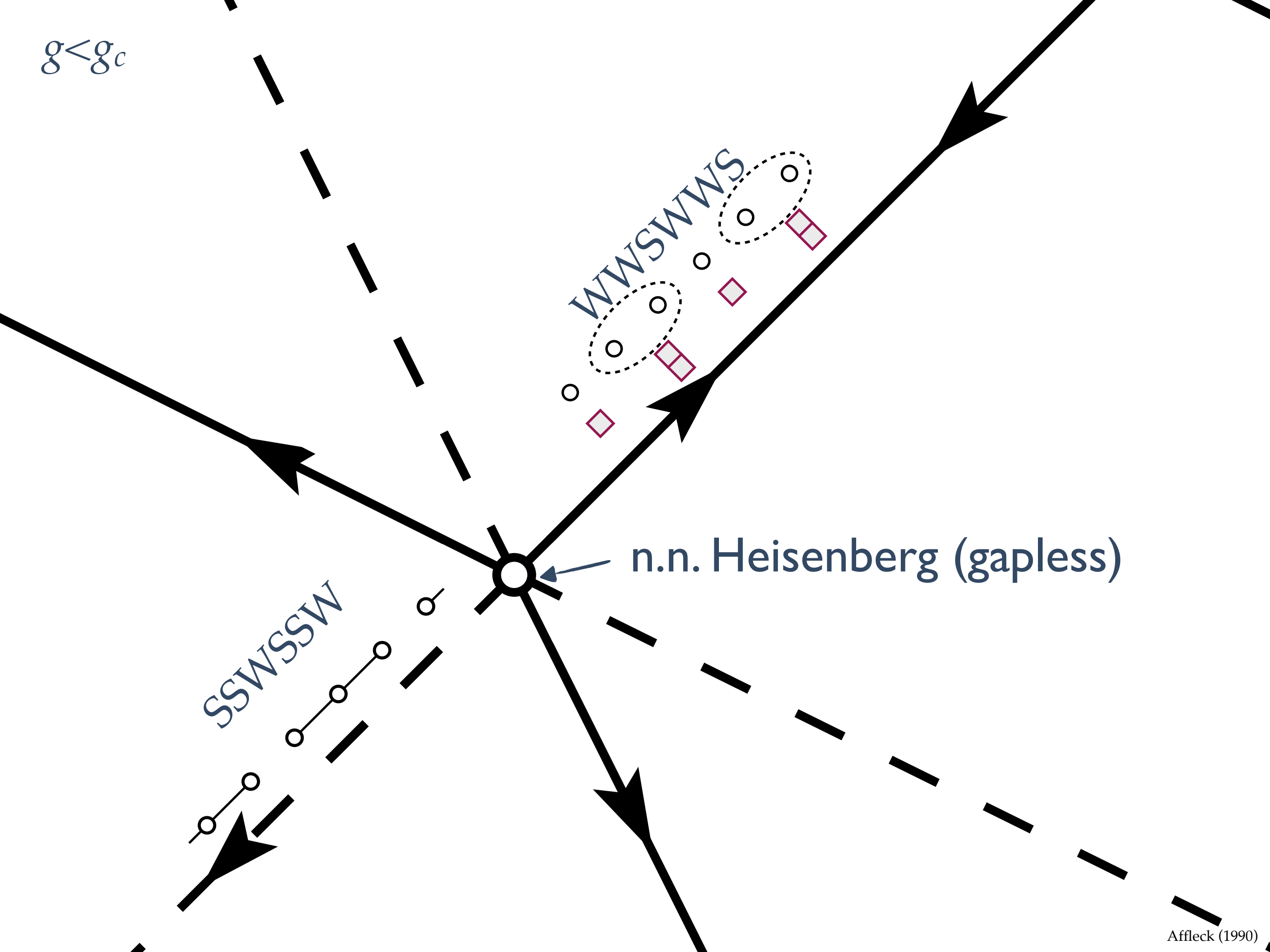


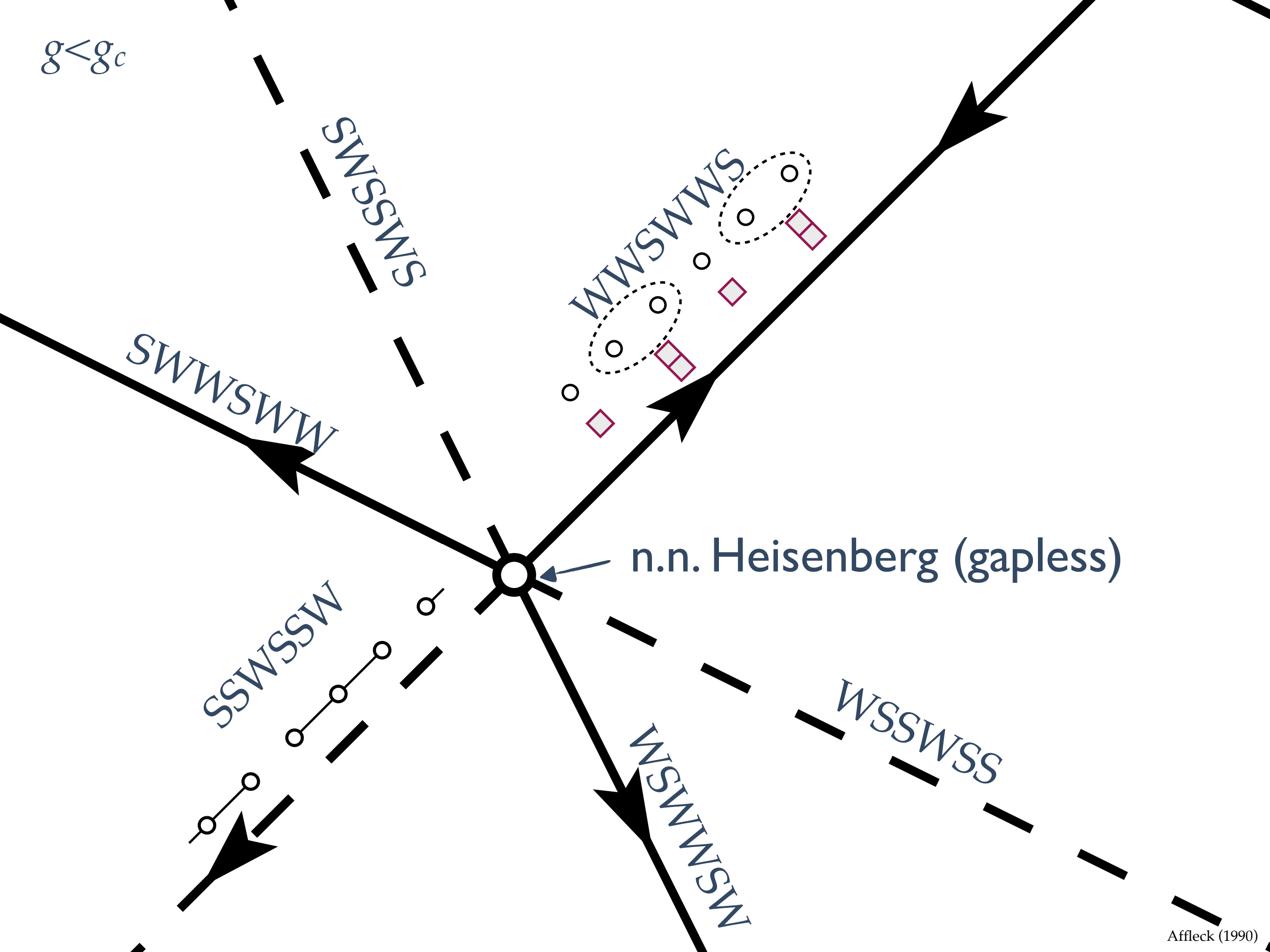
$g < g_c$

SSWSSW

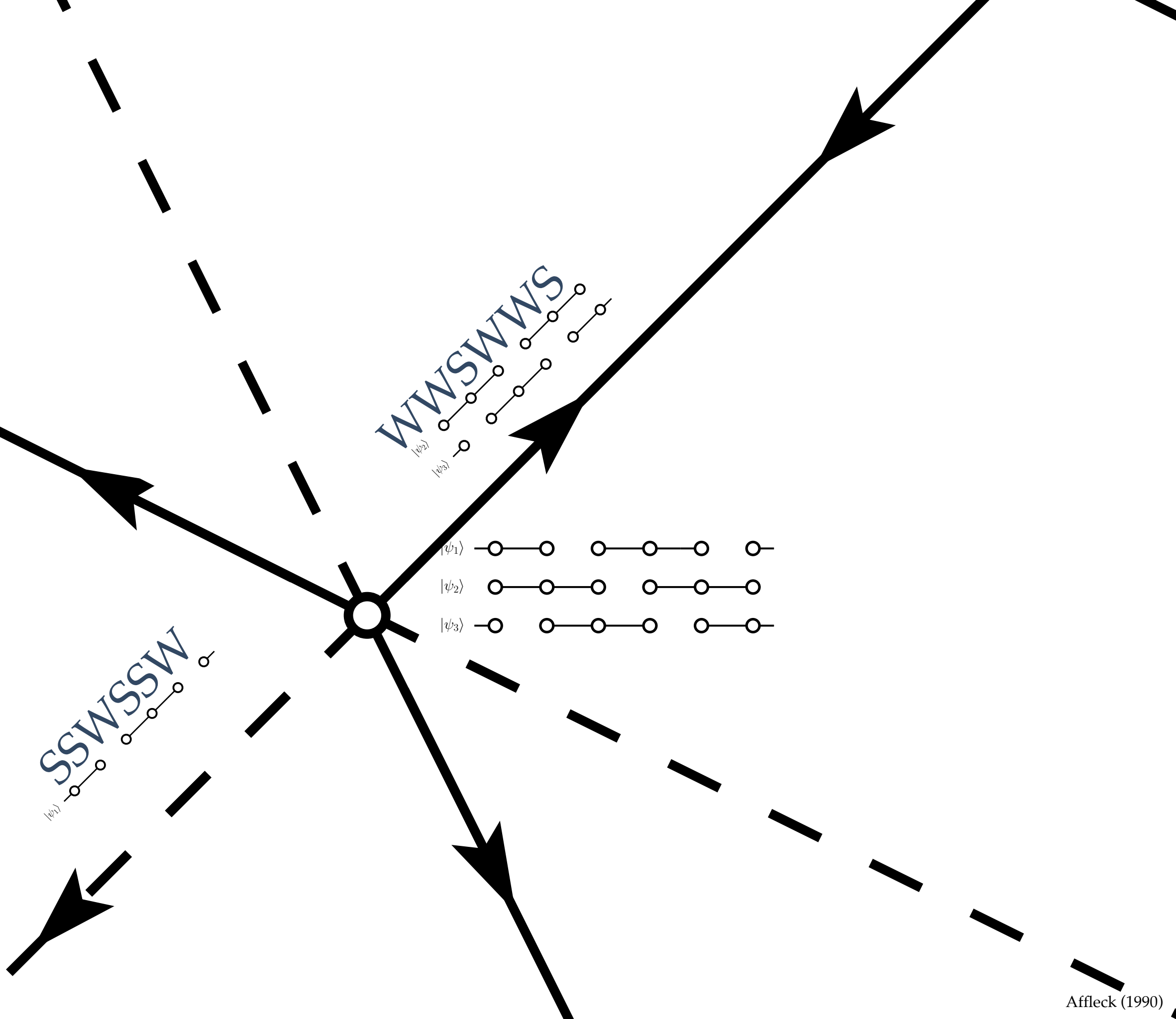
WWSWWS

n.n. Heisenberg (gapless)

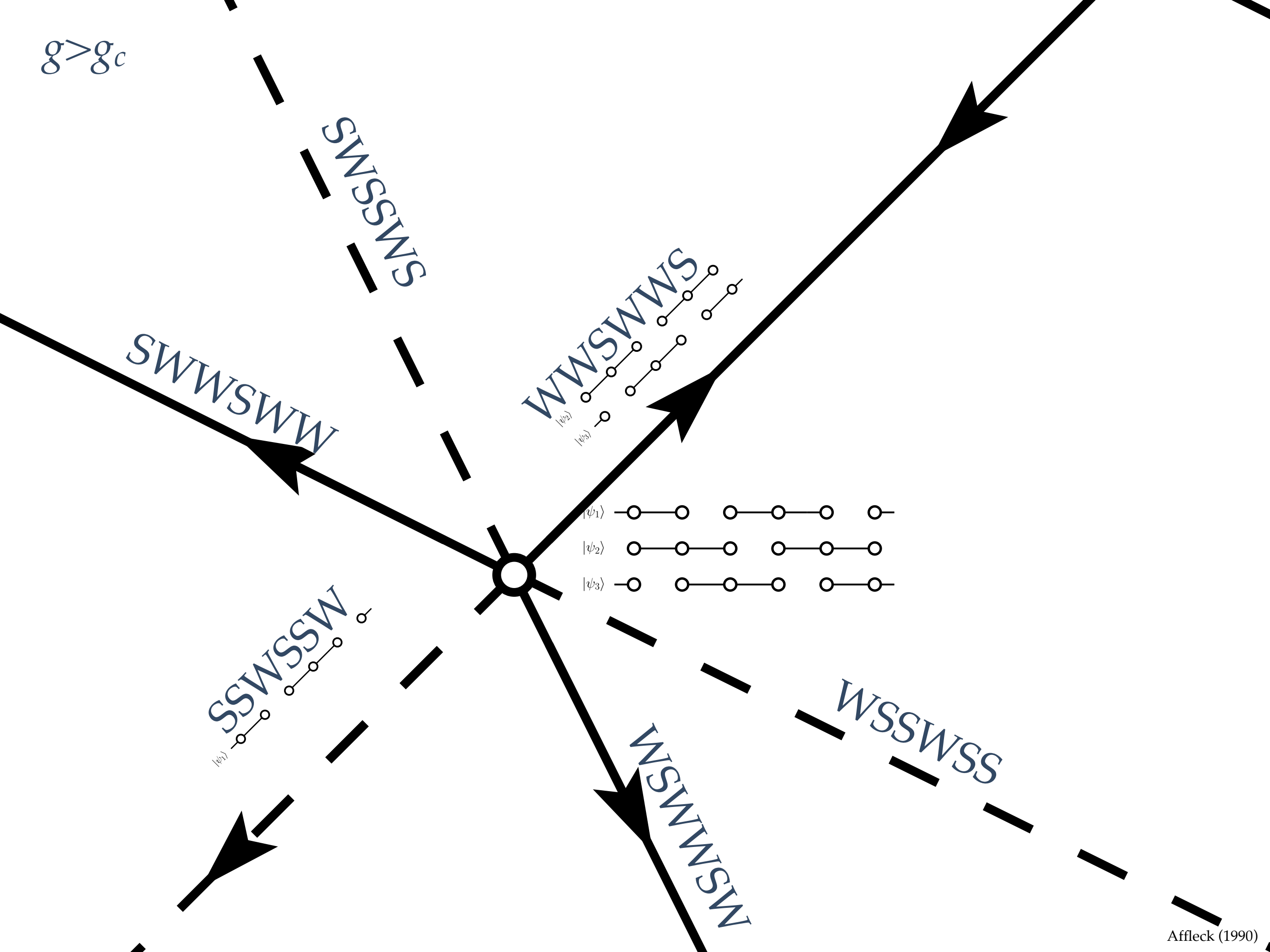


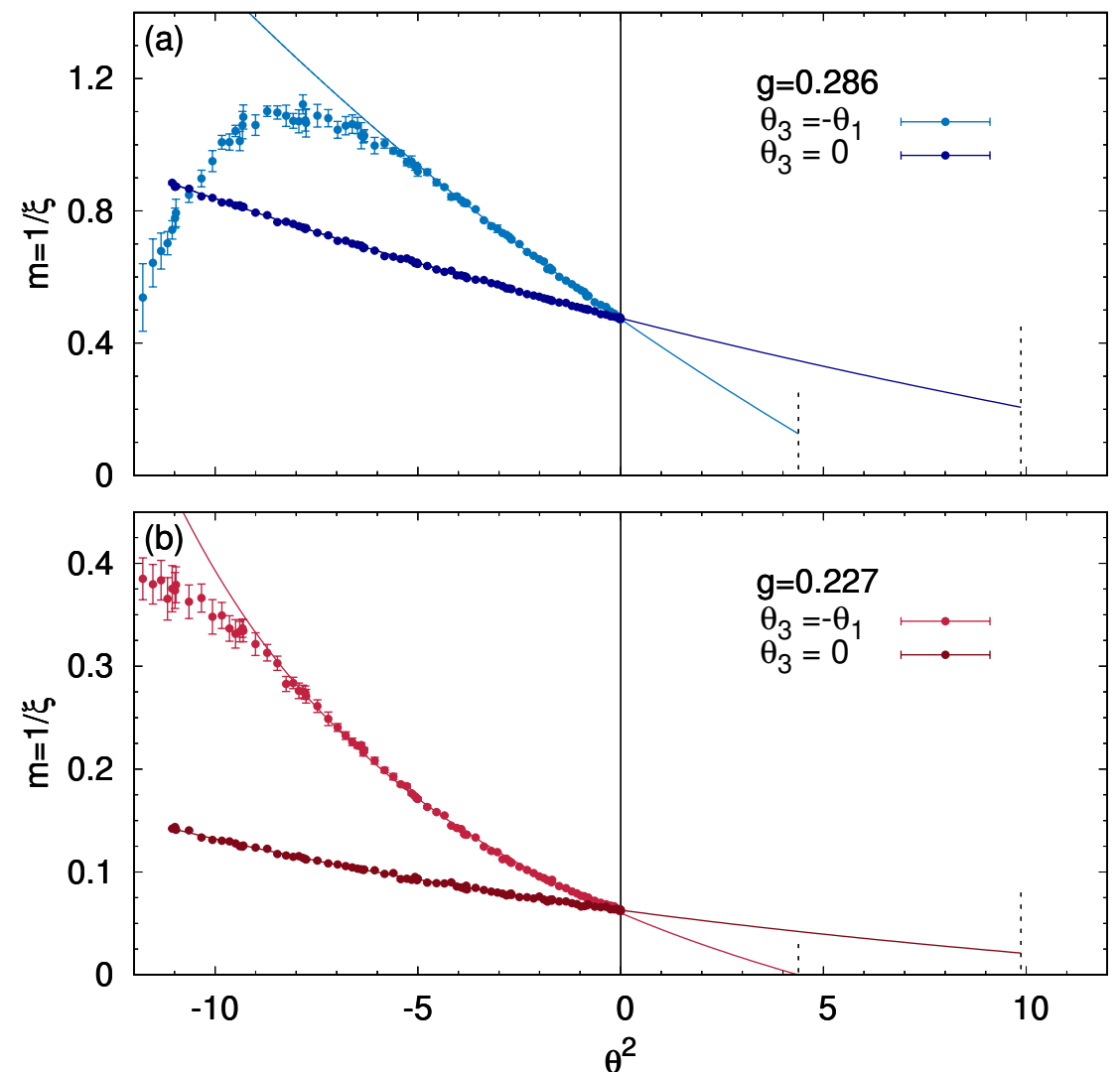
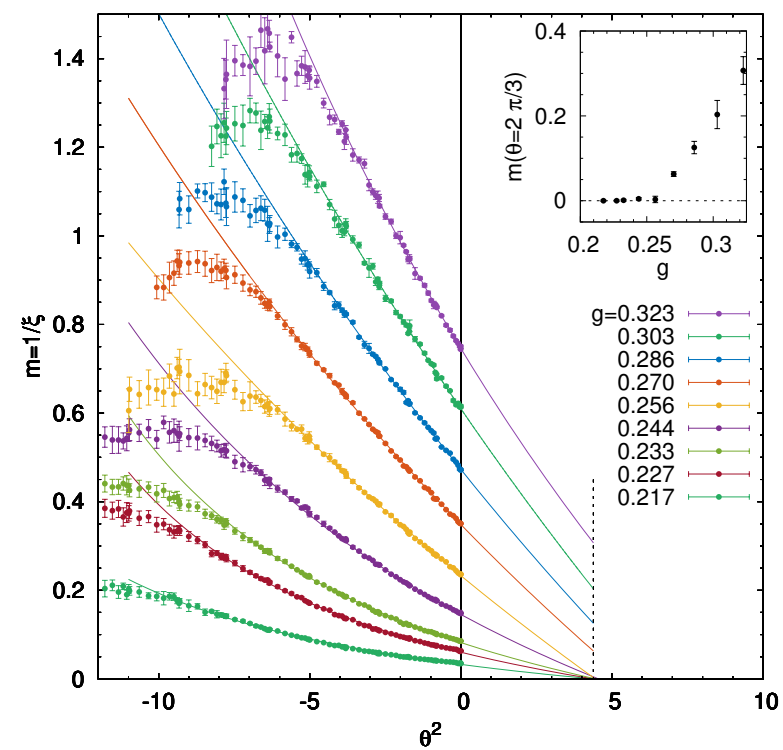
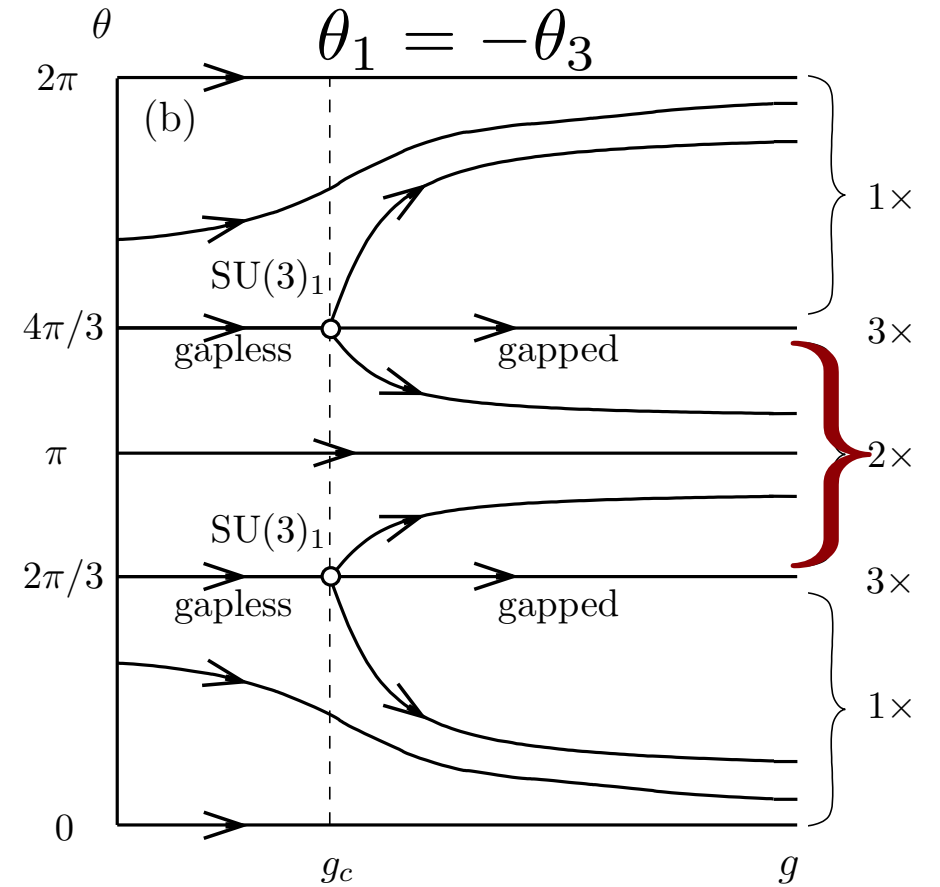
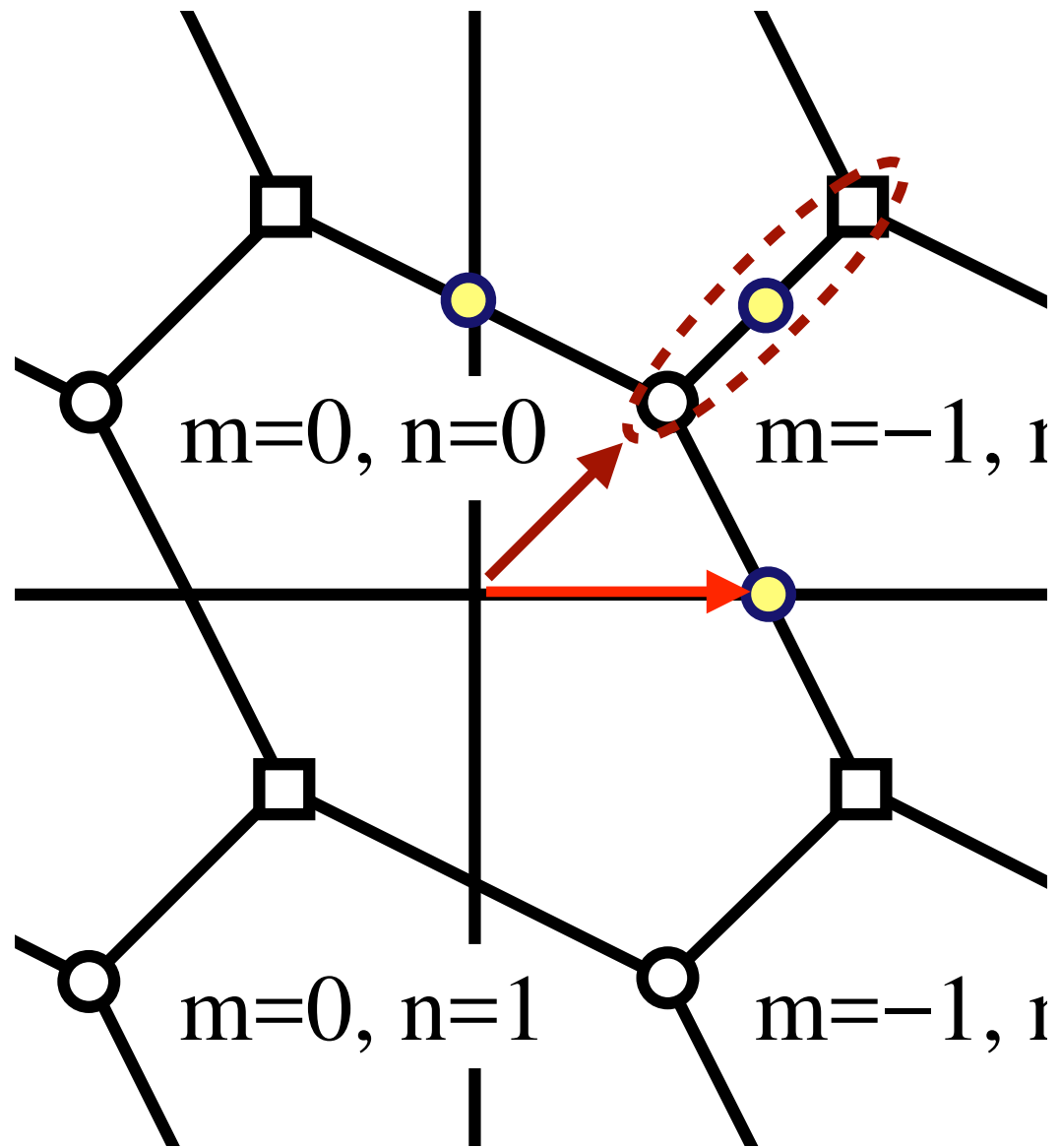


$g > g_c$

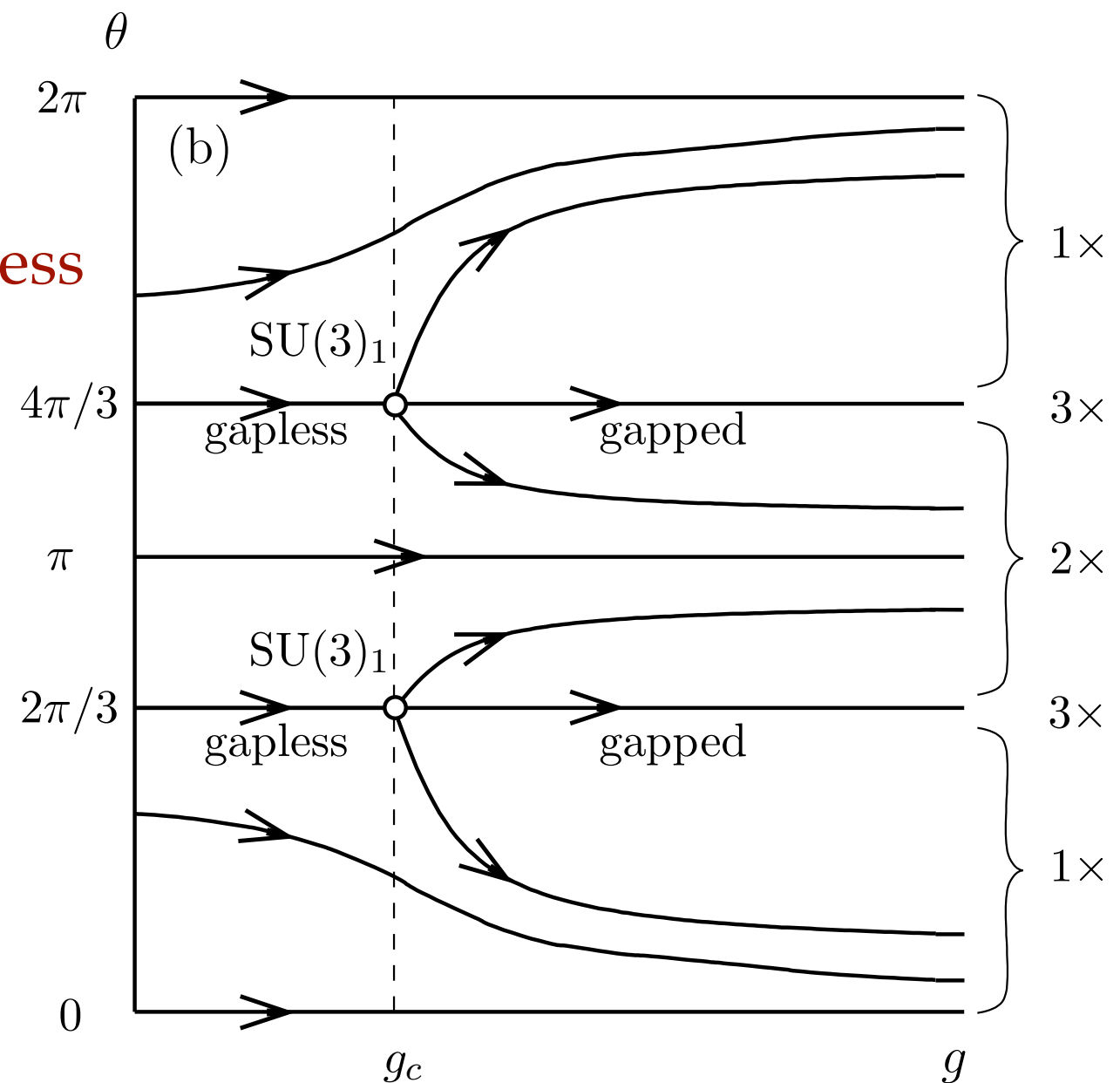
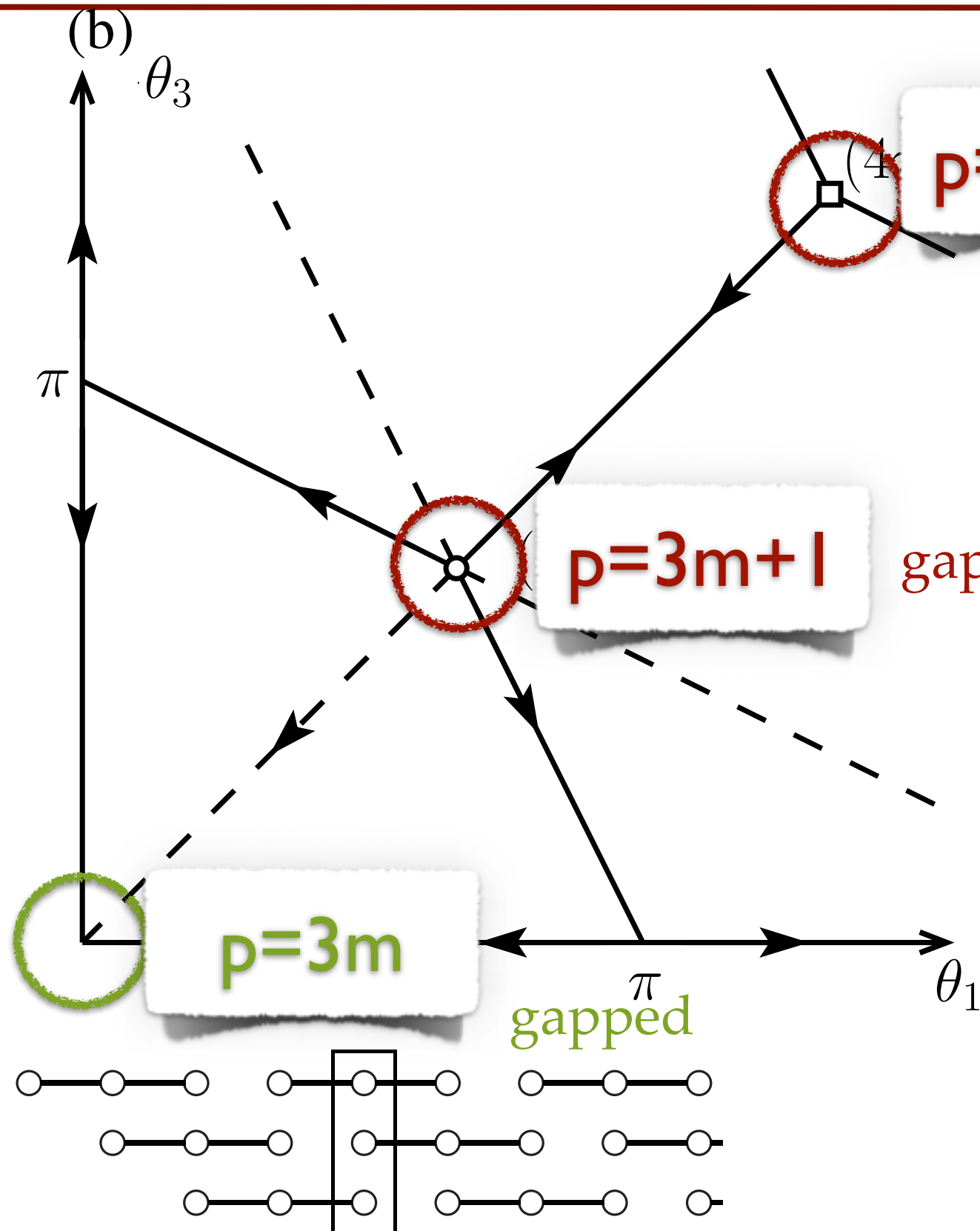


$g > g_c$





What we learned



Thank you for your attention