LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

2018 TAE School
September 2018, Benasque

QCD, Jets and Monte Carlo techniques

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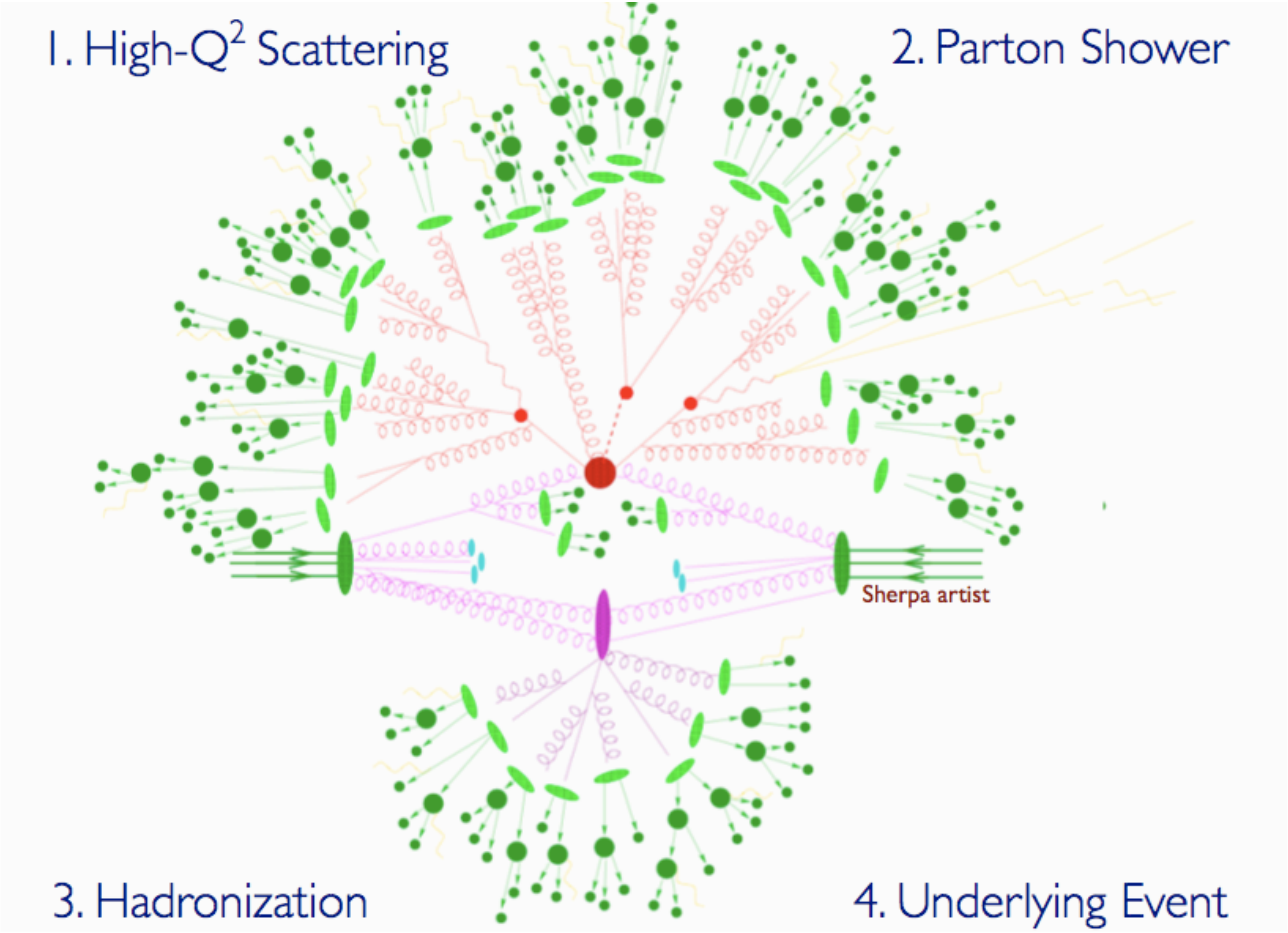
Lecture 1 - **Basics of QCD**
Lecture 2 - Higher orders and Monte Carlos
Lecture 3 - Jets

Strong interactions are complicated



"We are driven to the conclusion that the Hamiltonian method for strong interactions is dead and must be buried, although of course with deserved honor"

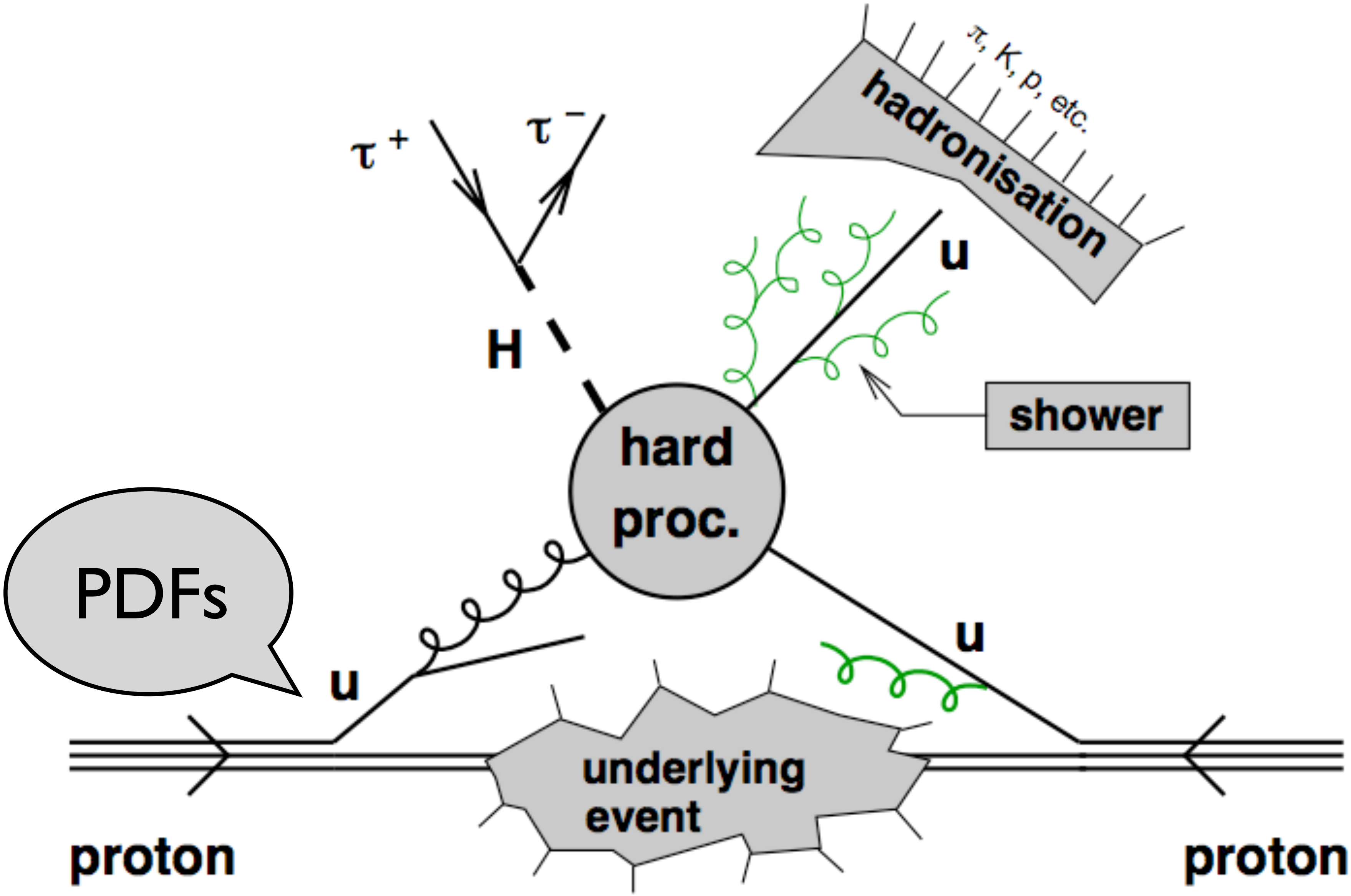
Lev Landau

"The correct theory [of strong interactions] will not be found in the next hundred years"

Freeman Dyson

We have come a long way towards disproving these predictions

A hadronic process



Final state

Hard process

Initial state

Books and "classics"...

- T. Muta, *Foundations of Quantum Chromodynamics*, World Scientific (1987)
- R.D. Field, *Applications of perturbative QCD*, Addison Wesley (1989)
  Great for specific examples of detailed calculations
- R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and Collider Physics*, Cambridge University Press (1996)
  Phenomenology-oriented
- G. Sterman, *An Introduction to Quantum Field Theory*, Cambridge University Press (1993)
  A QFT book, but applications tilted towards QCD
- Dokshitzer, Khoze, Muller, Troyan, *Basics of perturbative QCD*, http://www.lpthe.jussieu.fr/~yuri
  For the brave ones
- Dissertori, Knowles, Schmelling, *Quantum Chromodynamics: High Energy Experiments and Theory,* Oxford Science Publications
  One of the most recent QCD books
- M.L. Mangano, *Introduction to QCD*, http://doc.cern.ch//archive/cernrep//1999/99-04/p53.pdf
- S. Catani, *Introduction to QCD*, CERN Summer School Lectures 1999

...and recent lectures, slides and...videos

- Gavin Salam,
  - "Elements of QCD for Hadron Colliders", http://arxiv.org/abs/arXiv:1011.5131
  - http://gsalam.web.cern.ch/gsalam/teaching/PhD-courses.html
- Peter Skands
  - 2015 CERN-Fermilab School lectures, http://skands.physics.monash.edu/slides/
  - "Introduction to QCD", http://arxiv.org/abs/arXiv:1207.2389
- Fabio Maltoni
  - "QCD and collider physics", GGI lectures,
    https://www.youtube.com/playlist?list=PL1CFLtxeIrQqvt-e8C5pwBKG4PljSyouP

Outline of 'Basics of QCD'

- strong interactions
- QCD lagrangian, colour, ghosts
- running coupling
- radiation
- calculations of observables
    - theoretical uncertainties estimates
    - power corrections
    - infrared divergencies and IRC safety
    - factorisation

QED v. QCD

QED has a wonderfully simple lagrangian, determined by local gauge invariance

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}A\!\!\!/\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

In the same spirit, we build **QCD**:
a non abelian local gauge theory, based on SU(3)$_{\text{colour}}$,
with 3 quarks (for each flavour) in the fundamental representation of the group and 8 gluons in the adjoint

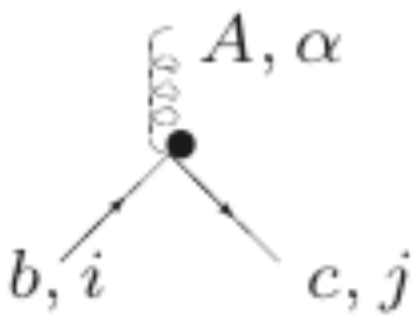
$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \sum_f \bar{\psi}_i^{(f)}(i\partial\!\!\!/ - m_f)\psi_i^{(f)} - \bar{\psi}_i^{(f)}(g_s t^a_{ij} A\!\!\!/_a)\psi_j^{(f)}$$

Gauge Fields and their interact. | Matter | Interaction

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

1. Colour

quark-gluon interaction



$$-ig\,(t^A)_{cb}\,(\gamma^\alpha)_{ji}$$

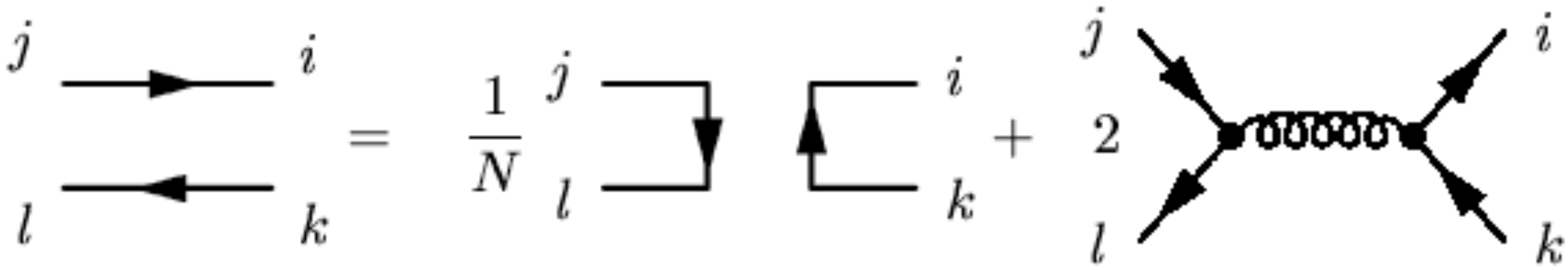
Index of the **adjoint** representation

colour matrix (generator of $SU(3)_{colour}$)

$$t^A{}_{cb}$$

Indices of the **fundamental** representation

A fundamental colour relation

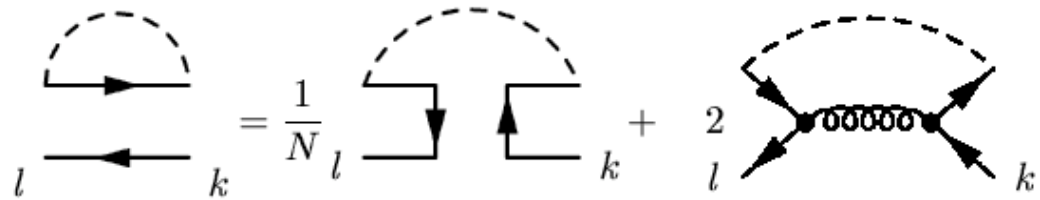


$$\delta_{ij}\delta_{lk} = \frac{1}{N}\delta_{ik}\delta_{lj} + 2\, t^A_{ik} t^A_{lj}$$

Take i=j in

$$\delta_{ij}\delta_{lk} = \frac{1}{N}\delta_{ik}\delta_{lj} + 2\, t^A_{ik} t^A_{lj}$$

↔



⇓

$$N\delta_{lk} = \frac{1}{N}\delta_{lk} + 2\, t^A_{ik} t^A_{li}$$

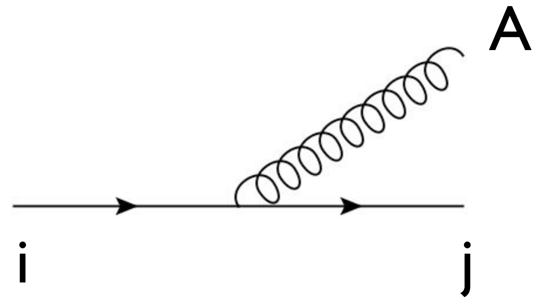
⇓

$$(t^A t^A)_{lk} = \frac{1}{2}\left(N - \frac{1}{N}\right)\delta_{lk} = \frac{N^2-1}{2N}\delta_{lk} \equiv C_F \delta_{lk}$$

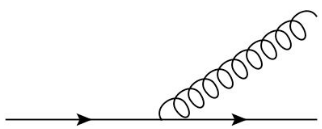
This defines C_F.
It is the Casimir of the fundamental representation of SU(N).
What is it, physically?

C_F

Gluon emission from a quark



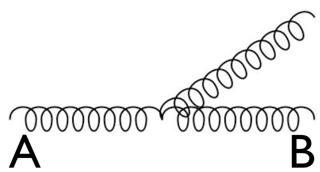
$$\sim \quad t^A_{ji}$$

Prob $\sim \sum_{jA} \Big|$  $\Big|^2$ $\sim \sum_{jA} t^A_{ij} t^A_{ji} = \sum_A (t^A t^A)_{ii} = C_F \delta_{ii}$

$C_F = (N^2-1)/(2N)$ is therefore the 'colour charge' of a quark, i.e. its probability of emitting a gluon (except for the strong coupling, of course)

C_A

Analogously, one can show that

Prob $\sim \sum_{BC} \left| \right.$ 

$\left. \right|^2 \sim C_A \delta_{AA}$

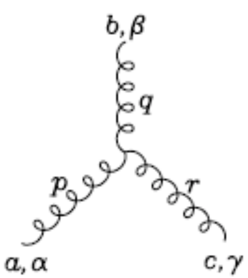
C_A = N is the 'colour charge' of a gluon, i.e.
its probability of emitting a gluon (except for the strong coupling, of course).
It is also the Casimir of the adjoint representation.

2. Gauge bosons self couplings

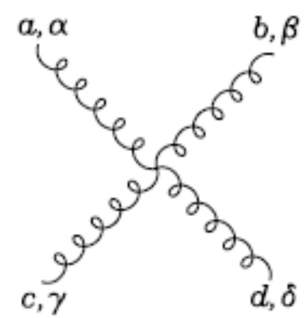
In QCD the gluons interact among themselves:

$$\mathcal{L}_{YM} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$



$$= g f^{abc}\left[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\right]$$



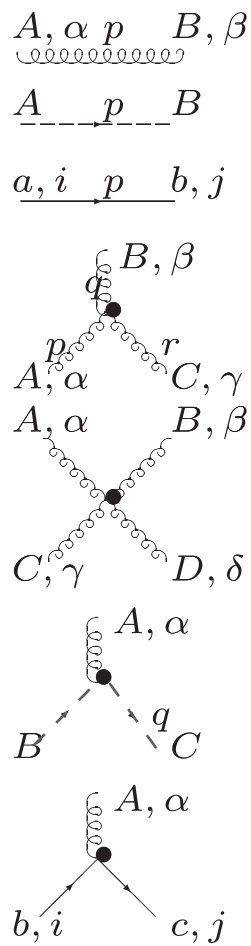
$$\begin{aligned} = & -ig^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ & -ig^2 f^{xad} f^{xbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ & -ig^2 f^{xab} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned}$$

New Feynman diagrams, in addition to the 'standard' QED-like ones

Direct consequence of non-abelianity of theory

3. Need for ghosts

Cancel unphysical degrees of freedom that would otherwise propagate in covariant gauges



gluon propagator	$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\dfrac{p^\alpha p^\beta}{p^2+i\varepsilon}\right]\dfrac{i}{p^2+i\varepsilon}$
ghost propagator	$\delta^{AB}\dfrac{i}{p^2+i\varepsilon}$
quark propagator	$\delta^{ab}\dfrac{i}{(\not{p}-m+i\varepsilon)_{ji}}$
three-gluon vertex	$-gf^{ABC}\left[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\right]$ (all momenta incoming)
four-gluon vertex	$-ig^2 f^{XAC}f^{XBD}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma})$ $-ig^2 f^{XAD}f^{XBC}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta})$ $-ig^2 f^{XAB}f^{XCD}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})$
gluon-ghost vertex	$gf^{ABC}q^\alpha$
quark-gluon vertex	$-ig\left(t^A\right)_{cb}(\gamma^\alpha)_{ji}$

ghost propagator

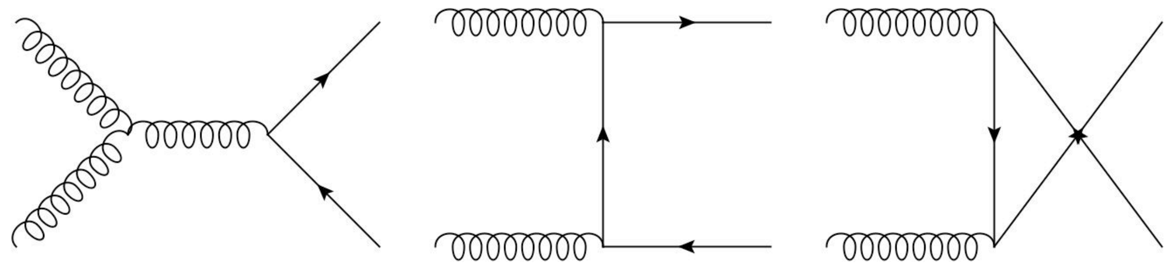
gauge parameter

gluon-ghost vertex

Table 1: Feynman rules for QCD in a covariant gauge.

Ghosts: an example

$gg \to q\bar{q}$



In QED (i.e. replacing gluons with photons) we'd only have the second and third diagram, and we would sum over the photon polarisations using

$$\sum_{pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu}$$

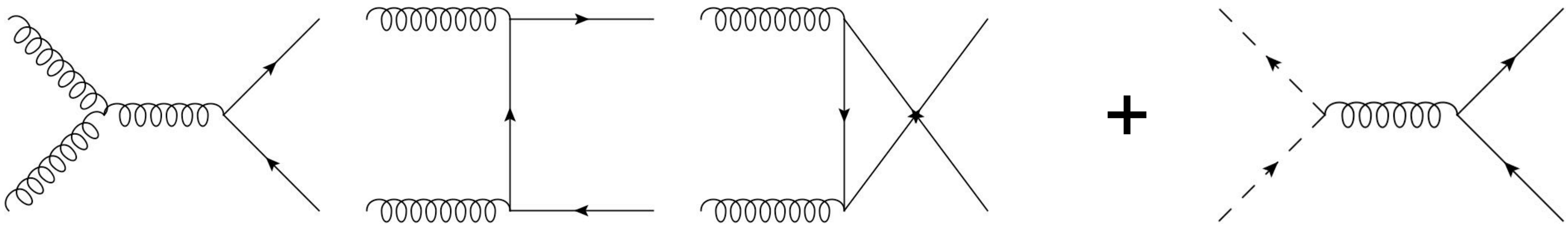
In QCD this would give the wrong result

We must use instead

$$\sum_{phys\,pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu} + \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k \cdot \bar{k}}$$

$\bar{k}$ is a light-like vector, we can use $(k_0,0,0,-k_0)$

An **alternative** approach is to include the ghosts in the calculation



Now we can safely use $$\sum_{pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu}$$

Macroscopic differences

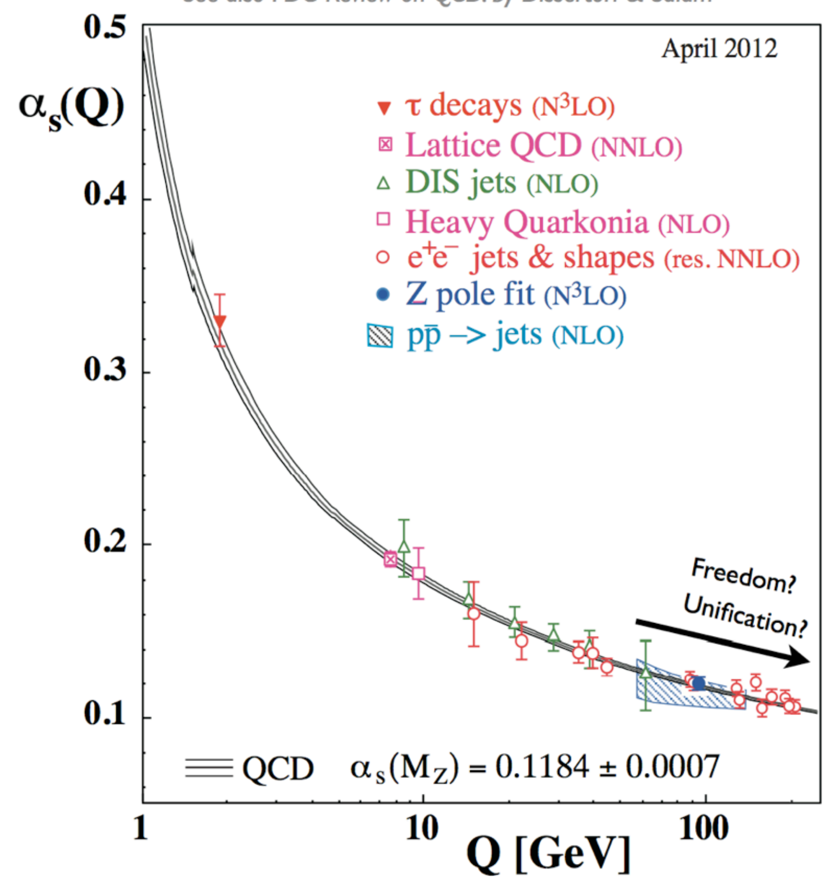
1. Confinement (probably -- no proof in QCD)

We never observe the fundamental degrees of freedom (quarks and gluons). They are always confined into hadrons.

2. Asymptotic Feedom

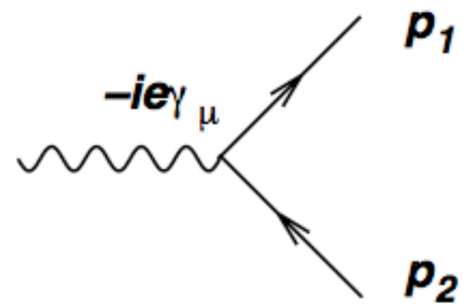
The running coupling of the theory, α_s, **decreases** at large energies

From S. Bethke, Nucl.Phys.Proc.Suppl. 234 (2013) 229
See also PDG Review on QCD. by Dissertori & Salam



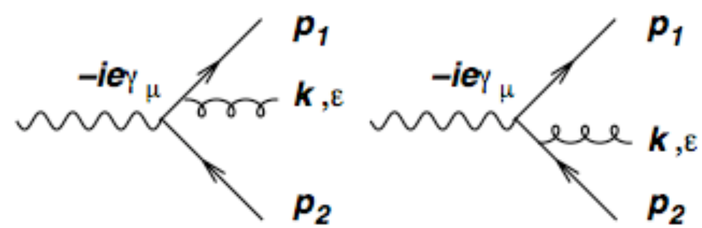
Start with $\gamma^* \to q\bar{q}$:

$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2)$$



Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$



In the **soft** limit , k << $p_{1,2}$

$$\mathcal{M}_{q\bar{q}g} \simeq \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

Squared amplitude, including phase space

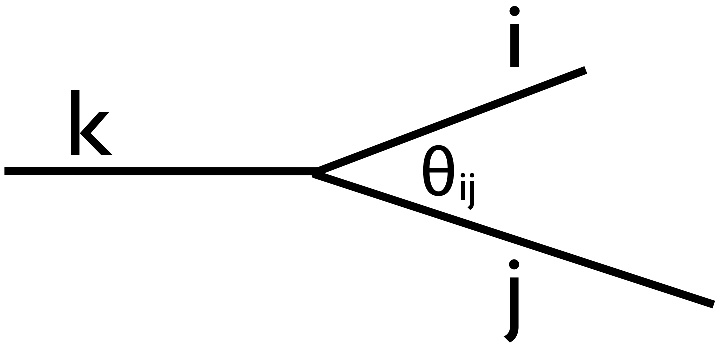
$$d\Phi_{q\bar{q}g}|M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}}|M^2_{q\bar{q}}|)\, \frac{d^3\vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

Factorisation: Born × radiation

Changing variables (use energy of gluon E and emission angle θ) we get for the radiation part

$$dS = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

QCD emission probability



$$\frac{dP_{k\to ij}}{dE_i d\theta_{ij}} \sim \frac{\alpha_s}{\min(E_i, E_j)\theta_{ij}}$$

Singular in the soft ($E_{i,j}\to 0$) and in the collinear ($\theta_{ij}\to 0$) limits.
Divergent upon integration.

The divergences can be cured by the addition of virtual corrections and/or **if** the definition of an observable is appropriate

Altarelli-Parisi kernel

Using the variables E=(1-z)p and k_t = Eθ we can rewrite

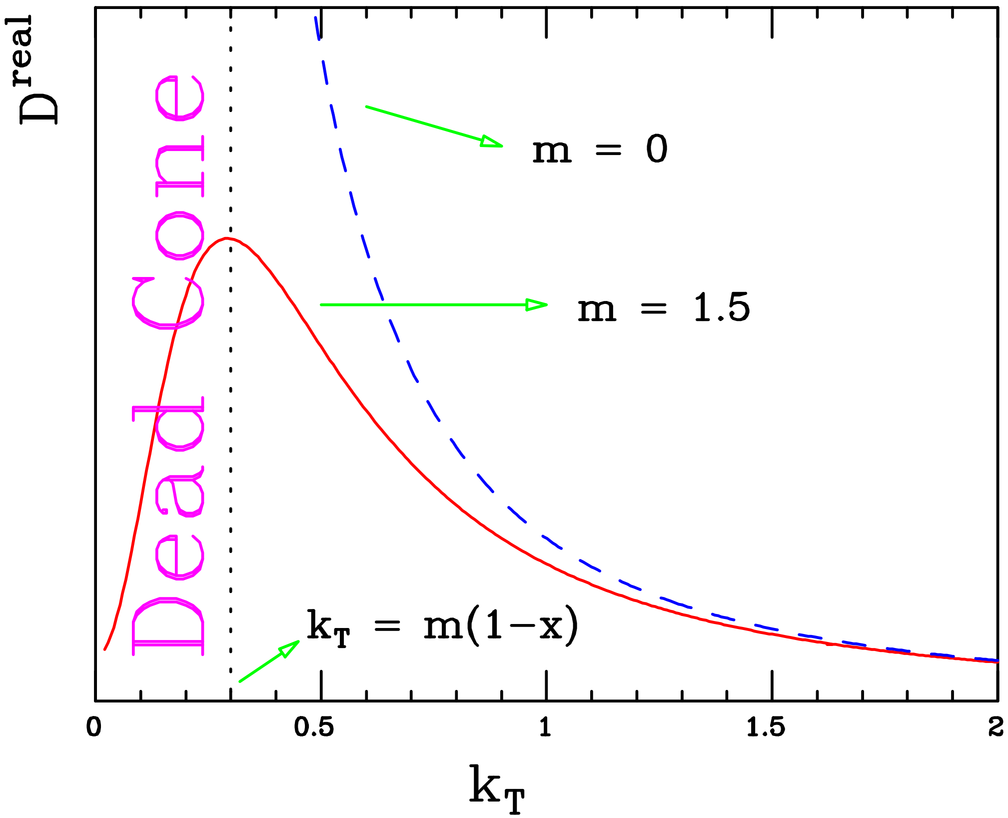
$$dS = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi} \quad \longrightarrow \quad \frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2}\frac{d\phi}{2\pi}$$

'almost' the Altarelli-Parisi splitting function P_{qq}

Massive quarks

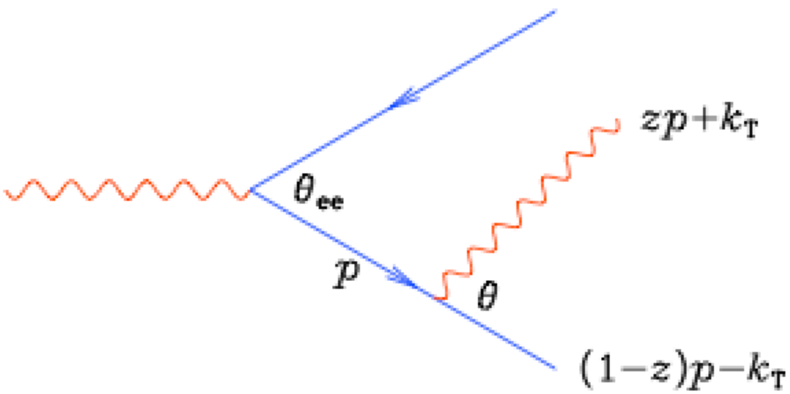
If the quark is massive the collinear singularity is **screened**

$$\frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2}\frac{d\phi}{2\pi} \to \frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2+(1-z)^2m^2}\frac{d\phi}{2\pi}+\cdots$$



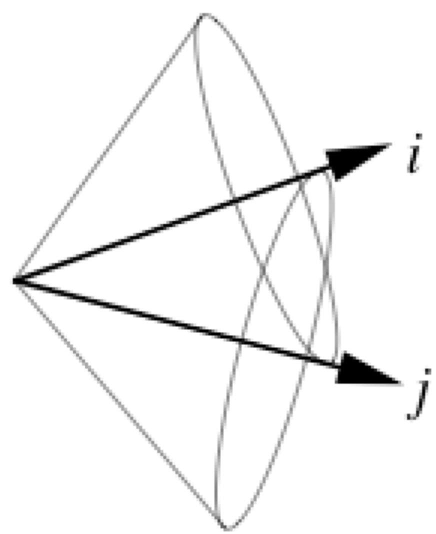
Angular ordering

The universal soft and collinear spectrum is not the only relevant characteristic of radiation. **Angular ordering** is another



Angular ordering means
$\theta < \theta_{ee}$

Soft radiation emitted by a dipole is restricted to cones smaller than the angle of the dipole



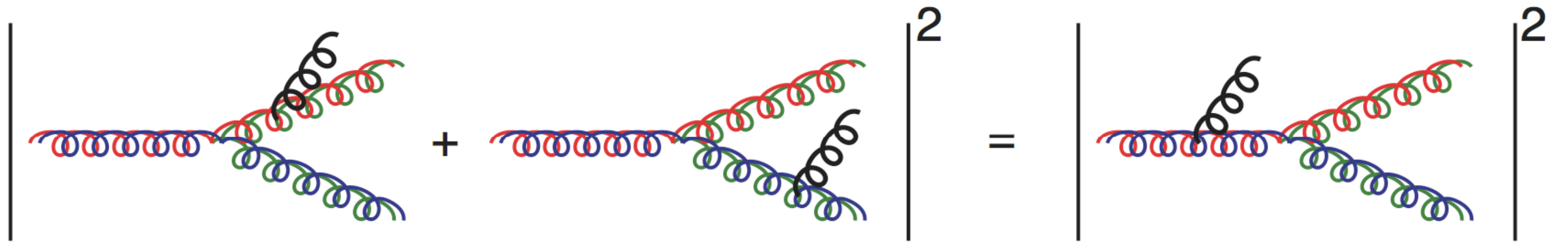
Coherence

Angular ordering is a manifestation of **coherence**,
a phenomenon typical of gauge theories

Coherence leads to the **Chudakov effect**,
suppression of soft bremsstrahlung from an e^+e^- pair.

"Quasi-classical" explanation: a soft photon cannot resolve a small-sized pair, and only sees its total electric charge (i.e. zero)

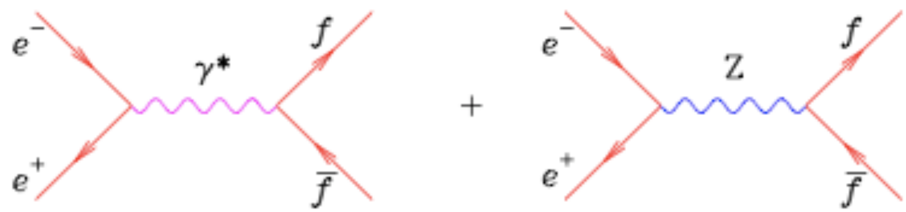
The phenomenon of coherence is preserved also in QCD.
Soft guon radiation off a coloured pair can be described as being emitted coherently by the colour charge of the parent of the pair



Drawing:
P. Skands

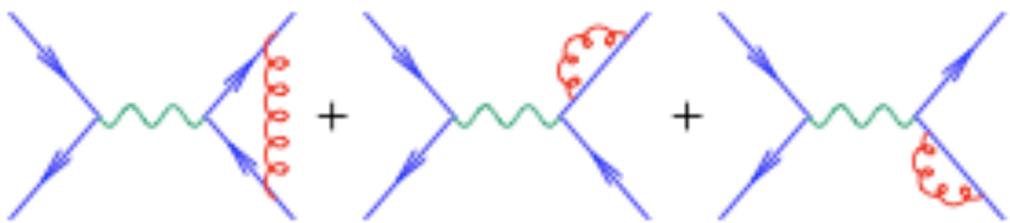
Easiest higher order calculation in QCD. Calculate $e^+e^- \to$ qqbar+X in pQCD

Born



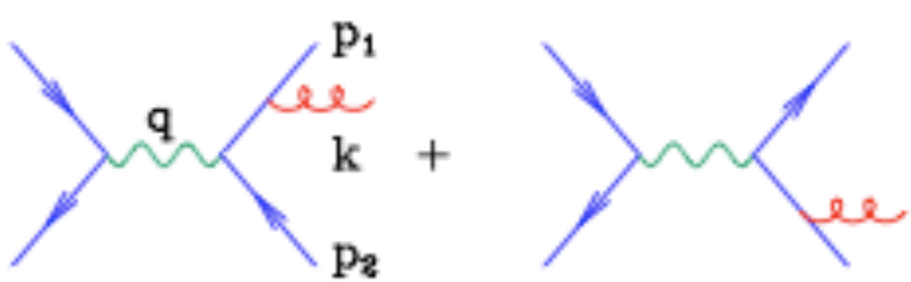
α_s^0

Virtual



α_s^1

Real



Regularize with dimensional regularization, expand in powers of ε

$$\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon)\right]$$

Real

$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{1 + \frac{2\alpha_S}{3\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]\right\}$$

Virtual

$$R = 3 \sum_q Q_q^2 \left\{1 + \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2)\right\}$$

Sum

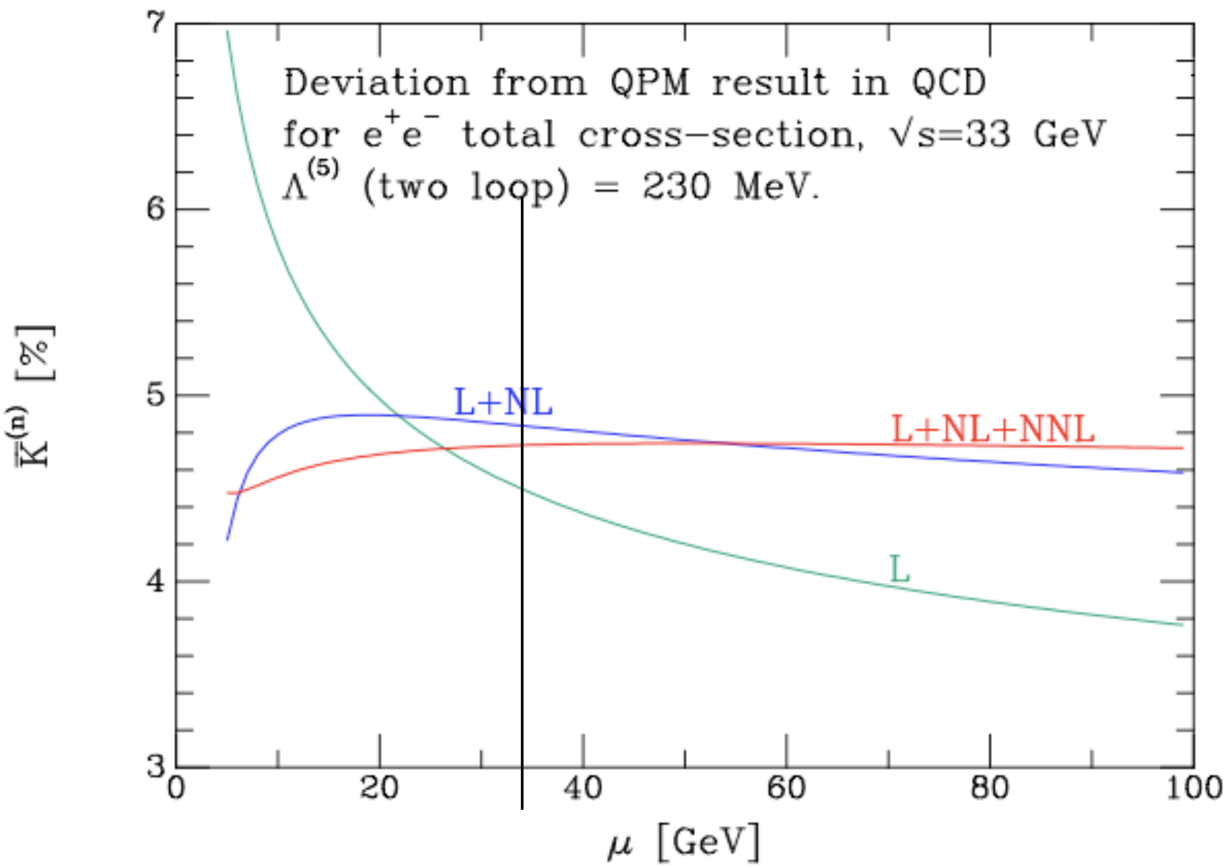
Real and virtual, separately divergent, 'conspire' to make total cross section finite

Scale dependence

In higher orders α_s must be renormalised and aquires a scale dependence.

$$K_{QCD} = 1 + \frac{\alpha_S(\mu^2)}{\pi} + \sum_{n\geq 2} C_n \left(\frac{s}{\mu^2}\right) \left(\frac{\alpha_S(\mu^2)}{\pi}\right)^n$$

C_n known up to C_3



Cross section prediction varies with renormalisation scale choice. **Which value do we pick for μ?**

μ cannot be uniquely fixed. It can however be exploited to **estimate the theoretical uncertainty of the calculation**

Theoretical uncertainties

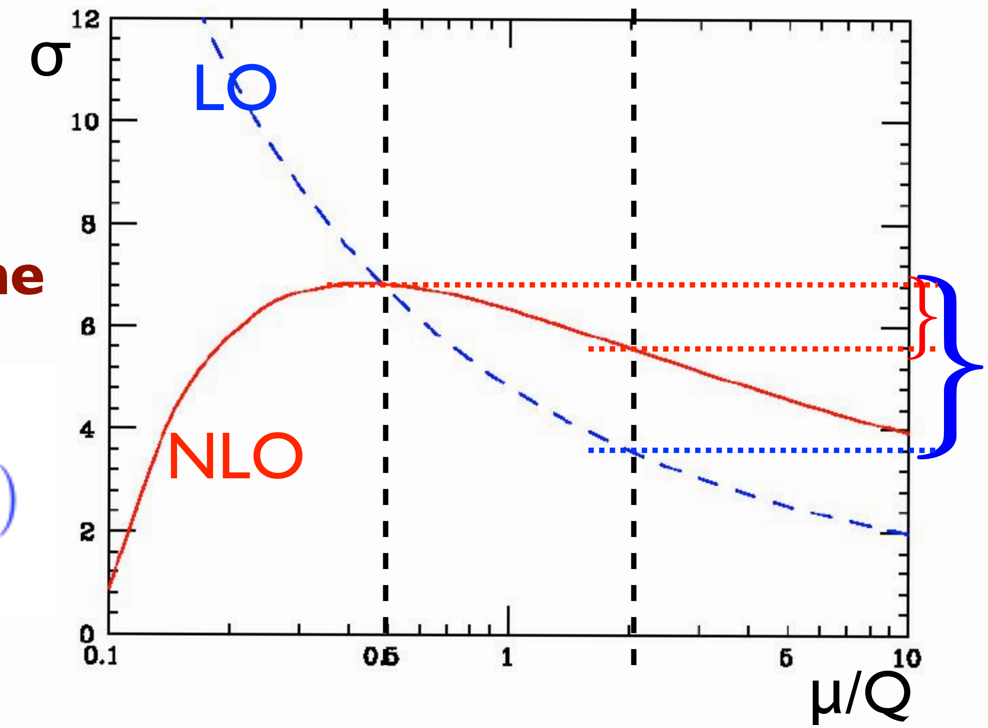
We wrote before: $$\frac{d}{d\ln\mu^2}\ln\sigma^{phys} = 0$$

i.e. independence of cross sections on artificial scales

Would only hold for all-orders calculations.

In real life: residual dependence at one order higher than the calculation

$$\frac{d}{d\log\mu}\sum_{n=1}^{N} c_n(\mu)\alpha_S^n(\mu) \sim \mathcal{O}\left(\alpha_S^n(\mu)^{N+1}(\mu)\right)$$



Vary scales (around a physical one) to **ESTIMATE** the uncalculated higher order

Non-perturbative contributions

We have calculated $\sum_q \sigma(e^+e^- \to q\bar{q})$ in **perturbative** QCD

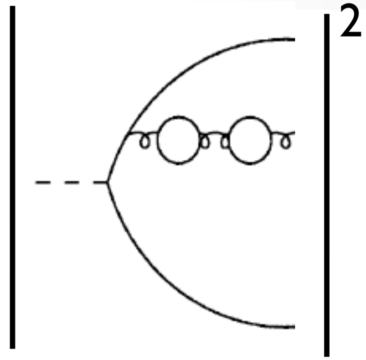
However

$$\sum_q \sigma(e^+e^- \to q\bar{q}) \neq \sigma(e^+e^- \to \mathrm{hadrons})$$

The (small) difference is due to hadronisation corrections, and is of non-perturbative origin

We cannot calculate it in pQCD, but in some cases we can get an idea of its behaviour from the incompleteness of pQCD itself

Suppose we keep calculating to higher and higher orders:



$$\rightarrow \alpha_s^{n+1} \beta_{0f}^n \, n!$$

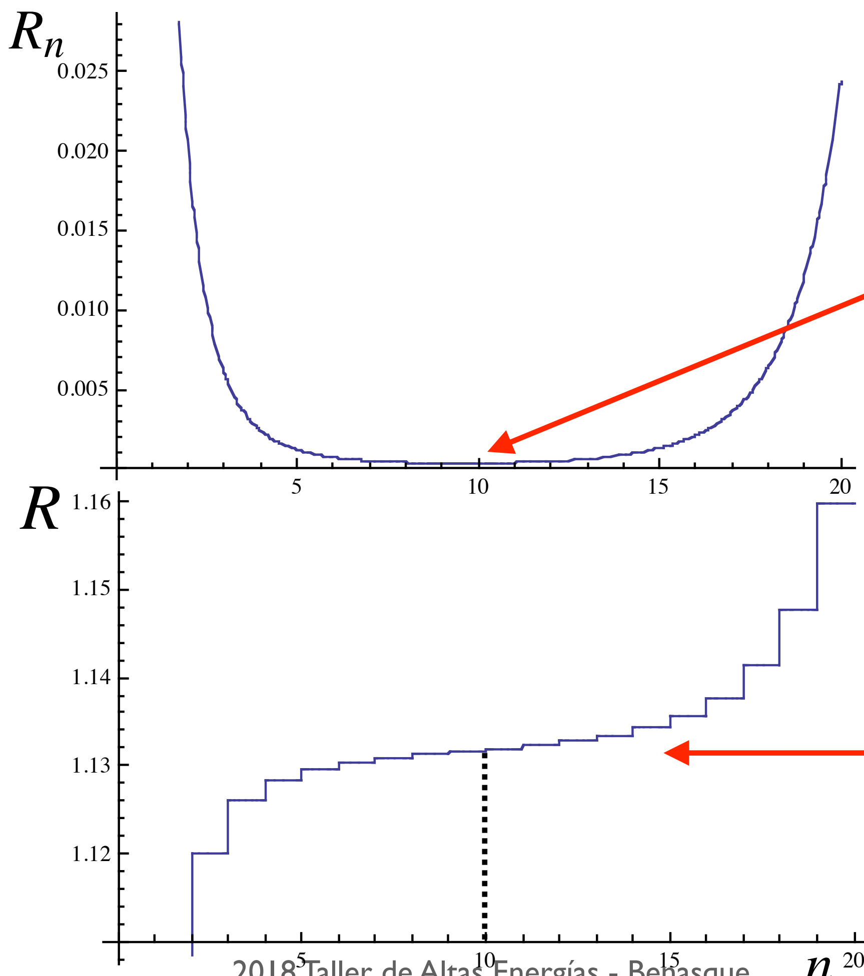
Factorial growth

This is big trouble: the series is not convergent, but only asymptotic

Evidence: try summing

$$R = \sum_{n=0}^{\infty} \alpha^n \, n!$$

$$(\alpha = 0.1)$$



minimal term

$n_{min} \simeq 1/\alpha$

Asymptotic value of the sum:

$$R^{asymp} \equiv \sum_{n=0}^{n_{min}} R_n$$

Power corrections

The renormalons signal the incompleteness of perturbative QCD

One can only define what the sum of a perturbative series is
(like truncation at the minimal term)

The rest is a genuine ambiguity, to be eventually lifted by non-perturbative corrections:

$$R^{true} = R^{pQCD} + R^{NP}$$

In QCD these non-perturbative corrections take the form of power suppressed terms:

$$R^{NP} \sim \exp\left(-\frac{p}{\beta_0 \alpha_s}\right) = \exp\left(-p \ln \frac{Q^2}{\Lambda^2}\right) = \left(\frac{\Lambda^2}{Q^2}\right)^p$$

The value of p depends on the process, and can sometimes be predicted by studying the perturbative series: pQCD - NP physics bridge

Cancellation of singularities

Block-Nordsieck theorem

IR singularities cancel in sum over soft unobserved photons in final state

(formulated for massive fermions ⇒ no collinear divergences)

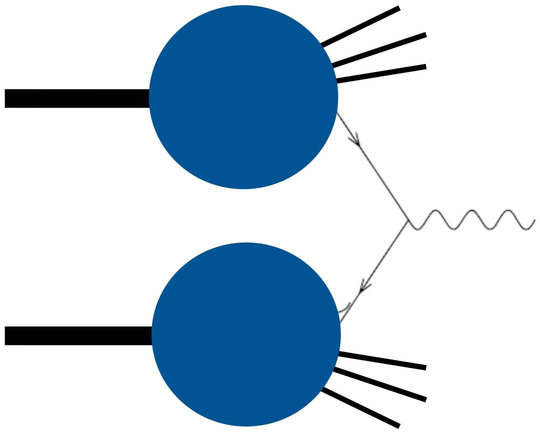
Kinoshita-Lee-Nauenberg theorem

IR and collinear divergences cancel in sum over degenerate initial and final states

These theorems suggest that the observable must be crafted in a proper way for the cancellation to take place

pQCD calculations: hadrons

Turn hadron production in e+e- collisions around: Drell-Yan.



Still easy in Parton Model: just a convolution of probabilities

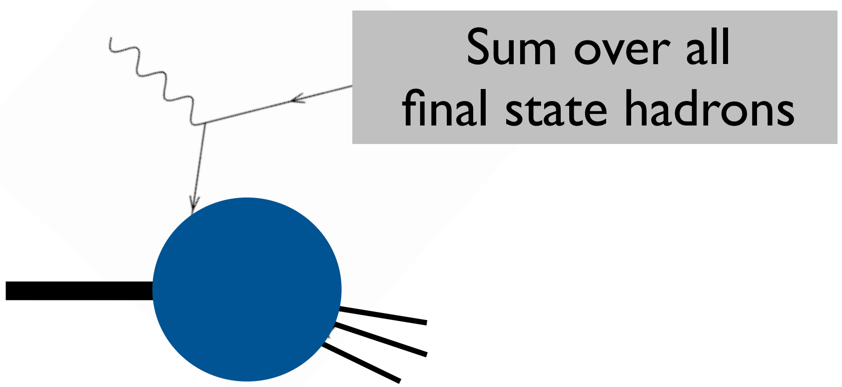
$$\frac{d\sigma_{NN\to\mu\bar{\mu}+X}(Q,p_1,p_2)}{dQ^2 d\ldots} \sim \int d\xi_1 d\xi_2 \sum_{a=\mathrm{q\bar{q}}} \frac{d\sigma^{\mathrm{EW,\,Born}}_{\mathrm{a\bar{a}}\to\mu\bar{\mu}}(Q,\xi_1 p_1,\xi_2 p_2)}{dQ^2 d\ldots}$$

$$\times(\text{probability to find parton a}(\xi_1)\text{ in }N)$$

$$\times(\text{probability to find parton }\bar{\mathrm{a}}(\xi_2)\text{ in }N)$$

This isn't anymore an **inclusive process** as far as hadrons are concerned:
I find them in the initial state, **I can't 'sum over all of them'**

Still, the picture holds at tree level (parton model)
The parton distribution functions can be roughly equated to those extracted from DIS



The non-inclusiveness of a general strong interaction process is a threat to calculability.

What do we do if we can't count on Bloch-Nordsieck and Kinoshita-Lee-Nauenberg?

- Infrared and collinear safe observables
  - less inclusive but still calculable in pQCD
- Factorisation
  - trade divergences for universal measurable quantities

A generic (not fully inclusive) observable *O* is **infrared and collinear safe** if

$$O(X; p_1, \ldots, p_n, p_{n+1} \to 0) \to O(X; p_1, \ldots, p_n)$$

$$O(X; p_1, \ldots, p_n \parallel p_{n+1}) \to O(X; p_1, \ldots, p_n + p_{n+1})$$

Infrared and collinear safety demands that, in the limit of a collinear splitting, or the emission of an infinitely soft particle, the observable remain **unchanged**

IRC safety: proof

Cancellation of divergences in **total cross section** (KLN)

$$\sigma_{tot} = \int_n |M_n^B|^2 d\Phi_n + \int_n |M_n^V|^2 d\Phi_n + \int_{n+1} |M_{n+1}^R|^2 d\Phi_{n+1}$$

A generic observable

$$\begin{aligned}\frac{dO}{dX} &= \int_n |M_n^B|^2 O(X; p_1, \ldots, p_n) d\Phi_n \\ &+ \int_n |M_n^V|^2 O(X; p_1, \ldots, p_n) d\Phi_n + \int_{n+1} |M_{n+1}^R|^2 O(X; p_1, \ldots, p_n, p_{n+1}) d\Phi_{n+1}\end{aligned}$$

In order to ensure the same cancellation existing in σ_{tot}, the definition of the observable must not affect the soft/collinear limit of the real emission term, because it is there that the real/virtual cancellation takes place

Drell-Yan: factorisation

In pQCD (i.e. with gluon emissions), life becomes more complicated

Non fully inclusive process (hadrons in initial state): non cancellation of collinear singularities in pQCD

Same procedure used for renormalising the coupling: reabsorb the divergence into bare non-perturbative quantities, the parton probabilities (collinear factorisation)

The factorisation theorem

$$\sigma^{phys} = F^{bare} F^{bare} \sigma^{divergent}(\varepsilon) = F(\mu) F(\mu) \hat{\sigma}(\mu)$$

infrared regulator

Parton Distribution Function

factorisation scale

short-distance cross section

and (schematically) $$F(\mu) = F^{bare}\left(1 + \alpha_s P \log\frac{\mu^2}{\mu_0^2}\right)$$

This factor universal

Drell-Yan: NLO result

$$\frac{d^2\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}(z,Q^2,\mu^2)}{dQ^2} = \sigma_0(Q^2)\left(\frac{\alpha_s(\mu)}{\pi}\right)\left\{2(1+z^2)\left[\frac{\ln(1+z^2)}{1-z}\right]_+ \right.$$

→ soft and collinear large log

$$\left. -\frac{[(1+z^2)\ln z\,]}{(1-z)} + \left(\frac{\pi^2}{3}-4\right)\delta(1-z)\right\}$$

$$+\,\sigma_0(Q^2)\,C_F\,\frac{\alpha_s}{\pi}\left[\frac{1+z^2}{1-z}\right]_+ \ln\left(\frac{Q^2}{\mu^2}\right)$$

→ residual of collinear factorisation

A prototype of QCD calculations: many finite terms but, more importantly, a few characteristic large logarithms

In many circumstances and kinematical situations the logs are much more important than the finite terms: hence in pQCD resummations of these terms are often phenomenologically more relevant than a full higher order calculation

Factorisation

$$\sigma^{phys} = F(\mu)\hat{\sigma}(\mu)$$

Evolution

$$F(\mu) = F^{bare}\left(1 + \alpha_s P \log\frac{\mu^2}{\mu_0^2}\right)$$

$$\frac{d}{d\ln\mu^2}\ln\sigma^{phys} = 0 \quad \Rightarrow \quad \frac{d\ln\hat{\sigma}(\mu)}{\ln\mu^2} = -\frac{d\ln F(\mu)}{\ln\mu^2} = -\alpha_s P$$

DGLAP evolution equations for PDF's

Resummation

Solution of evolution equations resums higher order terms

Responsible for scaling violations (for instance in DIS structure functions)

DGLAP equations

[Dokshitzer, Gribov, Lipatov, Altarelli, Parisi]

$$\frac{df_q(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}(z)\, f_q(\frac{x}{z},t) \,+\, P_{qg}(z) f_g(\frac{x}{z},t)\right]$$

$$\frac{df_g(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{gq}(z)\sum_{i=q,\bar{q}} f_i\left(\frac{x}{z},t\right) + P_{gg}(z) f_g\left(\frac{x}{z},t\right)\right]$$

The Altarelli-Parisi kernels control the evolution of the Parton Distribution Functions

Altarelli-Parisi kernels

[Altarelli-Parisi, 1977]

$$P_{gg} \to 2C_A \left\{ \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right\} + \delta(1-x) \left[\frac{11C_A - 2n_f}{6} \right]$$

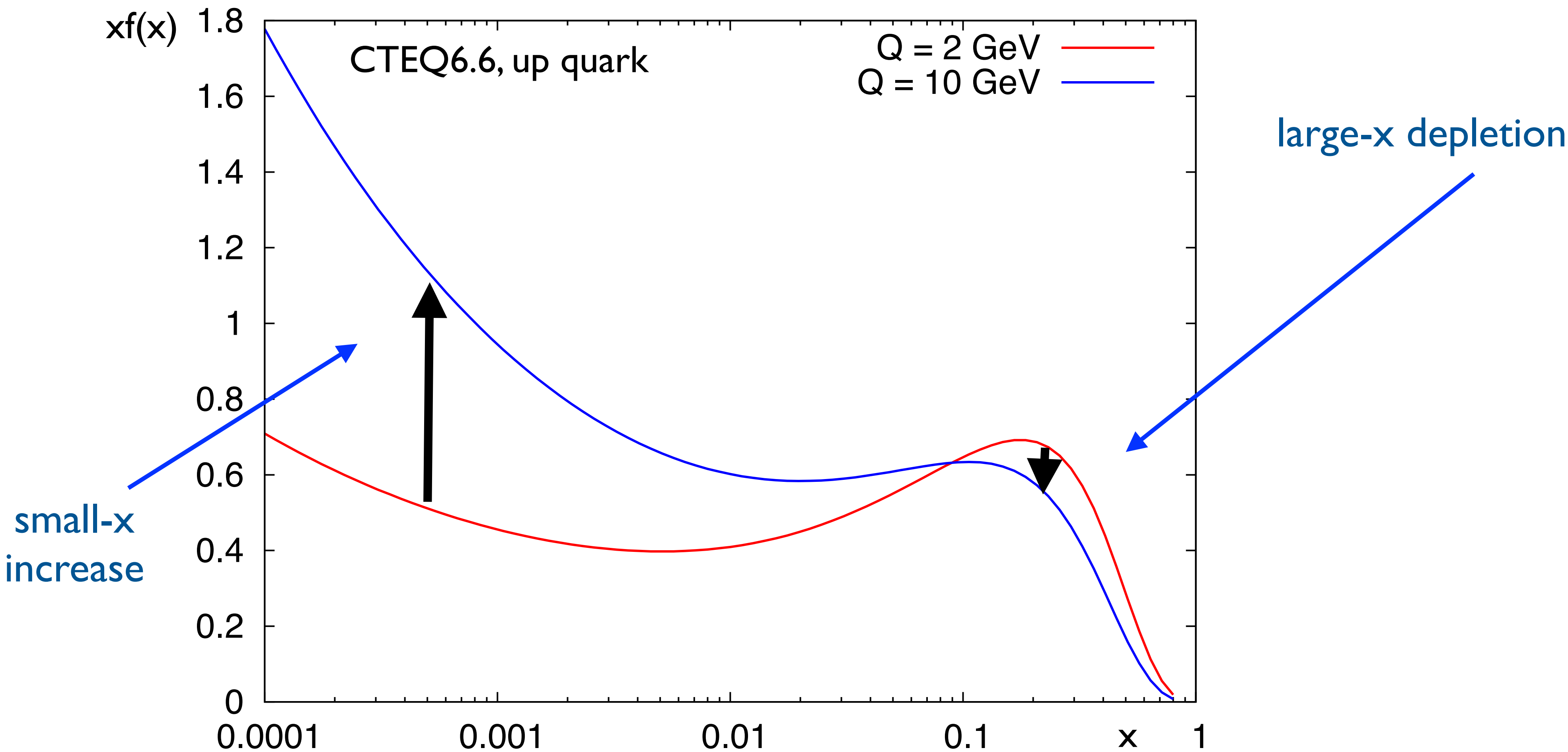
$$P_{qq}(z) \to \left(\frac{1+z^2}{1-z} \right)_+ \equiv \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dy \left(\frac{1+y^2}{1-y} \right)$$

$$P_{qg} = \frac{1}{2} \left[z^2 + (1-z)^2 \right]$$

$$P_{gq}(z) = \quad C_F \frac{1+(1-z)^2}{z}$$

Higher orders: Curci-Furmansky-Petronzio (1980), Moch, Vermaseren, Vogt (2004)

DGLAP evolution of PDFs



Evolution (i.e. higher momentum scale) produces more partons at small momentum fraction (because they lose energy by radiating)

As for the coupling, one can't predict PDF's values in pQCD, but only their evolution

Take-home points

- universal character of soft/collinear emission
- both real and virtual diagrams usually contribute to an observable (and are both needed to cancel divergences)
- not everything is calculable. Restrict to IRC-safe observables and/or employ factorisation

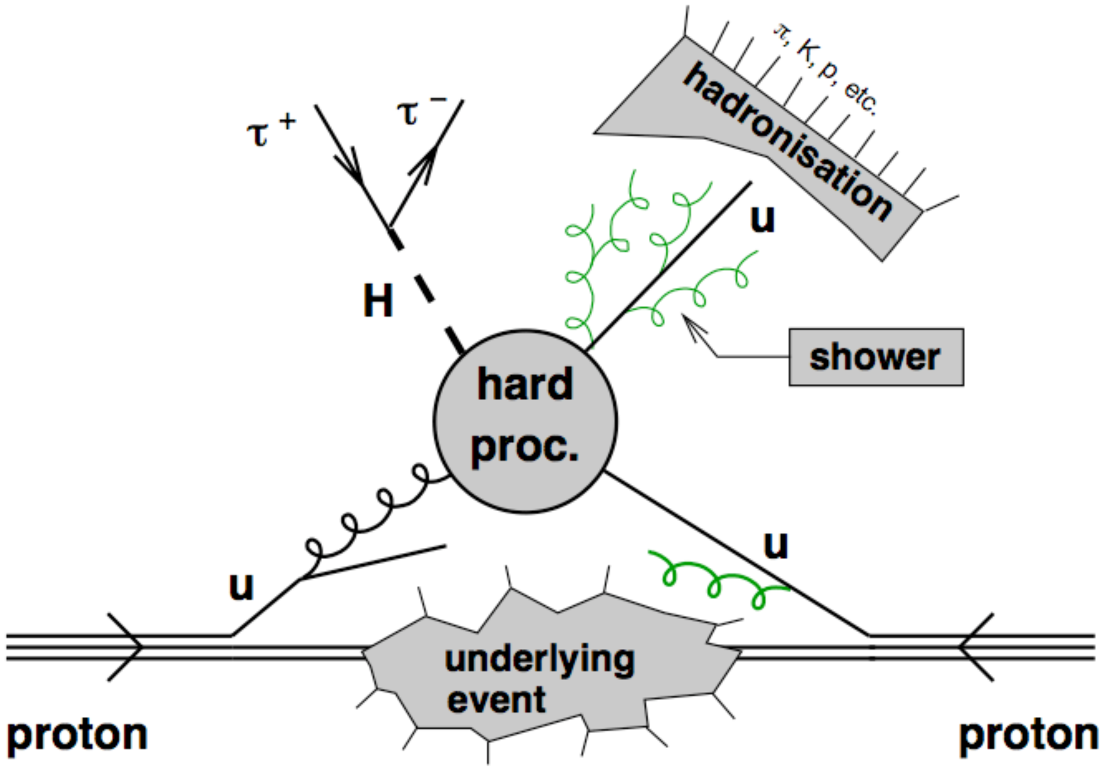
2018 TAE School
September 2018, Benasque

QCD, Jets and Monte Carlo techniques

Matteo Cacciari
LPTHE Paris and Université Paris Diderot

Lecture 1 - Basics of QCD
Lecture 2 - **Higher orders and Monte Carlos**
Lecture 3 - Jets

Ingredients and tools



- PDFs
- Hard scattering and shower
- Final state tools

Tools for the hard scattering

Can be divided in

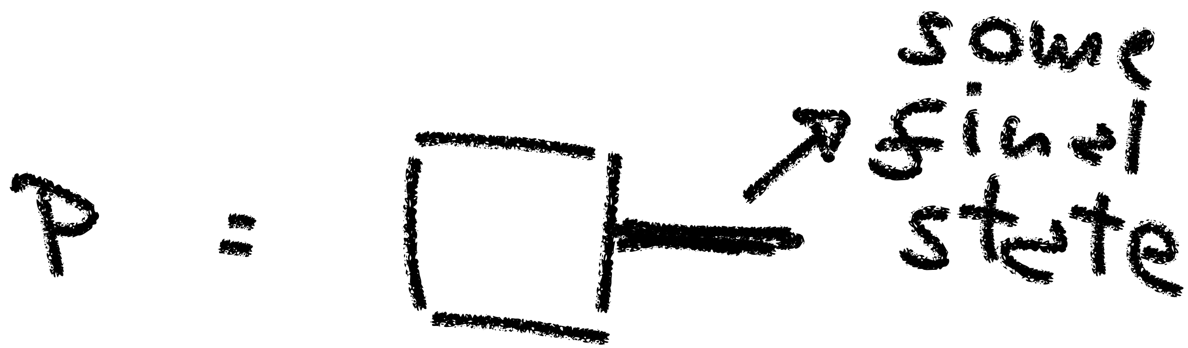
- **Integrators**
  - evaluate the (differential) cross section by integrating the calculation over the phase space, yielding (partly) inclusive quantities
  - Produce weighted events (the weight being the value of the cross section)
  - Calculations exist at LO, NLO, NNLO

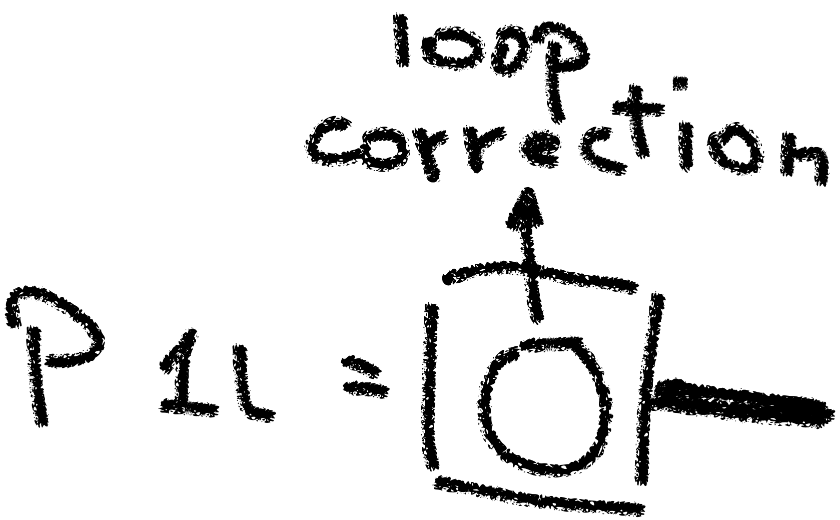
- **Generators**
  - generate fully exclusive configurations
  - Events are unweighted (i.e. produced with the frequency nature would produce them)
  - Easy at LO, get complicated when dealing with higher orders

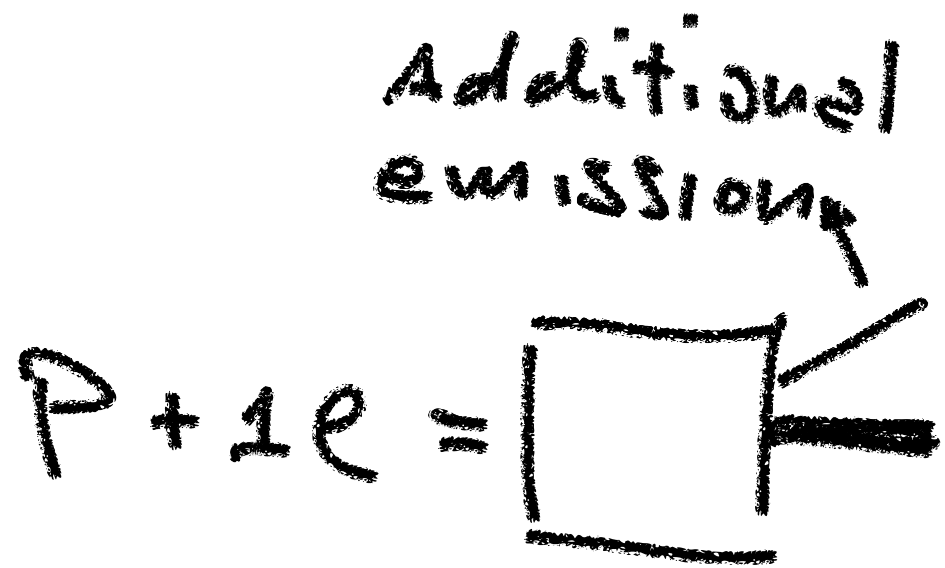
(Higher order) calculations

What goes into them ?

Nomenclature







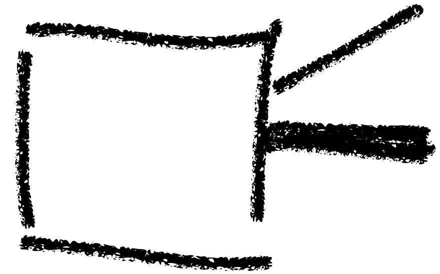
N.B.

$$P + \underline{1}e \neq P + \underline{1}\,\text{jet}$$

e = emission

or

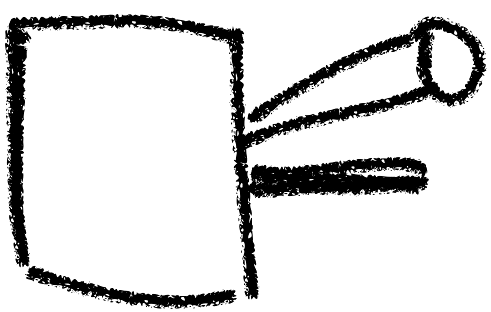
e = eeg

P+1e = 

Contributes to:

P+1 jet

P+X

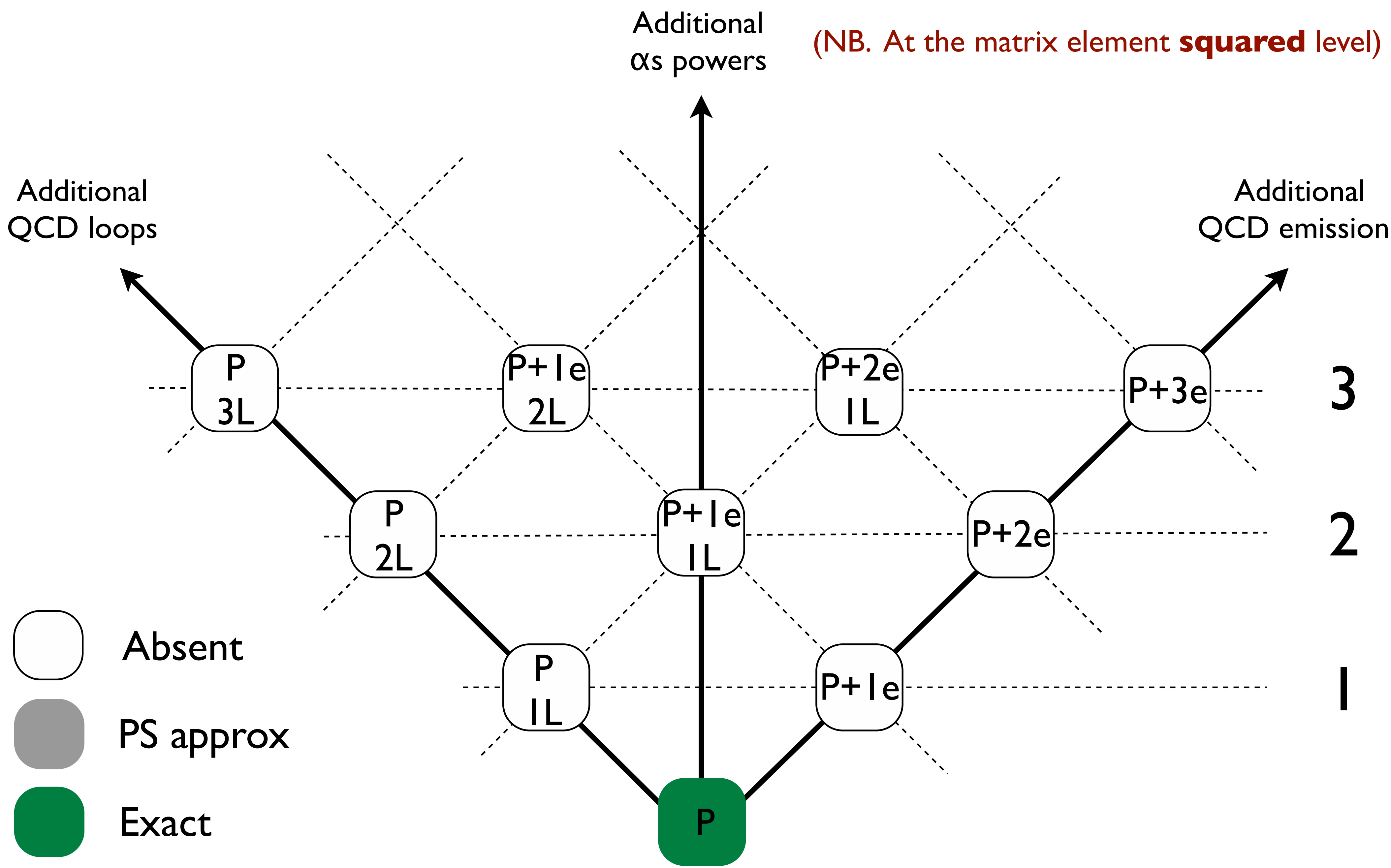


or

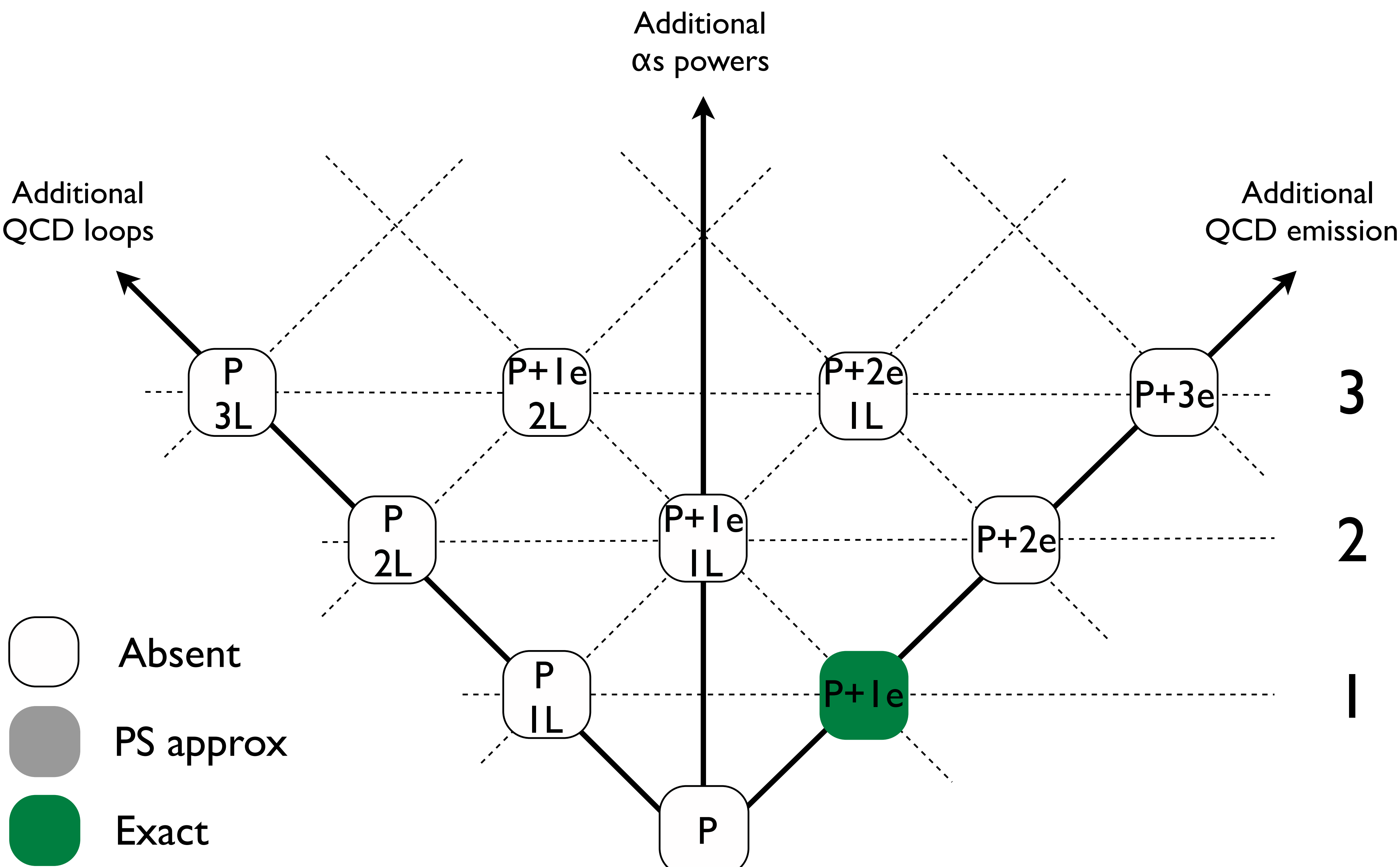
(if e is integrated over)

Process P exact at LO, nothing else

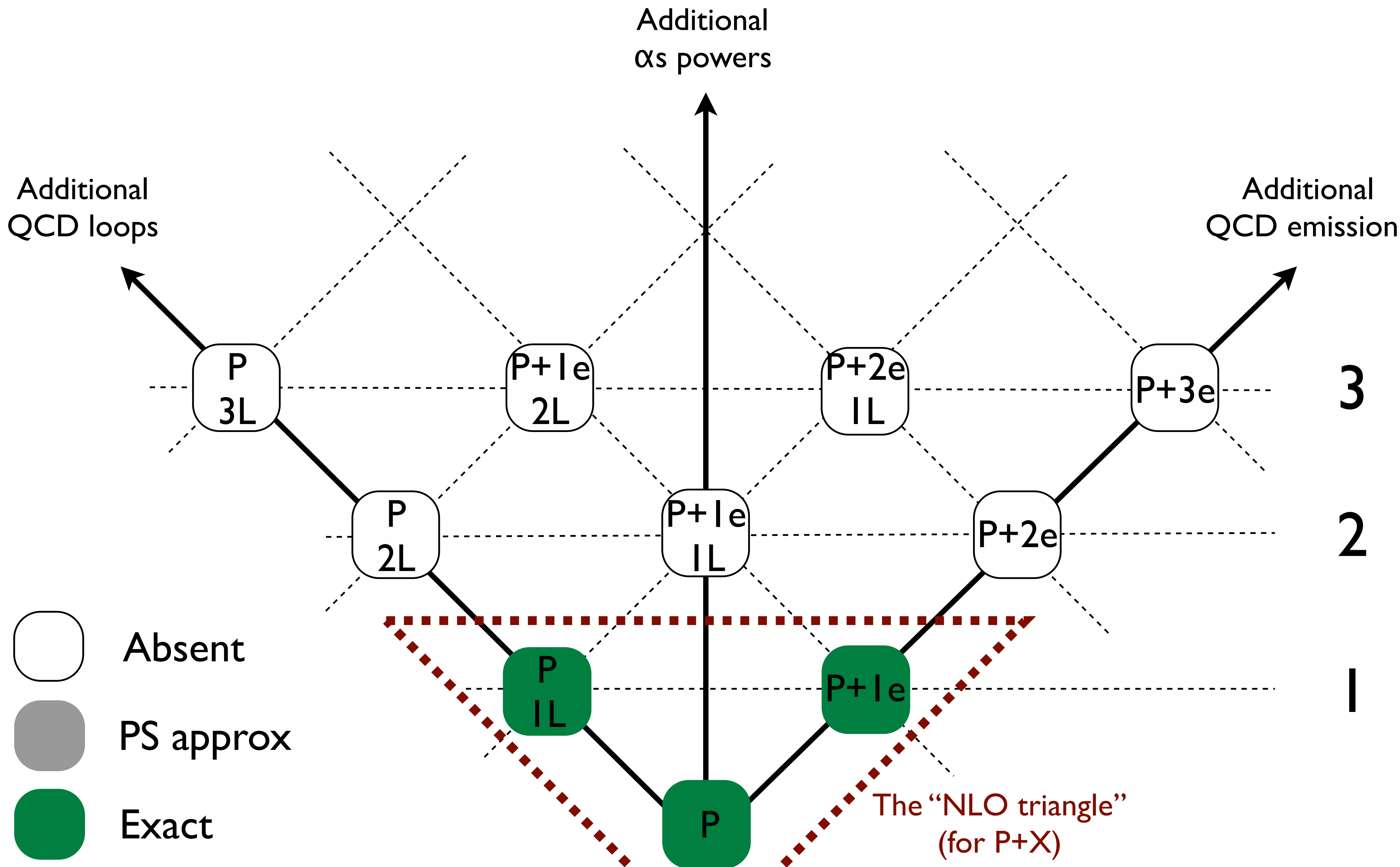
(NB. At the matrix element **squared** level)



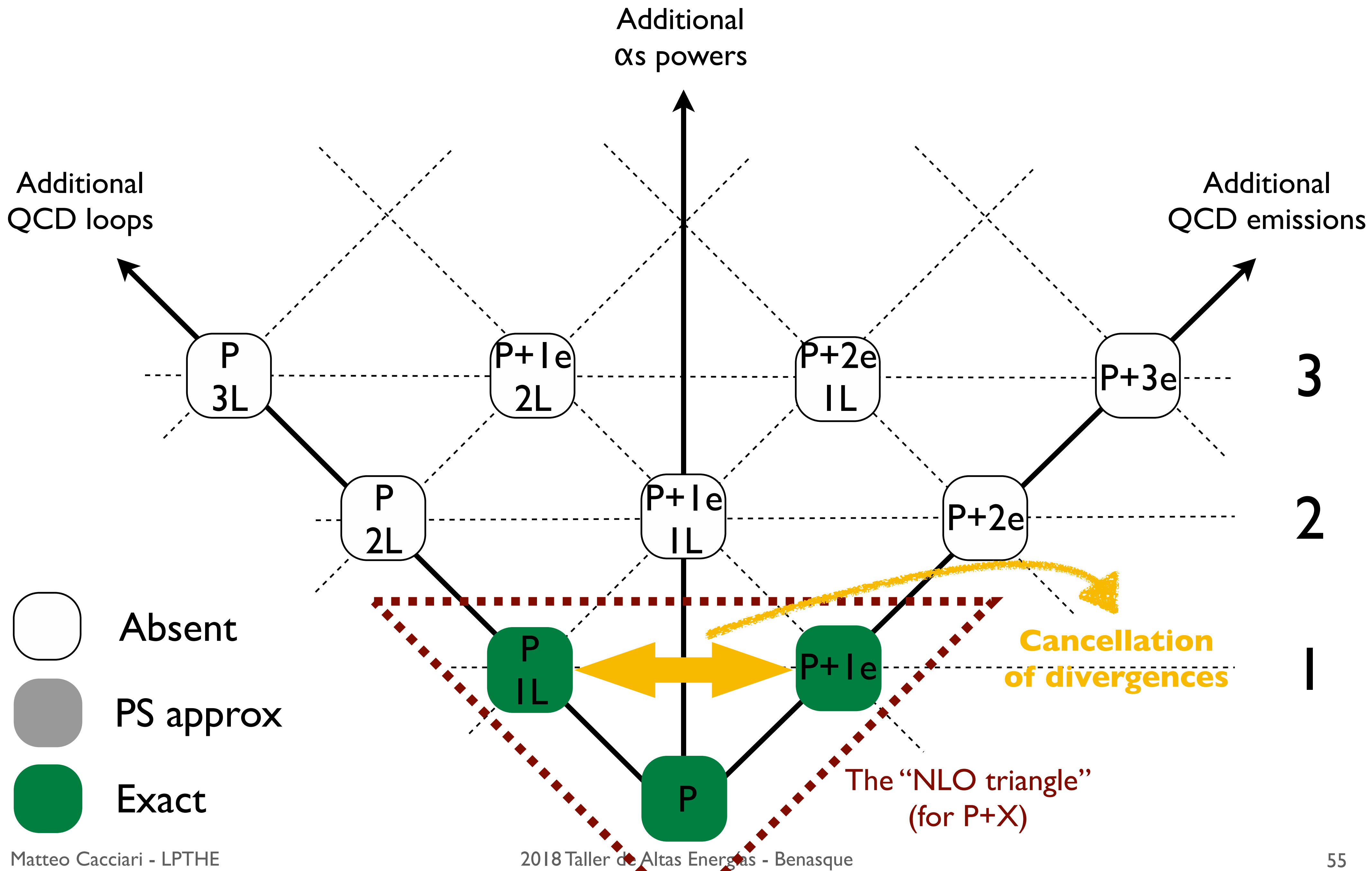
Process P+1j exact at LO, nothing else



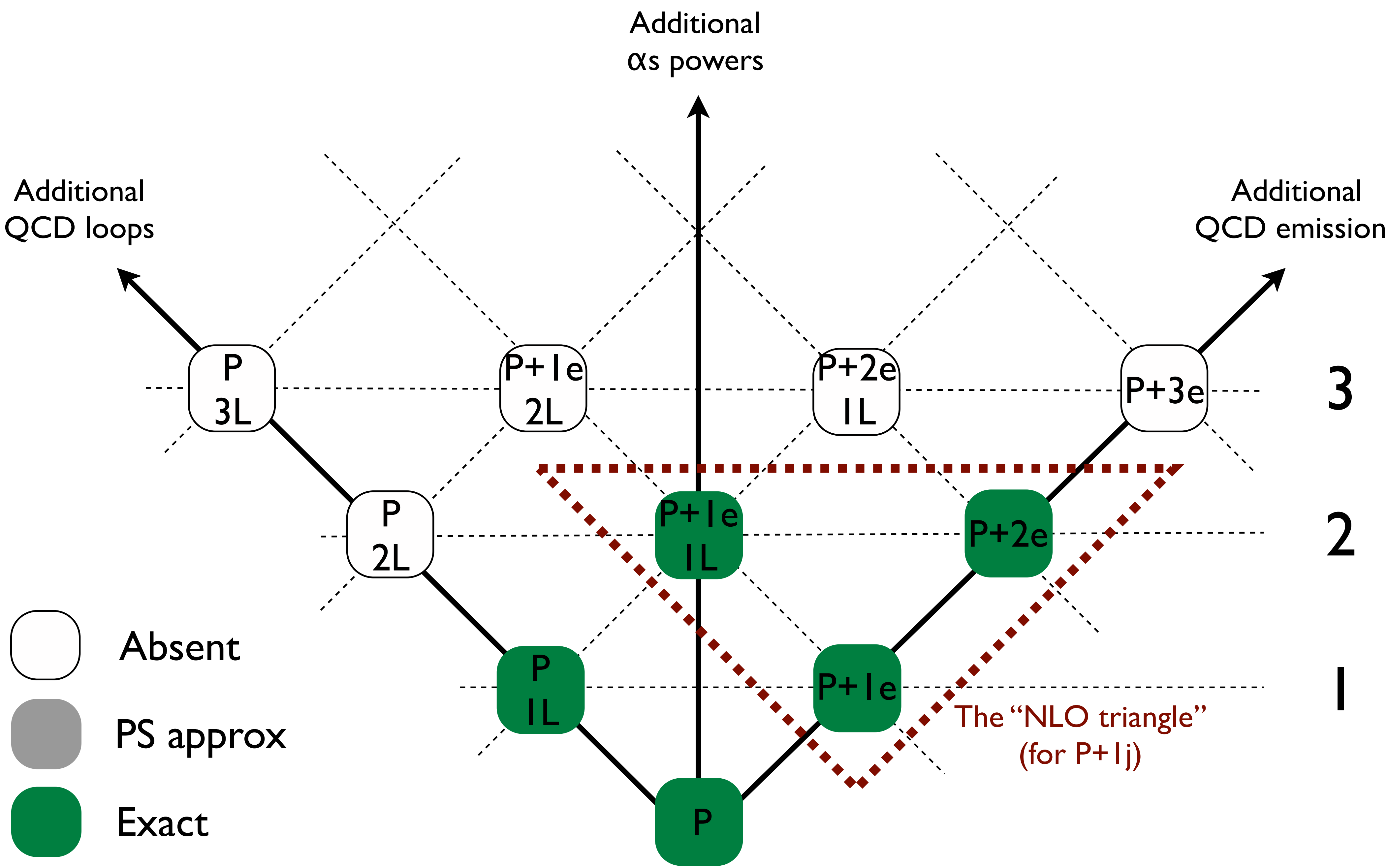
Process P exact at NLO, P+1j exact at LO, nothing else



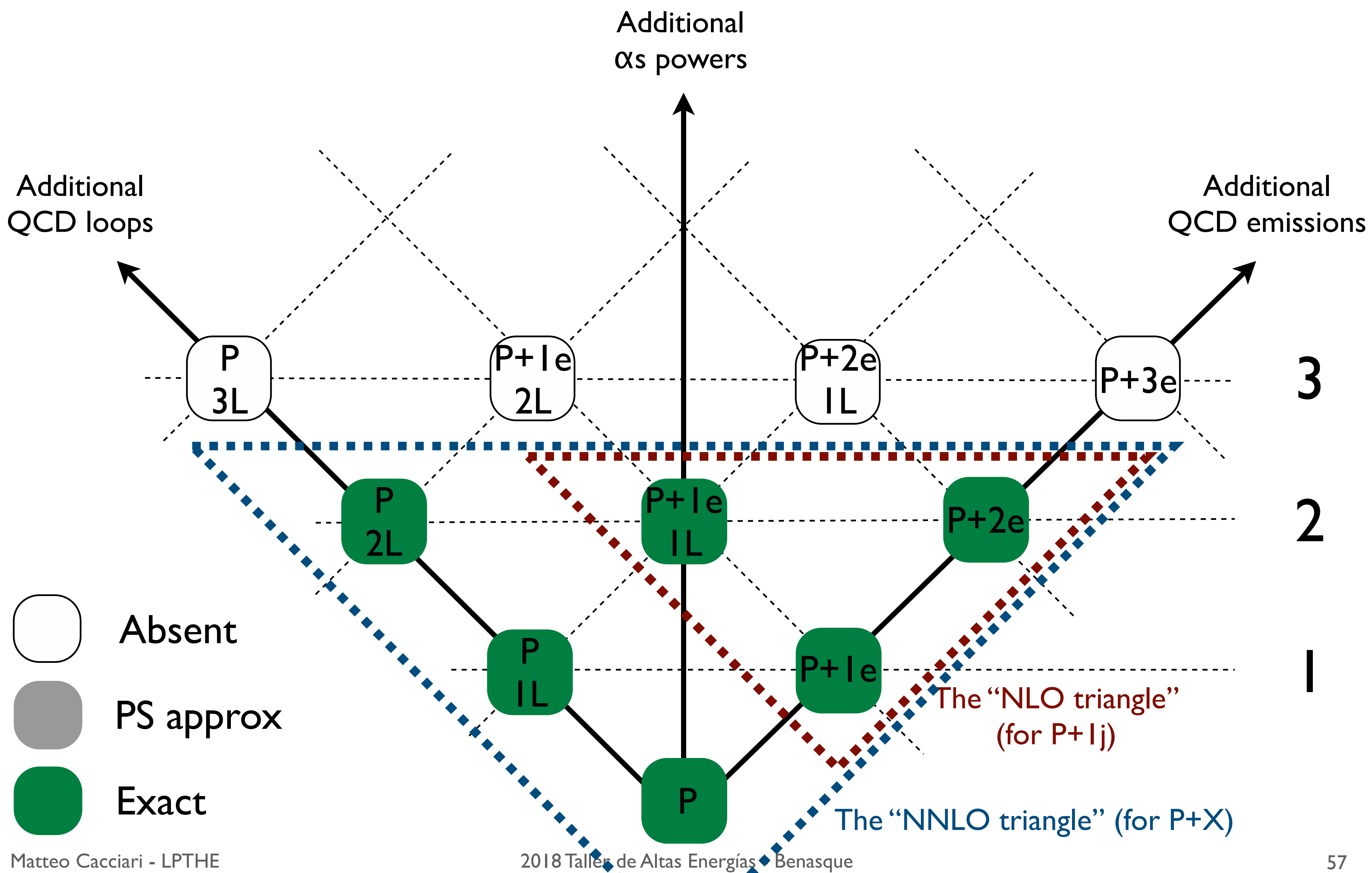
Process P exact at NLO, P+1j exact at LO, nothing else



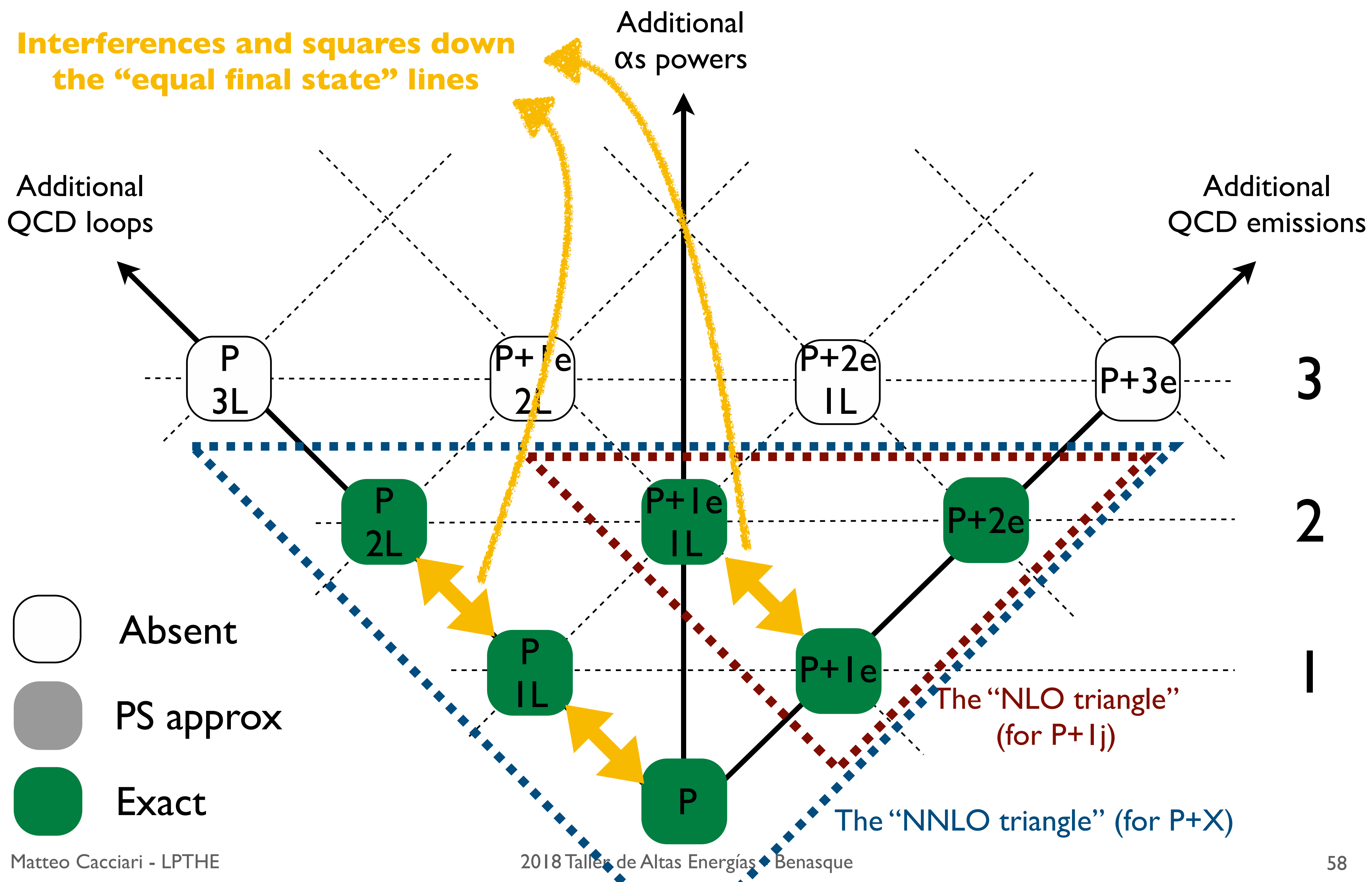
Process P and P+1j exact at NLO, P+2j at LO



Process P exact at NNLO, P+1j exact at NLO, P+2j at LO



Process P exact at NNLO, P+1j exact at NLO, P+2j at LO



Fixed order calculation

Born

$$d\sigma^{Born} = B(\Phi_B)d\Phi_B$$

NLO

$$d\sigma^{NLO} = [B(\Phi_B) + V(\Phi_B)]\, d\Phi_B + R(\Phi_R)d\Phi_R$$

$$d\Phi_R = d\Phi_B\, d\Phi_{rad}$$

$$d\Phi_{rad} = d\cos\theta\, dE\, d\phi$$

Problem:
$V(\Phi_B)$ and $\int R d\Phi_R$ are divergent

Subtraction terms

An observable O is **infrared and collinear safe** if

$$O(\Phi_R(\Phi_B, \Phi_{rad})) \to O(\Phi_B)$$

Soft or collinear limit

One can then write, with C→R in the soft/coll limit,

This integration performed analytically

$$\langle O \rangle = \int \left[B(\Phi_B) + V(\Phi_B) + \int C(\Phi_R) d\Phi_{rad} \right] O(\Phi_B) d\Phi_B$$

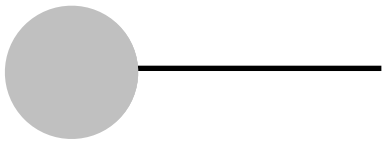
$$+ \left[R(\Phi_R) O(\phi_R) - C(\Phi_R) O(\Phi_B) \right] d\Phi_R$$

Separately finite

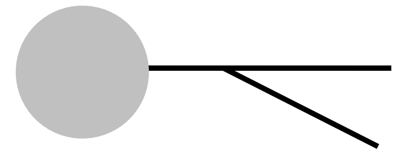
This (or a similar) cancellation will always be implicit in all subsequent equations

Parton Shower Monte Carlo

Exploit factorisation property of soft and collinear radiation



σ_n



σ_{n+1}

Factorisation

$$d\sigma_{n+1}(\Phi_{n+1}) = \mathcal{P}(\Phi_{\mathrm{rad}})\, d\sigma_n(\Phi_n)\, d\Phi_{\mathrm{rad}}$$

Emission probability

$$\mathcal{P}(\Phi_{\mathrm{rad}})\, d\Phi_{\mathrm{rad}} \approx \frac{\alpha_{\mathrm{S}}(q)}{\pi} \frac{dq}{q} P(z,\phi)\, dz \frac{d\phi}{2\pi}$$

Iterate emissions to generate higher orders (in the soft/collinear approximation)

Based on the **iterative emission of radiation** described in the **soft-collinear limit**

$$d\sigma^{(MC)}(\Phi_R)d\Phi_R = B(\Phi_B)d\Phi_B \mathcal{P}(\Phi_{rad})d\Phi_{rad}$$

Pros: soft-collinear radiation is resummed to all orders in pQCD

Cons: hard large-angle radiation is missing

Overall accuracy will be leading log (LL) for the radiation, and leading order (i.e. Born) for the integrated cross sections

A key ingredient of a parton shower Monte Carlo:

Sudakov form factor
$\Delta(t_1,t_2)$

Probability of **no emission** between the scales t_1 and t_2

Example:

- decay probability per unit time of a nucleus = c_N
- Sudakov form factor $\Delta(t_0,t) = \exp(-c_N(t-t_0))$

Probability that nucleus does **not** decay between t_0 and t

Sudakov form factor: derivation

Decay probability per unit time = $$\frac{dP}{dt} = c_N$$

Probability of **no** decay between t_0 and t = $\Delta(t_0,t)$ [with $\Delta(t_0,t_0) = 1$]

⇒ Probability of decay between t_0 and t = $1 - \Delta(t_0,t)$ [unitarity: either you decay or you don't]

Decay probability per unit time **at time t** can be written in two ways:

1. $$P^{\rm dec}(t) = \frac{d}{dt}\Big(1 - \Delta(t_0,t)\Big) = -\frac{d\Delta(t_0,t)}{dt}$$

2. $$P^{\rm dec}(t) = \Delta(t_0,t)\frac{dP}{dt}$$

No decay until t, probability per unit time to decay at t

Sudakov form factor: derivation

Equating the two expressions for $P^{dec}(t)$ we get

$$-\frac{d\Delta(t_0,t)}{dt} = \Delta(t_0,t)\frac{dP}{dt}$$

We can solve the differential equation using $dP/dt = c_N$ and we get

$$\Delta(t_0,t) = \exp(-c_N(t-t_0))$$

If the decay probability depends on t (and possibly other variables, call them z) this generalises to

$$\Delta(t_0,t) = \exp\left(-\int_{t_0}^{t} dt' \int dz\, c_N(t',z)\right)$$

Sudakov form factor in QCD

Emission probability

$$\mathcal{P}(\Phi_{\rm rad})\,{\rm d}\Phi_{\rm rad} \approx \frac{\alpha_{\rm S}(q)}{\pi}\,\frac{{\rm d}q}{q}\,P(z,\phi)\,{\rm d}z\,\frac{{\rm d}\phi}{2\pi}$$

Sudakov form factor = probability of **no emission**

from large scale q_1 to smaller scale q_2

$$\Delta_{\rm S}(q_1, q_2) = \exp\left[-\int_{q_2}^{q_1} \frac{\alpha_{\rm S}(q)}{\pi}\,\frac{{\rm d}q}{q}\int_{z_0}^{1} P(z)\,{\rm d}z\right]$$

Conventions for Sudakov form factor

$$\Delta_{\rm S}(q_1, q_2) = \exp\left[-\int_{q_2}^{q_1} \frac{\alpha_{\rm S}(q)}{\pi}\frac{{\rm d}q}{q}\int_{z_0}^{1} P(z)\,{\rm d}z\right]$$

Full expression,with details of soft-collinear radiation probability

$$\Delta(p_{\rm T}) = \exp\left[-\int_{p_{\rm T}}^{Q} \frac{\frac{{\rm d}\sigma^{\rm (MC)}}{{\rm d}y\,{\rm d}p'_{\rm T}}}{\frac{{\rm d}\sigma^{\rm (B)}}{{\rm d}y}}{\rm d}p'_{\rm T}\right]$$

Dropped upper limit, taken implicitly to be the hard scale Q

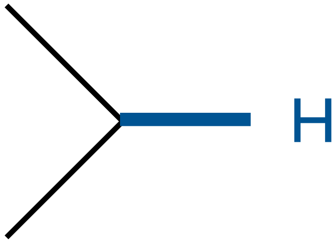
$$\Delta_R(p_T) = \exp\left[-\int \frac{R}{B}\Theta(k_T(\Phi_R) - p_T)d\Phi_{rad}\right]$$

Introduced suffix (R in this case) to indicate expression used to described radiation

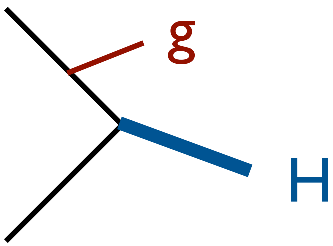
$$\Delta_R(p_T) = \exp\left[-\int_{p_T} \frac{R}{B}d\Phi_{rad}\right]$$

Integration boundaries only implicitly indicated

PS example: Higgs plus radiation



Leading order.
No radiation, Higgs $p_T = 0$



With emission of radiation
Higgs $p_T \neq 0$

Description of hardest emission in PS MC (either event is generated)

$$\frac{d\sigma^{(MC)}}{dy\,dp_T} = \frac{d\sigma^{(B)}}{dy}\delta(p_T)\Delta(Q_0) + \Delta(p_T)\frac{d\sigma^{(MC)}}{dy\,dp_T}$$

$$\Delta(p_T) = \exp\left[-\int_{p_T}^{Q} \frac{\frac{d\sigma^{(MC)}}{dy\,dp'_T}}{\frac{d\sigma^{(B)}}{dy}} dp'_T\right]$$

Sudakov form factor

- $\frac{d\sigma^{(B)}}{dy}$: x-sect for no emission
- $\delta(p_T)\Delta(Q_0)$: prob. of **no emission** (down to the PS cutoff)
- $\Delta(p_T)$: prob. of no emission down to p_T
- $\frac{d\sigma^{(MC)}}{dy\,dp_T}$: x-sect for **emission at p_T**, as described by the MC

Toy shower for the Higgs p_T

Gavin Salam has made public a 'toy shower' that generates the Higgs transverse momentm via successive emissions controlled by the Sudakov form factor

$$\Delta(p_T) = \exp\left[-\frac{2\alpha_s C_A}{\pi}\ln^2\frac{p_{T,\max}^2}{p_T^2}\right]$$

You can get the code at
https://github.com/gavinsalam/zuoz2016-toy-shower

NB. In order to get more realistic results you need at least at the code in v2

Shower unitarity

It holds

$$\int_0^Q \left[\delta(p_T)\Delta(Q_0) + \frac{\Delta(p_T)\frac{d\sigma^{(MC)}}{dy\,dp_T}}{\frac{d\sigma^{(B)}}{dy}}\right] dp_T = \Delta(Q_0) + \int_{Q_0}^Q \frac{d\Delta(p_T)}{dp_T} dp_T = \Delta(Q) = 1$$

Shower unitarity

so that

$$\int_0^Q dp_T \frac{d\sigma^{(MC)}}{dy\,dp_T} = \frac{d\sigma^{(B)}}{dy} \int_0^Q \left[\delta(p_T)\Delta(Q_0) + \frac{\Delta(p_T)\frac{d\sigma^{(MC)}}{dy\,dp_T}}{\frac{d\sigma^{(B)}}{dy}}\right] dp_T = \frac{d\sigma^{(B)}}{dy}$$

A parton shower MC correctly reproduces the Born cross section for integrated quantities

PS MC in different notation

Writing the real cross section as described by the Monte Carlo (i.e. with the parton shower) simply as R^{MC}, we can rewrite

$$d\sigma^{MC} = Bd\Phi_B \left[\Delta_{MC}(Q_0) + \Delta_{MC}(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right]$$

with $$\Delta_{MC}(p_T) = \exp\left[-\int_{p_T} \frac{R^{MC}}{B} d\Phi_{rad} \right]$$

as our Master Formula for a Parton Shower Monte Carlo.

Thanks to the shower unitarity, it holds

$$\Rightarrow \int d\sigma^{MC} = \int Bd\Phi_B = \sigma^{LO}$$

Matrix Element corrections

In a PS Monte Carlo $R^{(MC)}(\Phi_R) = B(\Phi_B)\mathcal{P}(\Phi_{rad})$ soft-collinear approximation

Replace the MC description of radiation with the **correct** one: $\mathcal{P}(\Phi_{rad}) \rightarrow \dfrac{R}{B}$

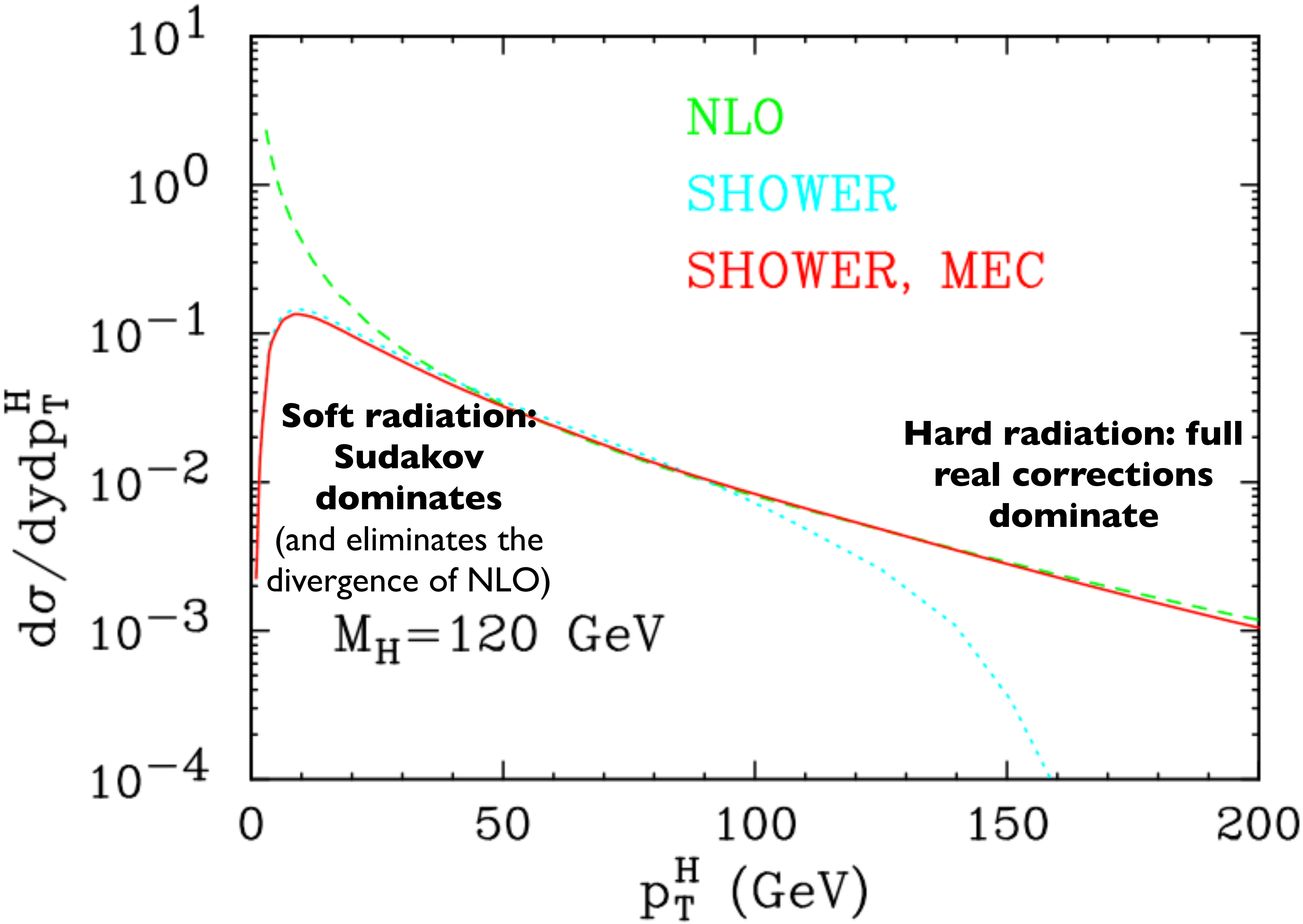
The Sudakov becomes

$$\Delta(p_{\rm T}) = \exp\left[-\int_{p_{\rm T}}^{Q} \frac{\frac{{\rm d}\sigma^{\rm (MC)}}{{\rm d}y\,{\rm d}p'_{\rm T}}}{\frac{{\rm d}\sigma^{\rm (B)}}{{\rm d}y}} {\rm d}p'_{\rm T}\right] \longrightarrow \Delta_R(p_T) = \exp\left[-\int \frac{R}{B}\Theta(k_T(\Phi_R) - p_T)d\Phi_{rad}\right]$$

and the x-sect formula for the hardest emission

$$d\sigma^{MEC} = Bd\Phi_B\left[\Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B}d\Phi_{rad}\right]$$

Matrix Element corrections



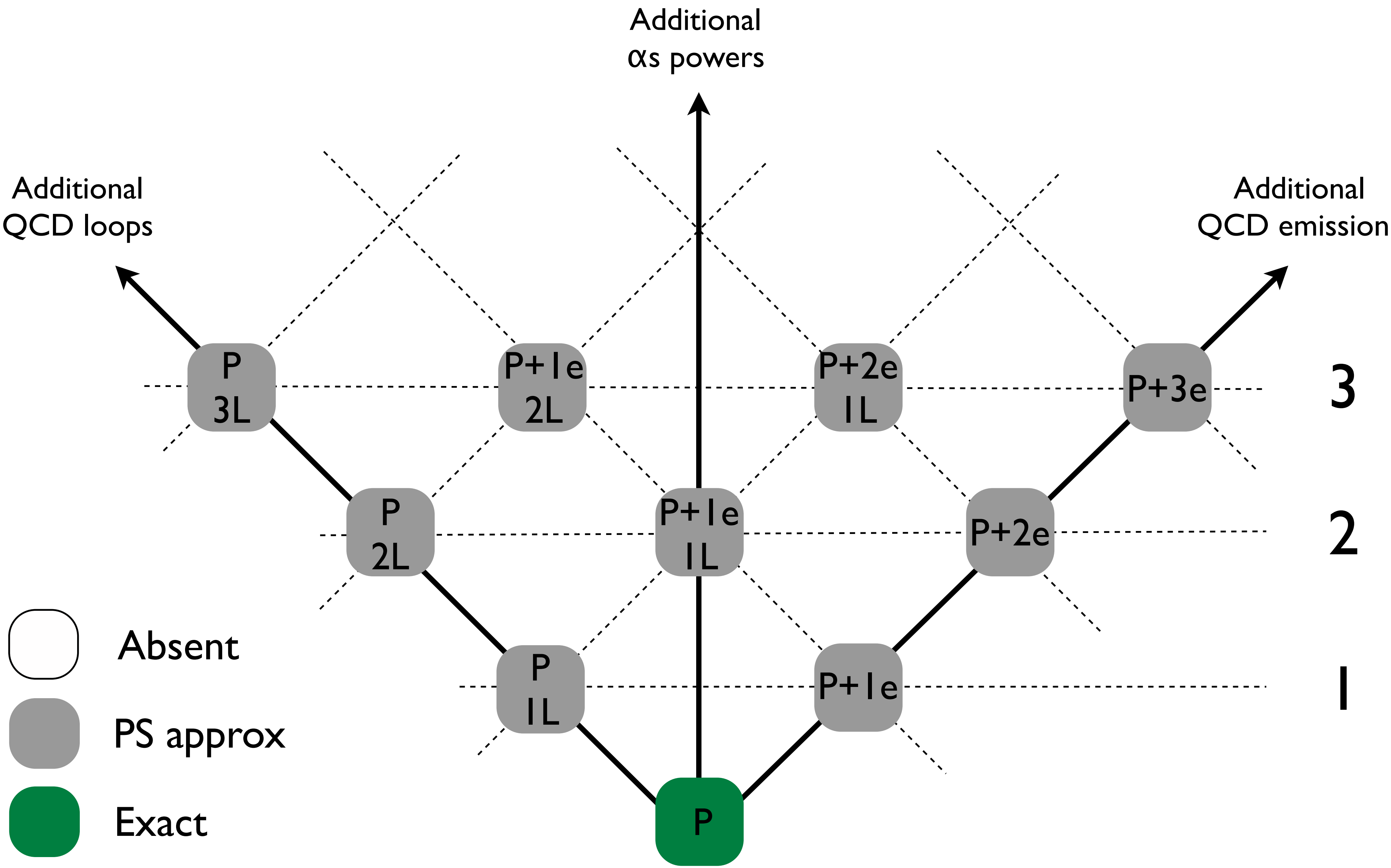
Beyond PS MC

We wish to go beyond a Parton Shower (+MEC) Monte Carlo, so that

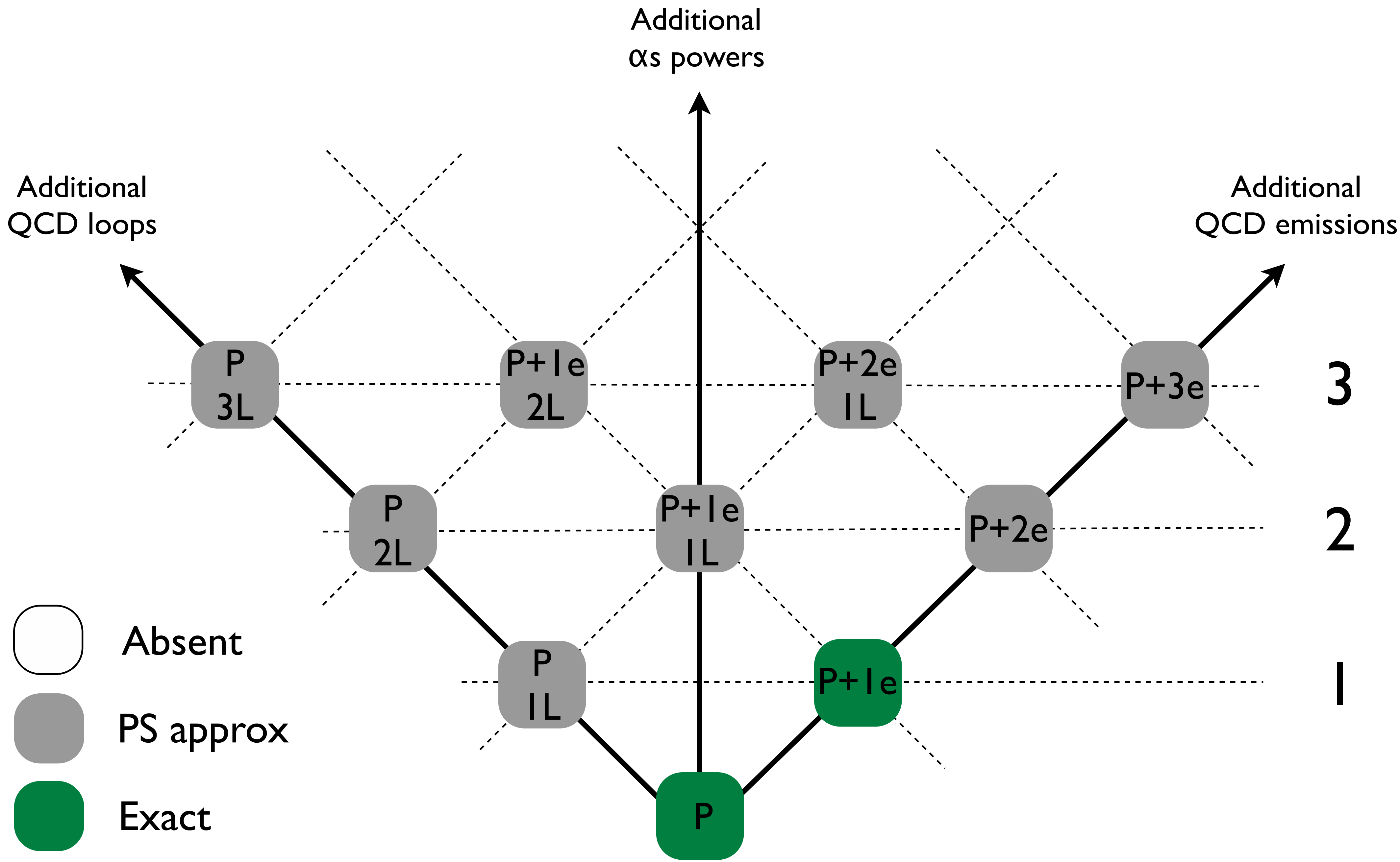
- we can successfully interface **matrix elements for multi-parton production with a parton shower**
- we can successfully interface a **parton shower with a NLO calculation**

It's a **quest for exactness** of ever more complex processes

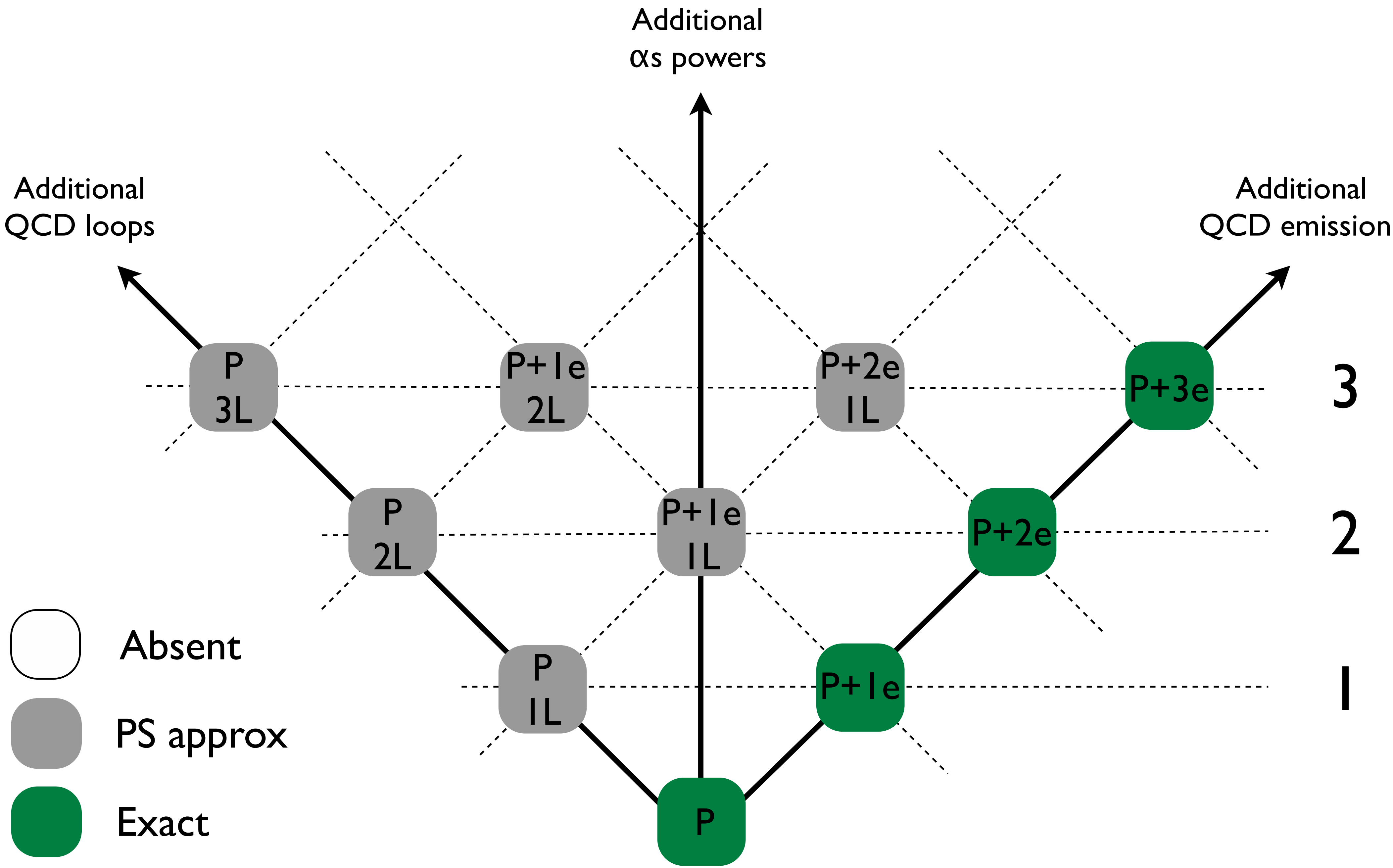
Process P exact at LO, the rest PS approximation



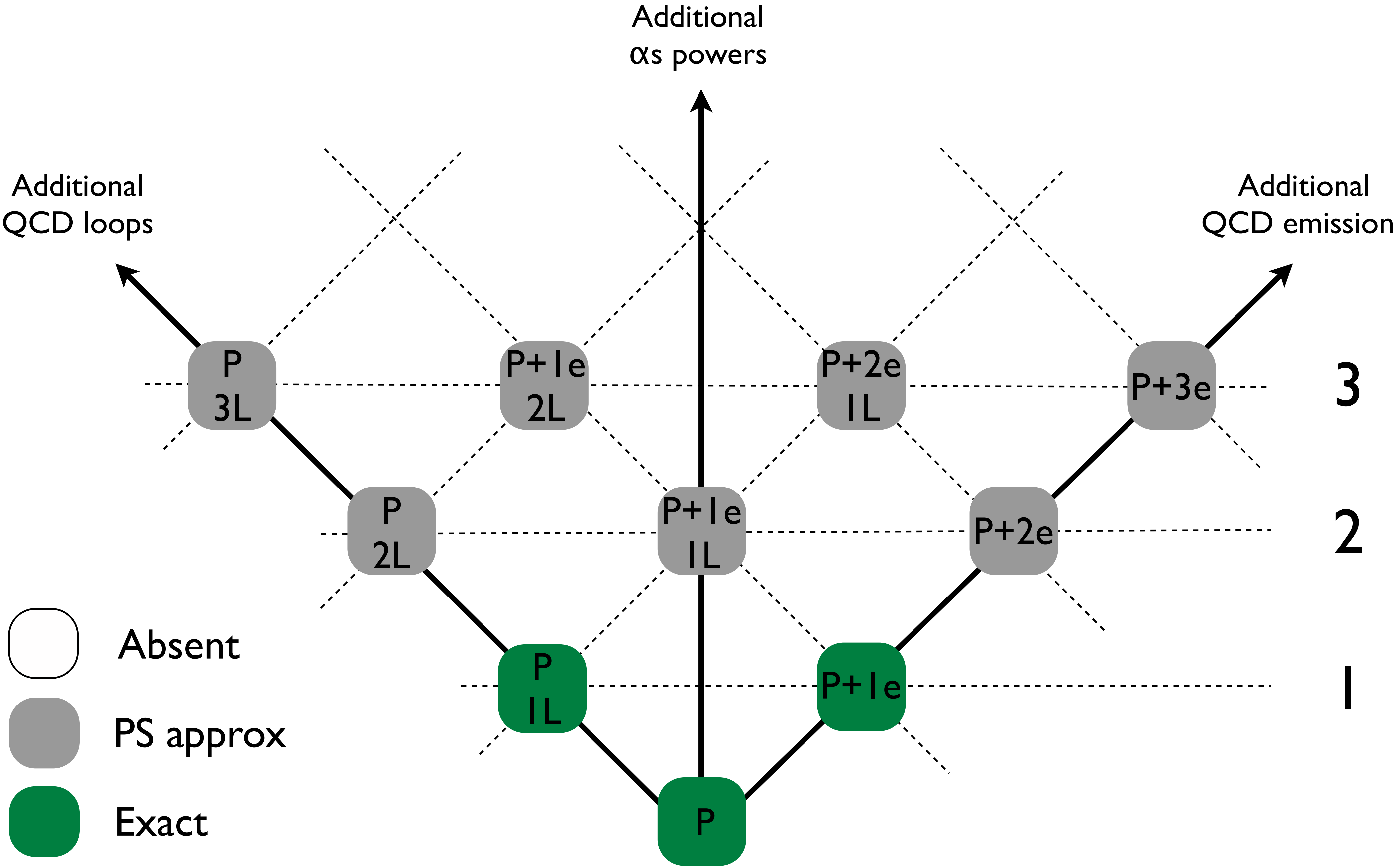
Process P and P+1j exact at LO, the rest PS approximation
[PS+MEC or PS from ME for P+1e]



Process P, P+1j, P+2j, ... exact at LO, the rest PS approx. [PS+Matrix Element (CKKW, MLM,....)]

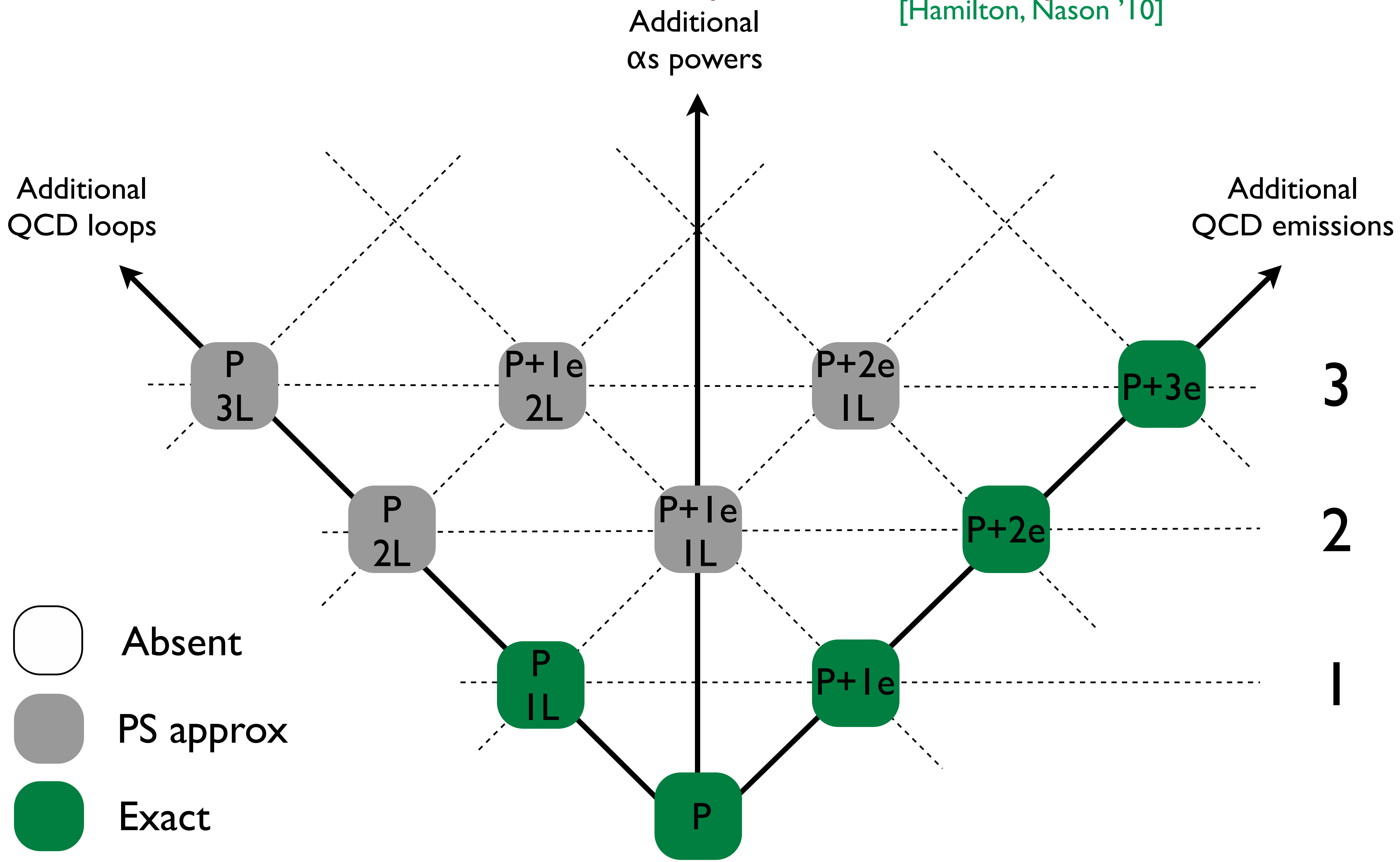


Process P exact at NLO, the rest PS approximation [PS+NLO (MC@NLO, POWHEG,...)]

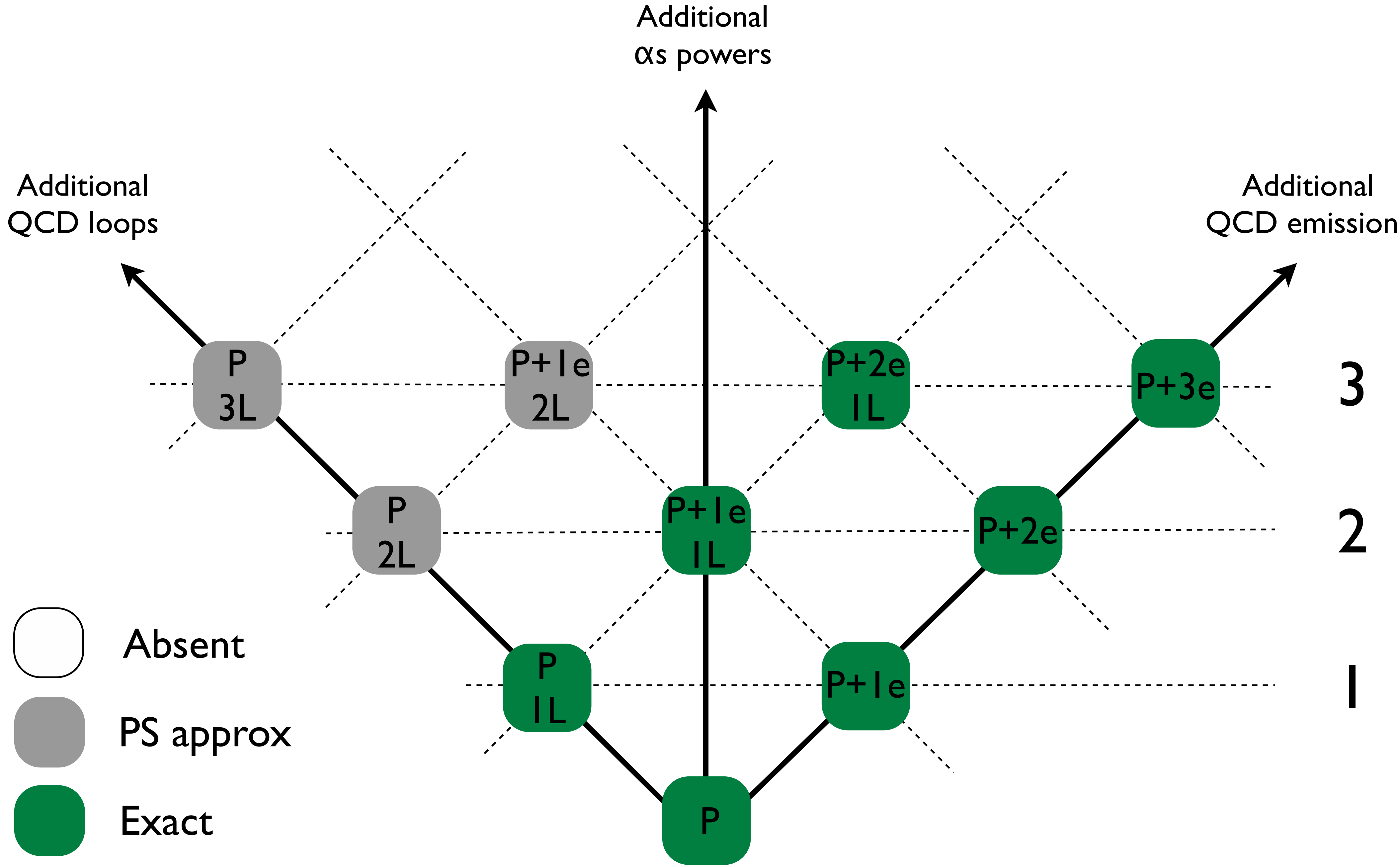


Process P exact at NLO, P+1j, P+2j,... at LO, the rest PS [PS+NLO+ME (MENLOPS,...)]

[Hamilton, Nason '10]



Process P, P+1j, P+2j,... exact at NLO, the rest PS [PS+NLO+ME$_{NLO}$ (MEPS@NLO,...)]



Existing 'MonteCarlos at NLO':

- MC@NLO [Frixione and Webber, 2002]
- POWHEG [Nason, 2004]

NB. MC@NLO is a code, POWHEG is a method

Evolving into (semi)automated forms:

- The POWHEG BOX [powhegbox.mib.infn.it 2010]
- aMC@NLO [amcatnlo.cern.ch 2011]

Matrix-element corrected shower Monte Carlos still have leading order accuracy for the total rates

$$d\sigma^{MEC} = Bd\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B}d\Phi_{rad}\right] \quad \text{and} \quad \Delta_R(Q_0) + \int \Delta_R(p_T)\frac{R}{B}d\Phi_{rad} = 1$$

$$\Rightarrow \int d\sigma^{MEC} = \int Bd\Phi_B = \sigma^{LO}$$

We want to do better, and **merge** PS and NLO, so that

$$\int d\sigma^{PS+NLO} = \int (B+V)d\Phi_B + \int Rd\Phi_R = \sigma^{NLO}$$

Idea: remove from the NLO the terms that are already generated by the parton shower (NB. MC-specific)

$$d\sigma^{MC@NLO} = \bar{B}_{MC} d\Phi_B \left[\Delta_{MC}(Q_0) + \Delta_{MC}(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right] + [R - R^{MC}] d\Phi_R$$

$$\bar{B}_{MC} = B + \left[V + \int R^{MC} d\Phi_{rad} \right]$$

'soft' event — MC shower — 'hard' event

It is easy to see that, as desired,

$$\int d\sigma^{MC@NLO} = \int d\sigma^{NLO}$$

Idea: generated hardest radiation first, then pass event to MC for generation of subsequent, softer radiation

$$d\sigma^{POWHEG} = \bar{B} d\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right]$$

$$\bar{B} = B + \left[V + \int R \, d\Phi_{rad} \right]$$

NLO x-sect

MC shower

It is easy to see that, as desired,

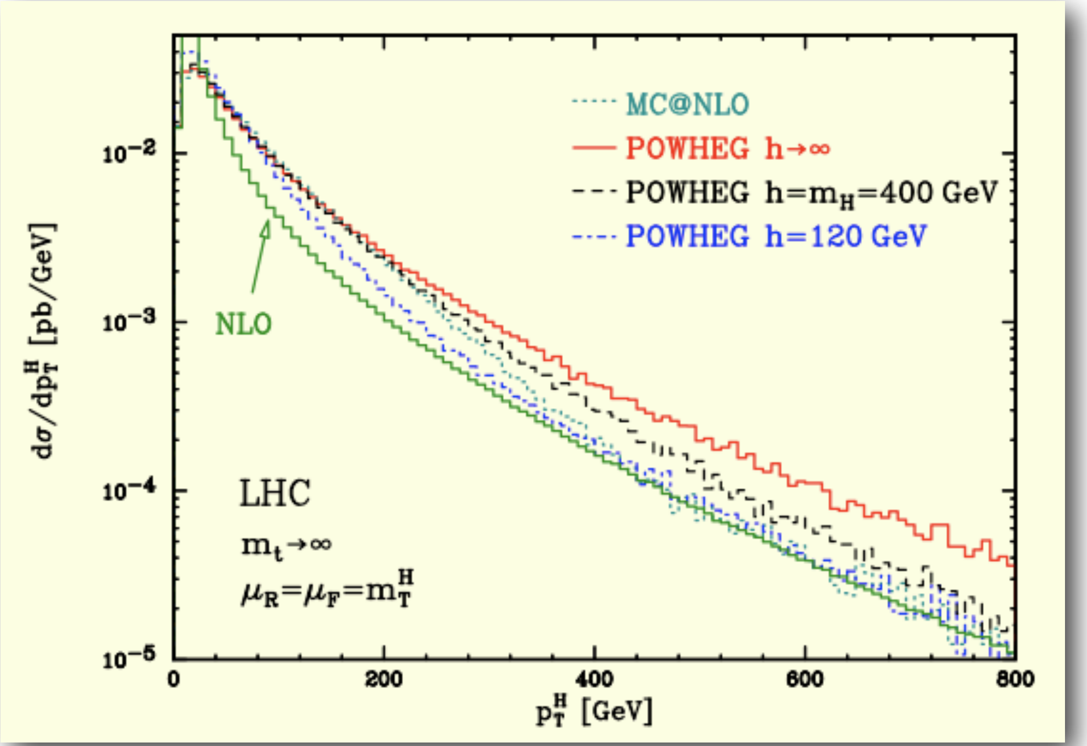
$$\int d\sigma^{POWHEG} = \int d\sigma^{NLO}$$

Large p_T enhancement in POWHEG

The 'naive' formulation for POWHEG is

$$d\sigma^{POWHEG} = \bar{B} d\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right]$$

In this form $\bar{B} d\Phi_B$ provides the NLO K-factor (order 1+ O(α_s)) , but also associates it to large p_T radiation, where the calculation is already O(α_s) (but only LO accuracy).



This generates an effective (but not necessarily correct) O(α_s^2) term (i.e. NNLO for the total cross section)

OK because beyond nominal accuracy, but one may feel uncomfortable with such large numerical factors

Modified POWHEG

The 'problem' with the naive POWHEG comes from the hard radiation being enhanced by spurious higher orders. In order to suppress this effect, we split

$$R = R^S + R^F \qquad R^S \equiv \frac{h^2}{h^2 + p_T^2} R \qquad R^F \equiv \frac{p_T^2}{h^2 + p_T^2} R$$

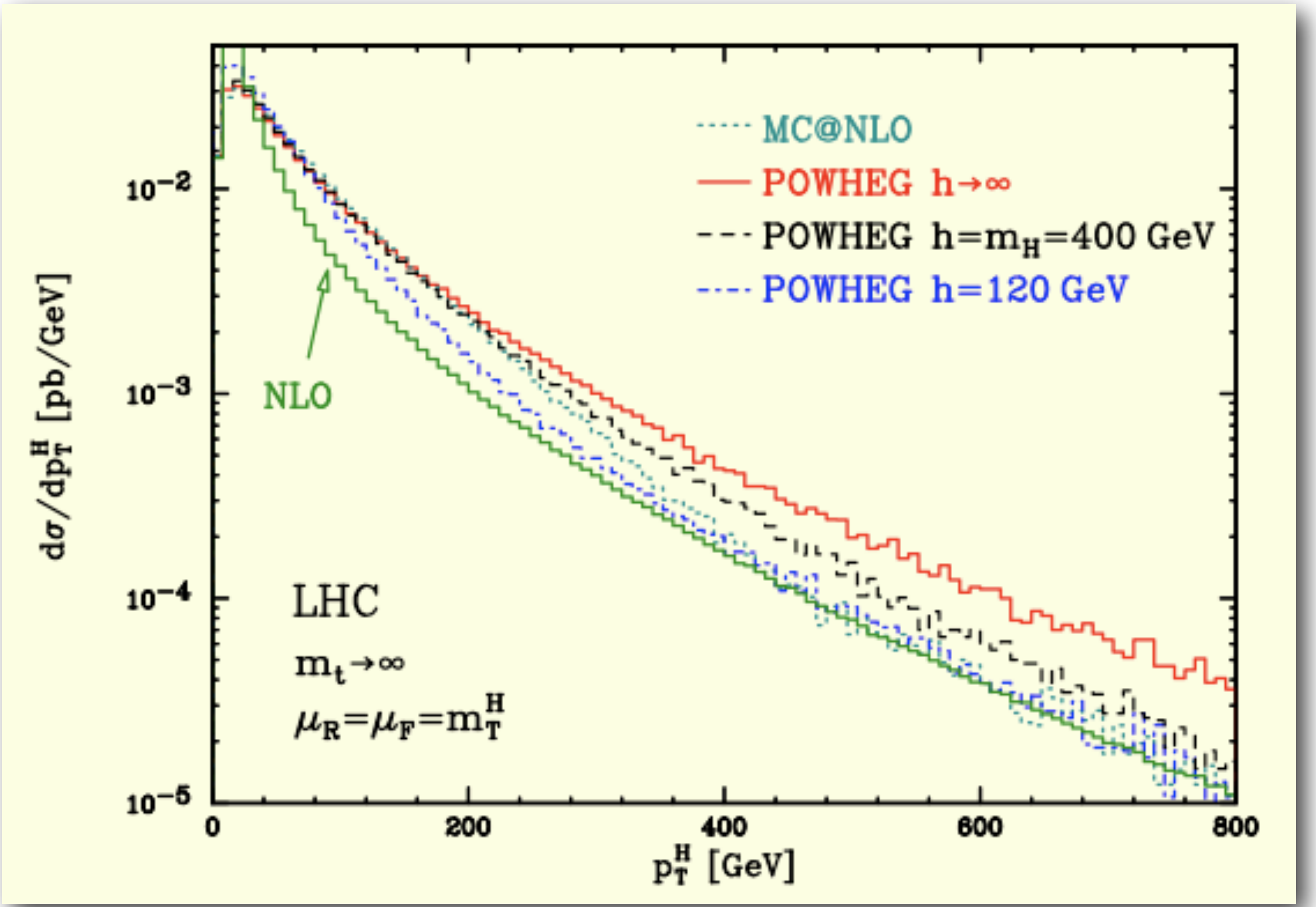
R^S: Contains singularities

R^F: Regular in small p_T region

$$d\sigma^{POWHEG} = \bar{B}^S d\Phi_B \left[\Delta_S(Q_0) + \Delta_S(p_T) \frac{R^S}{B} d\Phi_{rad}\right] + R^F d\Phi_R$$

$$\bar{B}^S = B + \left[V + \int R^S \, d\Phi_{rad}\right] \qquad \Delta_S(p_T) = \exp\left[-\int_{p_T} \frac{R^S}{B} d\Phi_{rad}\right]$$

In the $h\to\infty$ limit the exact NLO result is recovered



Comparisons

$$d\sigma^{MC} = B d\Phi_B \left[\Delta(Q_0) + \Delta(p_T)\frac{R^{MC}}{B} d\Phi_{rad}\right]$$

$$d\sigma^{MEC} = B d\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B} d\Phi_{rad}\right]$$

$$d\sigma^{NLO} = [B + V]\, d\Phi_B + R d\Phi_R$$

$$d\sigma^{MC@NLO} = \bar{B}_{MC} d\Phi_B \left[\Delta_{MC}(Q_0) + \Delta_{MC}(p_T)\frac{R^{MC}}{B} d\Phi_{rad}\right] + [R - R^{MC}] d\Phi_R$$

$$d\sigma^{POWHEG} = \bar{B}^S d\Phi_B \left[\Delta_S(Q_0) + \Delta_S(p_T)\frac{R^S}{B} d\Phi_{rad}\right] + R^F d\Phi_R$$

POWHEG approaches MC@NLO if $R^S \to R^{MC}$

Take home messages

Monte Carlos in QCD are complicated. I only scratched the surface here and gave almost no details. If interested, check lectures of real MC people (Sjostrand, Skands, Nason, Maltoni, Frixione, Krauss, Richardson, Webber,.....)

Monte Carlos exploit property of universality of soft/collinear radiation to resum its effects to all orders (within some approximations)

Effects of multi-parton, hard, large-angle radiation can be included via exact calculations and proper (and delicate) mergings

The result is a detailed description of the final state, covering as much phase space as possible. Accurate descriptions of data are usually achieved