Atomtronics
May 7, 2019

Some Empirical Implementations of the Multi-Dimensional Reflection Groups

Maxim Olshanii (Olchanyi) UMass Boston

Thibault Scoquart (ENS), Dmitry Yampolsky (UMB), Zajong Hwang (UMB), Joseph Seaward (UMB), Joanna Ruhl (UMB) ,Yuri Styrkas (Princeton HS), Vanja Dunjko (UMB), Juan Polo (U Grenoble), Nathan Harshman (American U), Anna Minguzzi (U Grenoble), Steven G. Jackson (UMB Mathematics)

Kaleidoscopes ... and the reflection groups generated thereby

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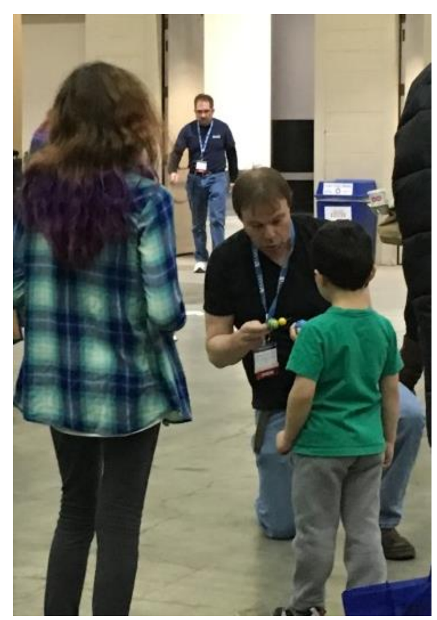
Meet the Scientists!
at the AAAS 2017 Annual Meeting
in Boston February 18 and 19, 2017



Kaleidoscopes are the systems of mirrors where one does not know where one mirror ends and another begins.

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Kaleidoscopes are the systems of mirrors where one does not know where one mirror ends and another begins. Tightly linked to Bethe Ansatz solvability.

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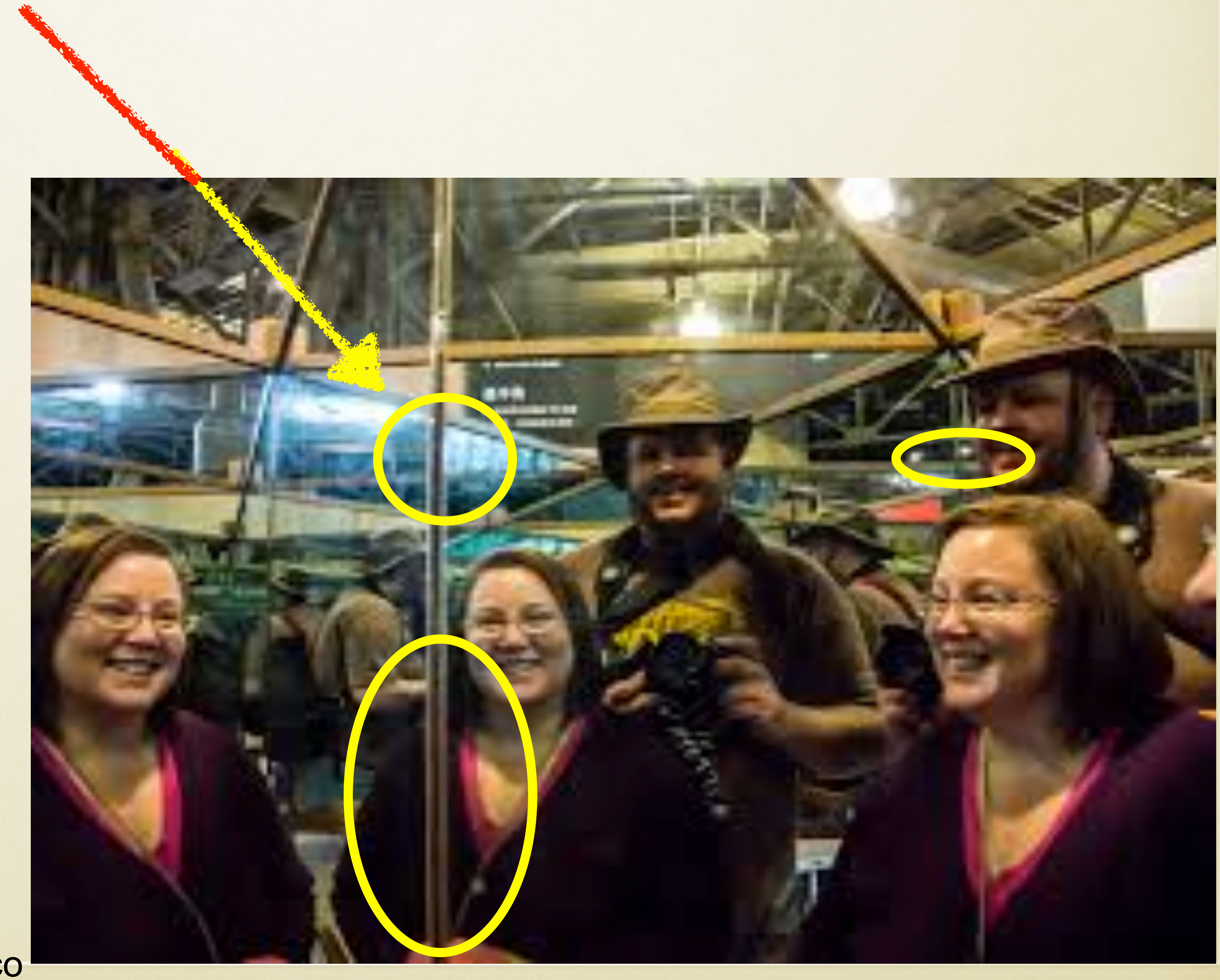
Attention! Next two sentences constitute a formula-less
crash course on Bethe Ansatz

Kaleidoscopes are the systems of mirrors where one does not know where one mirror ends and another begins. Tightly linked to Bethe Ansatz solvability. No sharp transitions



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Kaleidoscopes are the systems of mirrors where one does not know where one mirror ends and another begins. Tightly linked to Bethe Ansatz solvability. No sharp transitions -> no tails in Fourier transform -> can hope to have a solution with a finite number of plane waves = Bethe Ansatz

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Kaleidoscopes are the systems of mirrors where one does not know where one mirror ends and another begins. Tightly linked to Bethe Ansatz solvability. No sharp transitions -> no tails in Fourier transform -> can hope to have a solution with a finite number of plane waves = Bethe Ansatz. Kaleidoscopes are classified using the reflection groups.

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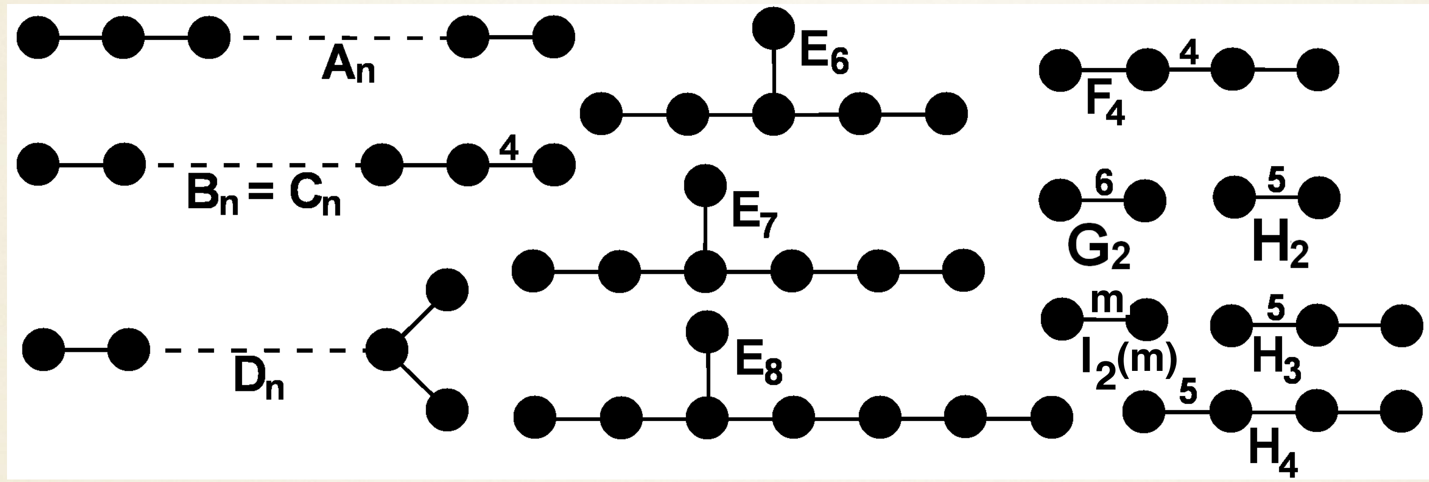
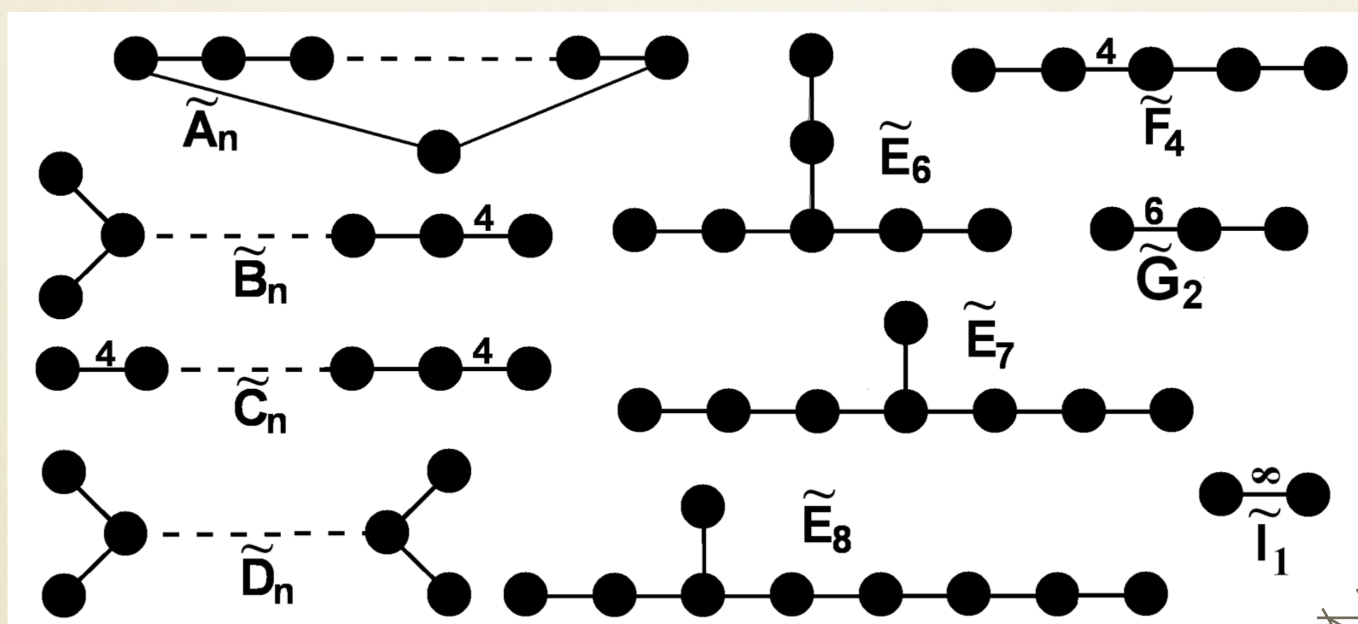

A_n
E_6
F_4
4
4
$B_n = C_n$
E_7
6
G_2
5
H_2
m
$I_2(m)$
5
H_3
D_n
E_8
5
H_4


$\tilde{A}_n$
$\tilde{E}_6$
4
$\tilde{F}_4$
4
$\tilde{B}_n$
6
$\tilde{G}_2$
4
4
$\tilde{C}_n$
$\tilde{E}_7$
$\tilde{D}_n$
$\tilde{E}_8$
∞
$\tilde{I}_1$

Reflection groups generate solvable wave problems with δ-functional slabs along the mirrors of the group [Gutkin (1982); Emsiz-Opdam-Stokman (2006)].

Those, in turn, may, potentially, generate integrable problems with pair-wise δ-interacting particles [Girardeau, Lieb-Lineger, McGuire, Yang, Gaudin, … (1960s - early 1970s)].

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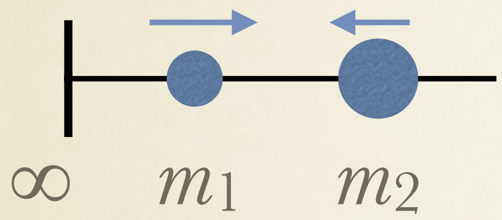
Difficulty: number of mirrors typically far exceeds the number of particle pairs

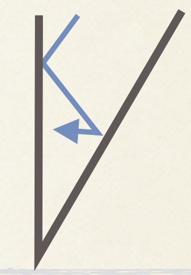
Must have a way to disable mirrors!

Ideas with hard-core particles: hiding mirrors behind hard-core walls

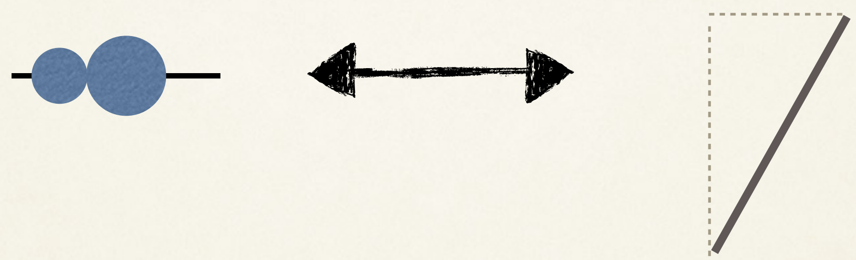
d hard-cores on a half-line ↔ *d*-dimensional billiard



$$x_i = y_i/\sqrt{m_i}$$

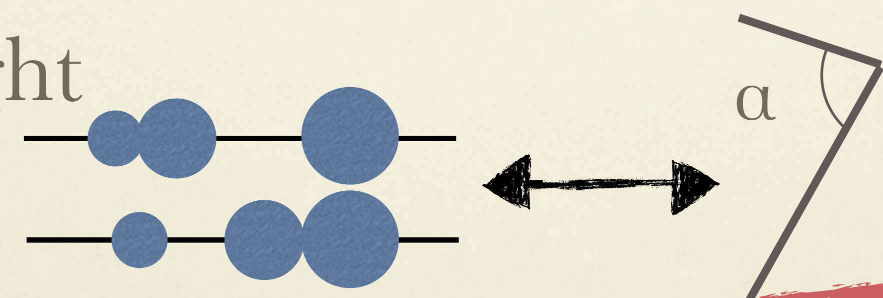


inter-particle contact



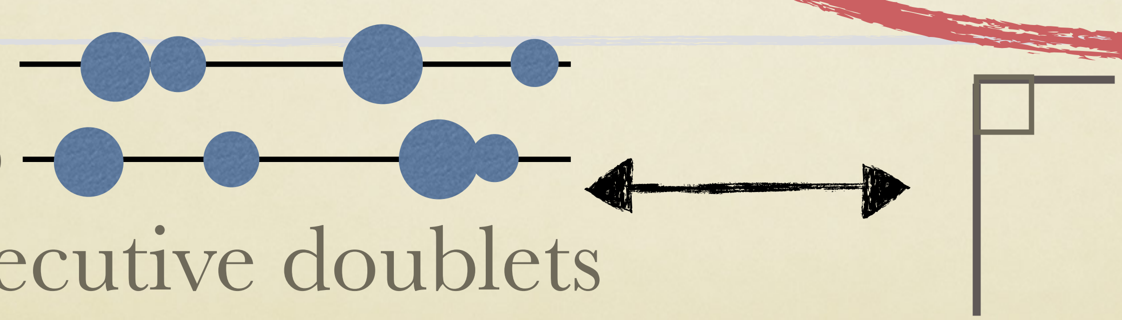
particle-wall contact

left-mid and mid-right contacts in a consecutive triplet



$$\alpha = \arctan\left[\frac{m_2(m_1+m_2+m_3)}{m_1 m_3}\right]$$

contacts in two unrelated consecutive doublets



Hard-core particle systems generate simplex-shaped billiards.

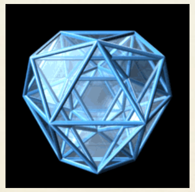
In particular, alcoves (i.e. a system of generating mirrors) of all reflection group with a non-forking Coxeter diagram can be built. They, in turn, generate **integrable hard-core particle systems…**

Hard-core particle systems generate simplex-shaped billiards.

In particular, alcoves (i.e. a system of generating mirrors) of all reflection group with a non-forking Coxeter diagram can be built. They, in turn, generate **integrable hard-core particle systems…**

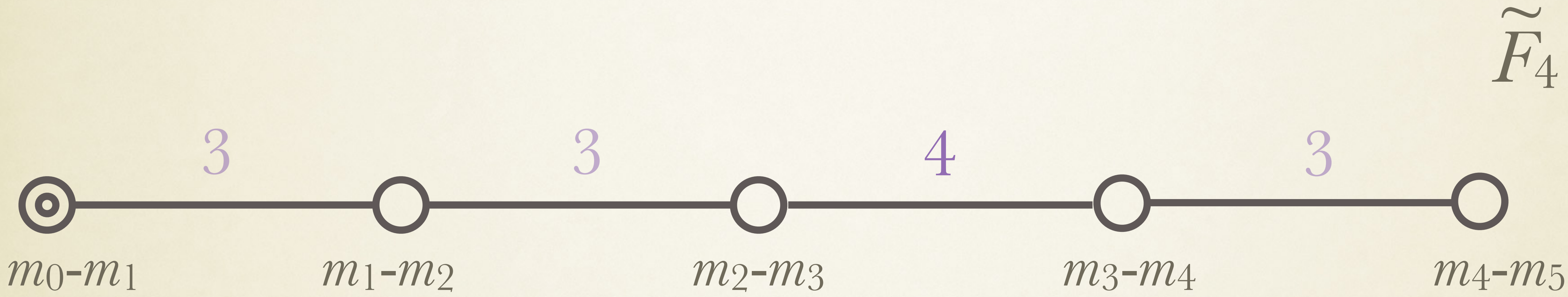
..10 of them

A proof of principle



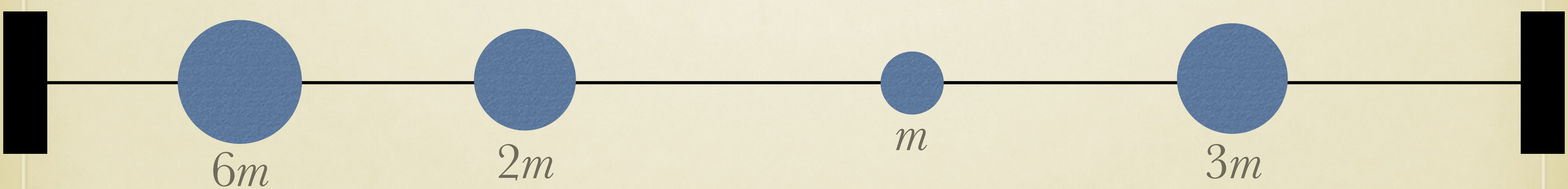
$\widetilde{F}_4$

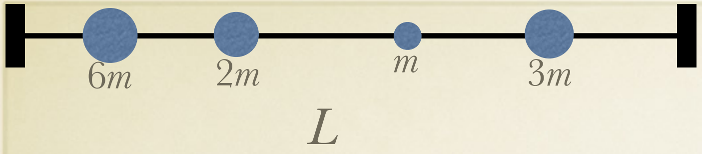
Building a particle system from the $\widetilde{F}_4$ Coxeter diagram



Single solution:

$m_0=\infty$, $m_1=6m$, $m_2=2m$, $m_3=m$, $m_4=3m$, $m_5=\infty$



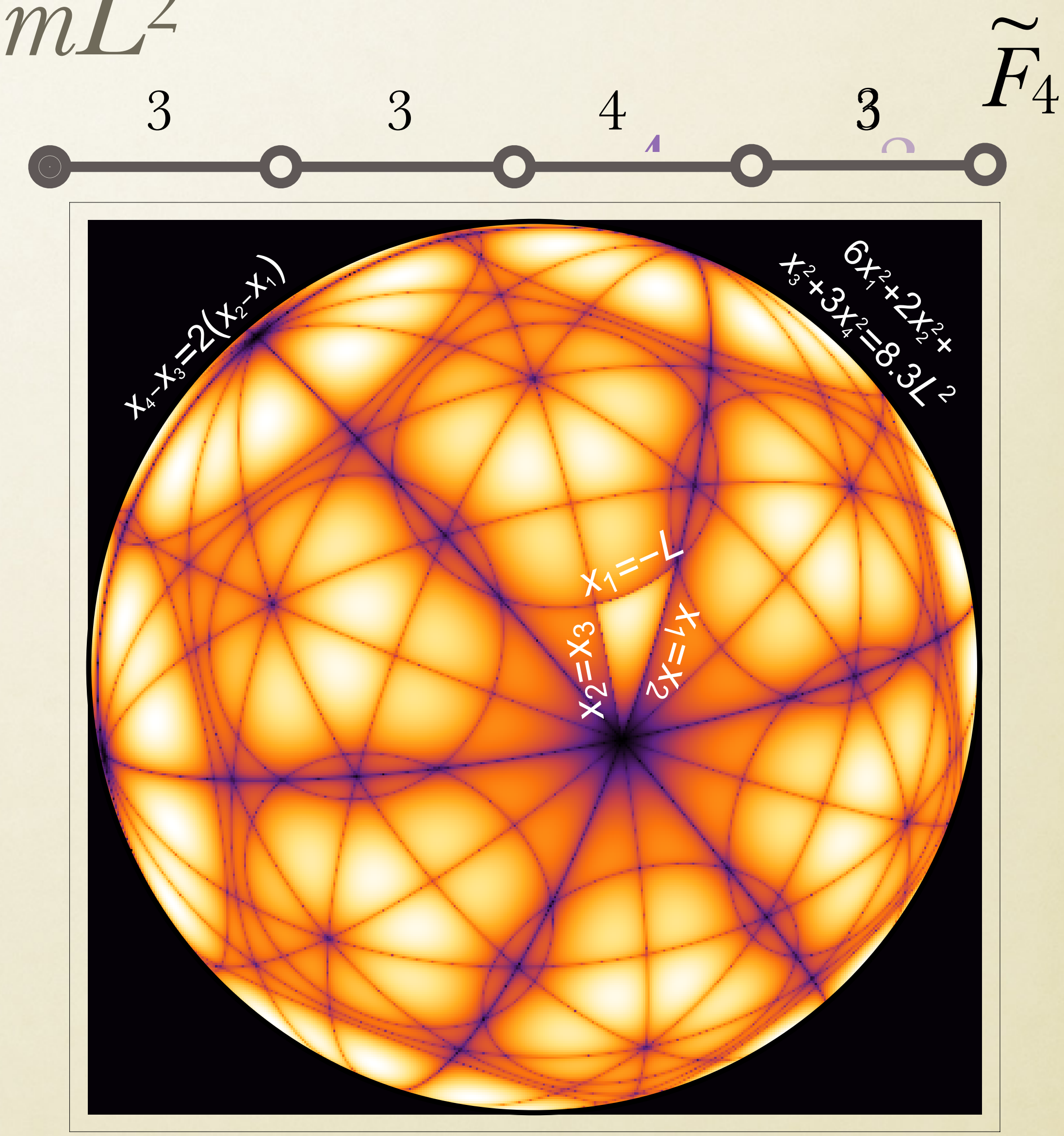


Results

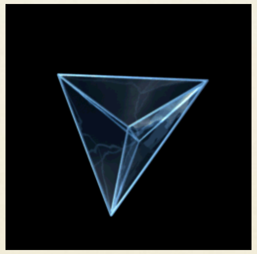
Ground state energy:

$$E_{1,1,2,3} = \frac{13\pi^2\hbar^2}{2mL^2}$$

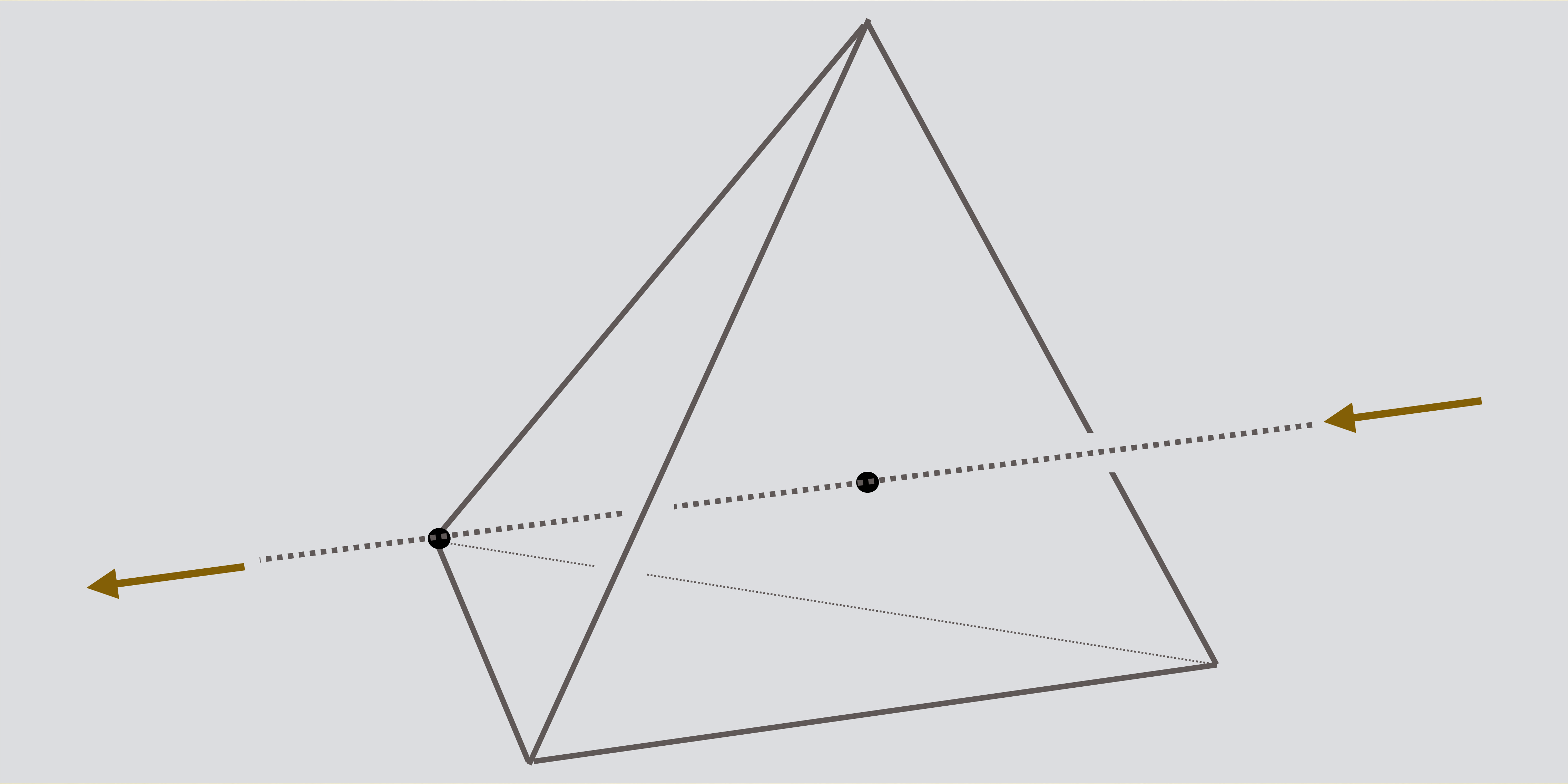
Ground state wavefunction: consists of 1152 plane waves (the same for any other eigenstate)



Quantum Galilean Cannon as an entanglement amplifier

 A_N

A practically important property of the Platonic solids

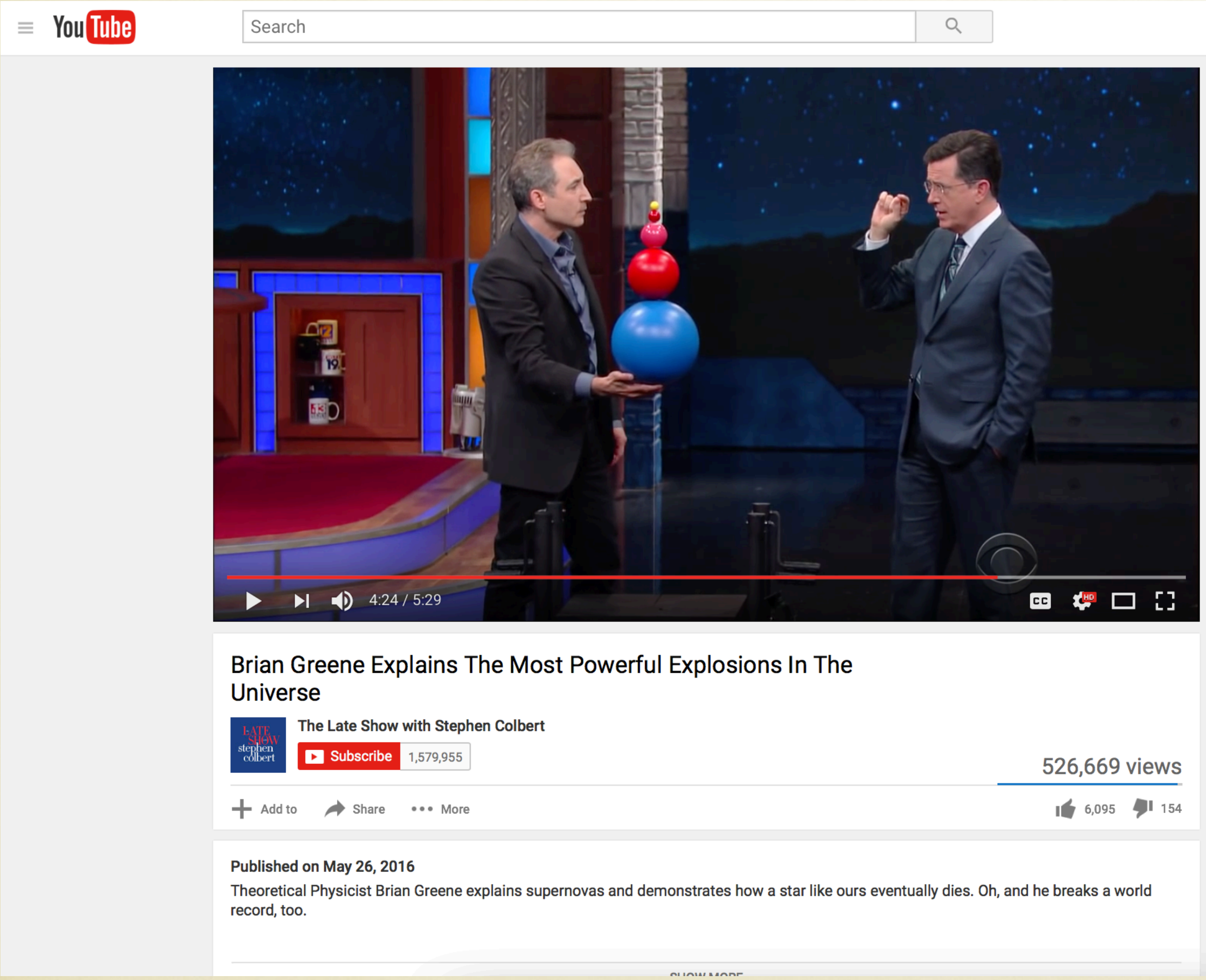


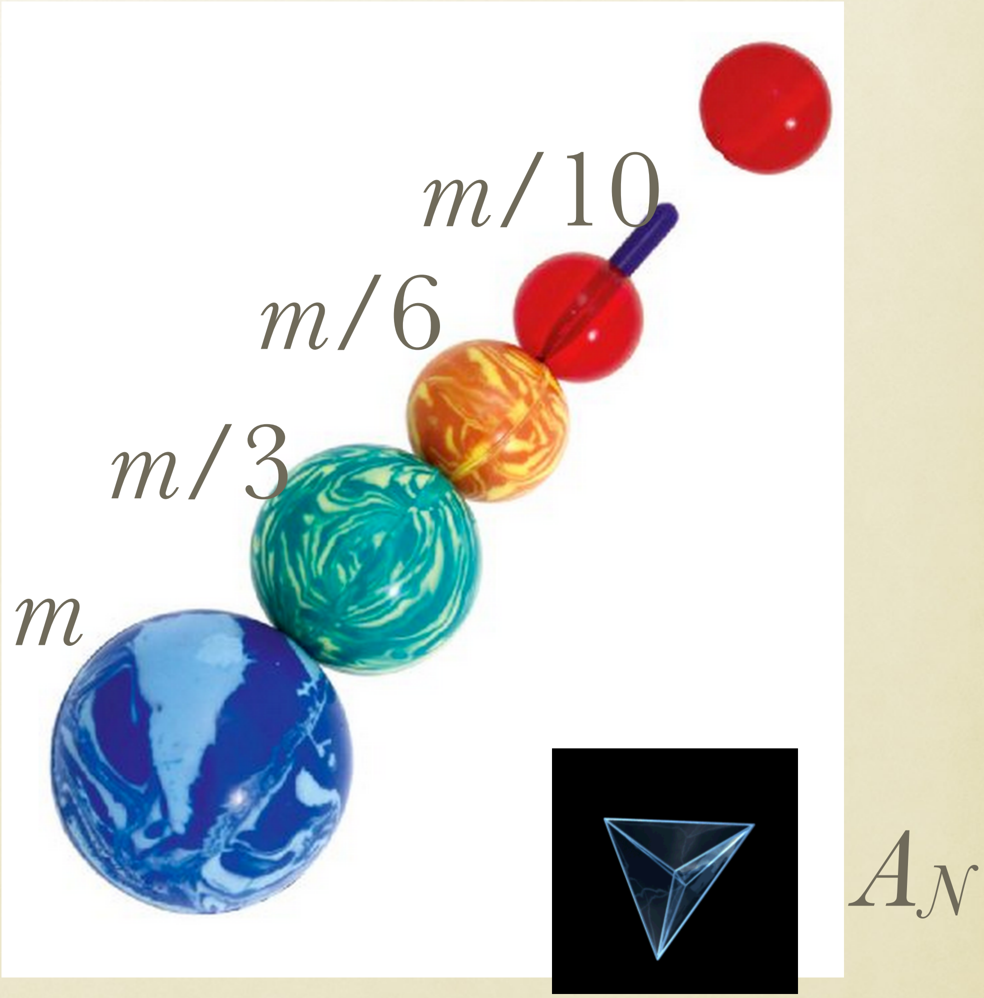
A light ray sent through a <u>center of a face</u> of a Platonic solid, perpendicular thereto, will leave through either a <u>center of another face</u> or through a <u>vertex</u>.

That is:
“Special point in, special point out”

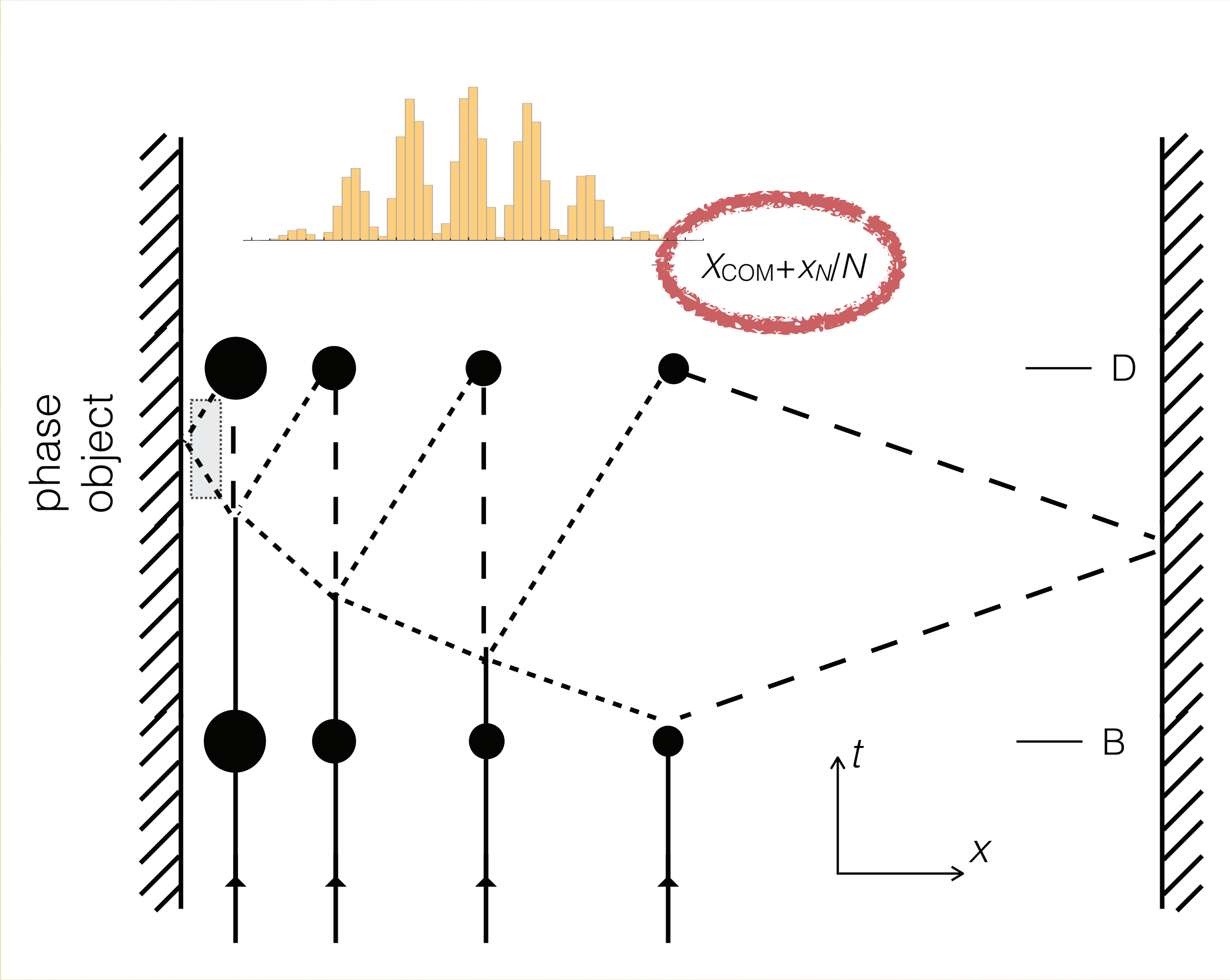
“Special point in, special point out”

Example: Galilean Cannon

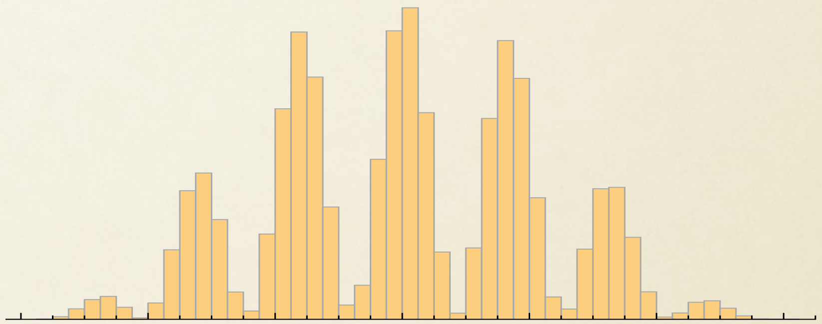




A scheme for an entanglement amplifier

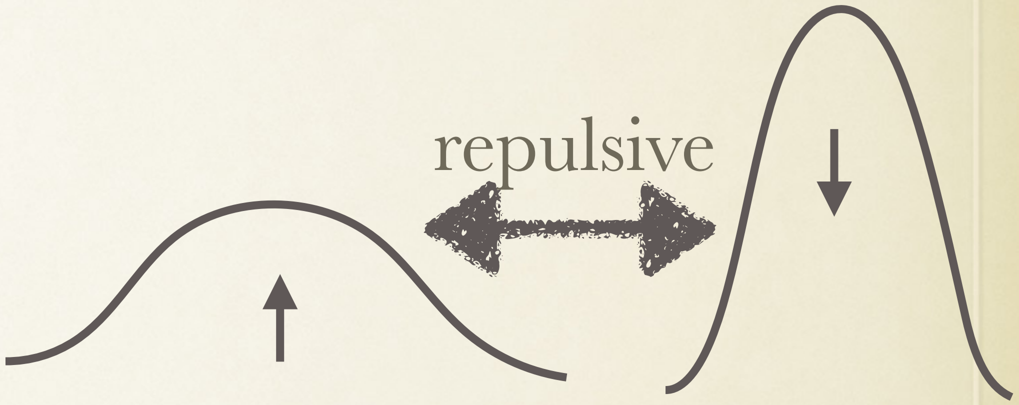


80,000 realizations

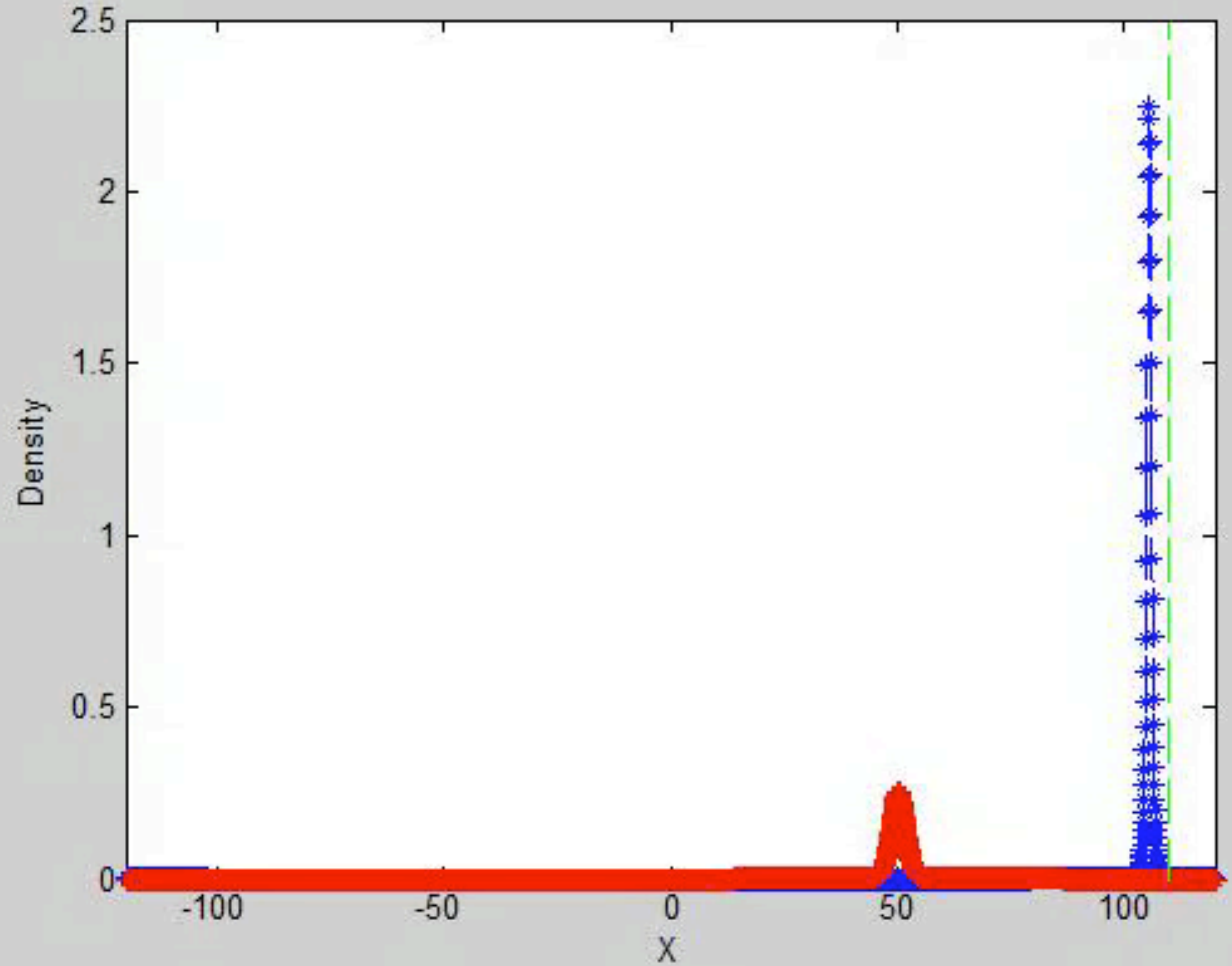


The same wavelength as for a single particle of a *total* mass of the system with the same velocity as the out-velocity

Implementation note:

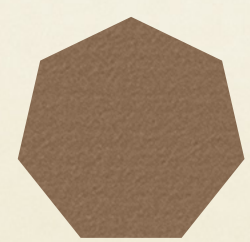


- quasi-1D atom guide
- two species or two internal states
- attractive intra-specie interaction
  - -> NLS **solitons as particles**
- alternating order of species
- **hard-core inter-soliton** (hence inter-specie) repulsion
- E.g. Li^7, at 855 G, mixture of $(mF = -1)$–$(mF = 0)$:
  - $a_{-1,-1} \approx -0.5\ a_B$, $a_{0,0} \lessapprox -10\ a_B$, $a_{-1,0} \approx +1.0\ a_B$
- massive particles of different mass are emulated by the solitons of different length
- Need: $E_{\text{kinetic, total}} << \mu$ (within reach)

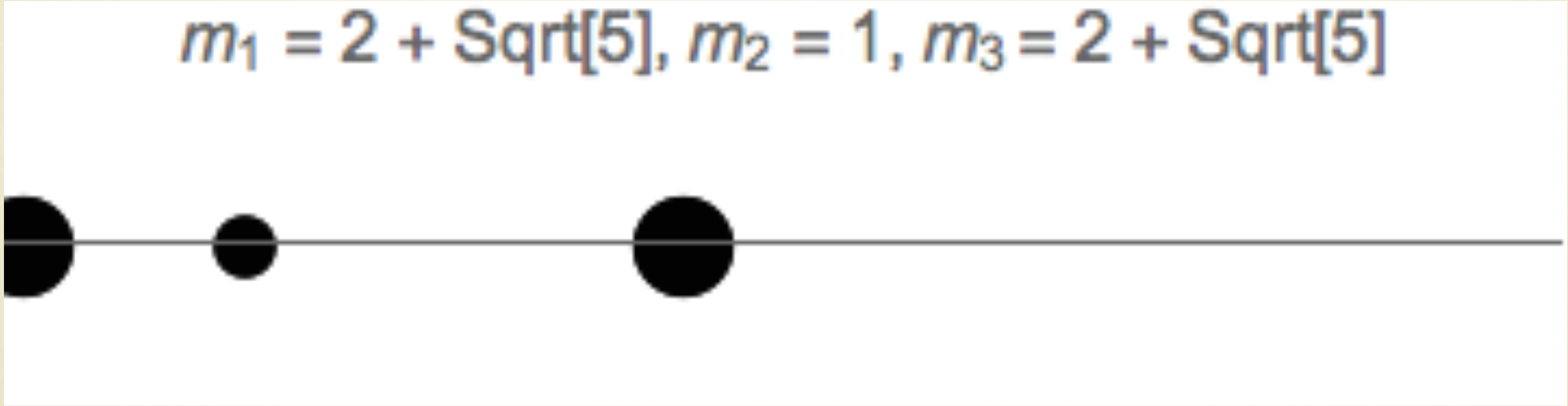

2.5
2
1.5
1
0.5
0
Density
-100
-50
0
50
100
X

Integrability = maximal light-to-heavy energy transfer and protected channels in phase space to protect entanglement

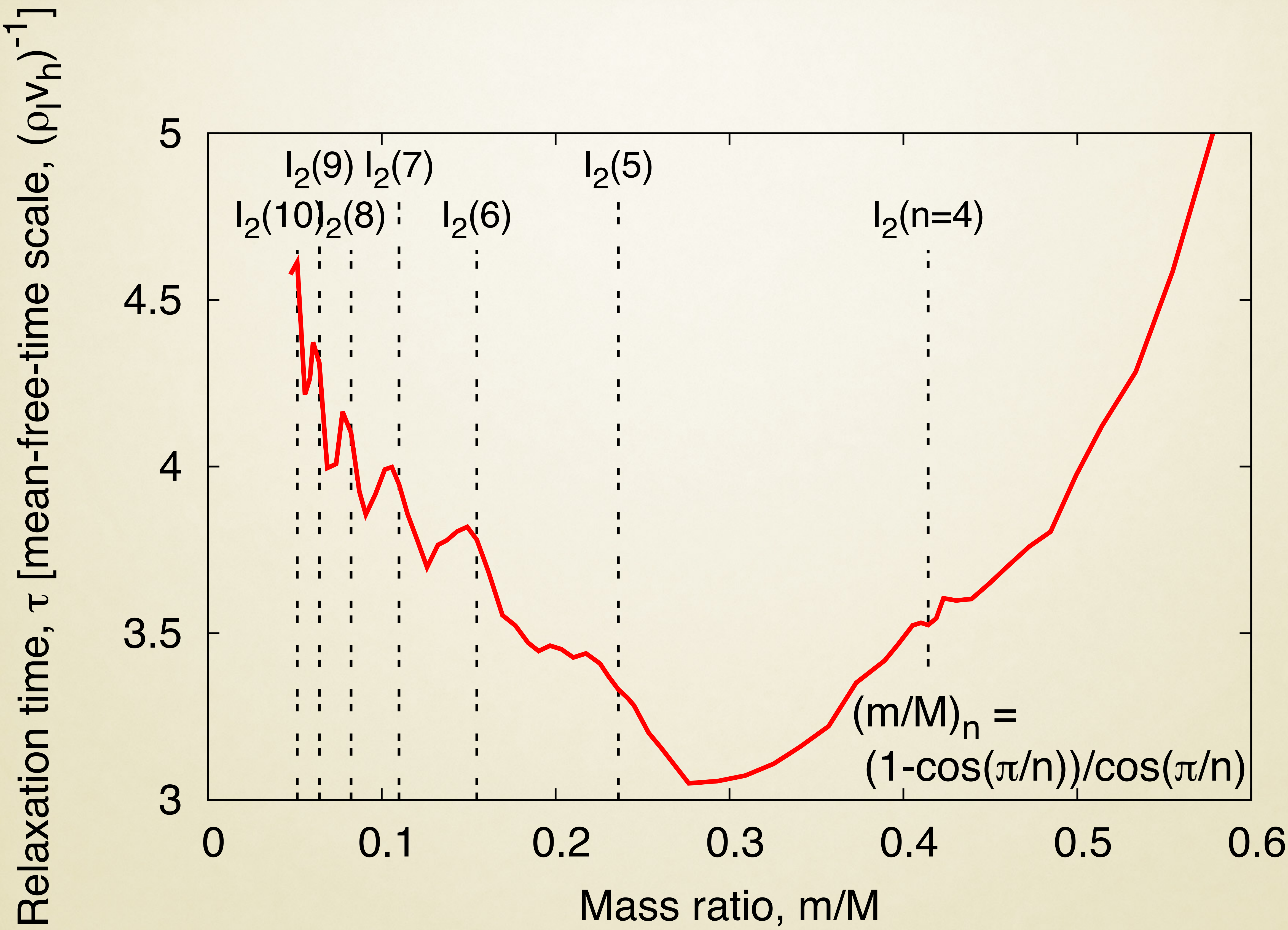
Another idea: slow-down of relaxation



$I_2(m)$



$I_2(5)$



Relaxation time, τ [mean-free-time scale, $(\rho_l v_h)^{-1}$]
5
4.5
4
3.5
3
0
0.1
0.2
0.3
0.4
0.5
0.6
Mass ratio, m/M
$I_2(9)$ $I_2(7)$
$I_2(5)$
$I_2(10)$ $I_2(8)$
$I_2(6)$
$I_2(n=4)$
$(m/M)_n$ =
$(1-\cos(\pi/n))/\cos(\pi/n)$

Integrability = slowdown of relaxation

[6] N.L. Harshman, Maxim Olshanii, A.S. Dehkharghani, A.G. Volosniev, Steven Glenn Jackson, N.T. Zinner, **Integrable families of hard-core particles with unequal masses in a one-dimensional harmonic trap**, Phys. Rev. X 7, 041001 (2017)

[5] X. M. Aretxabaleta, M. Gonchenko, N. L. Harshman, S. G. Jackson, M. Olshanii, G. E. Astrakharchik, **Two-ball billiard predicts digits of the number PI in non-integer numerical bases**, arXiv:1712.06698 (2017), submitted to JPA

[4] M. Olshanii, T. Scoquart, Dmitry Yampolsky, V. Dunjko, S. G. Jackson, **Creating entanglement using integrals of motion**, PRA 97, 013630 (2018)

[3] T. Scoquart, J. J. Seaward, S. G. Jackson, M. Olshanii, **Exactly solvable quantum few-body systems associated with the symmetries of the three-dimensional and four-dimensional icosahedra**, SciPost Phys. 1(1), 005 (2016) (inaugural issue)

[2] Maxim Olshanii & Steven G. Jackson, **An exactly solvable quantum four-body problem associated with the symmetries of an octacube**, *NJP* **17**, 105005 (2015)

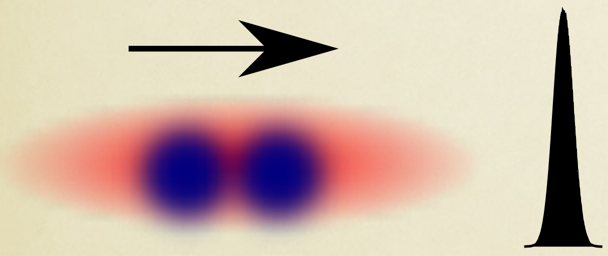
[1] Zaijong Hwang, Frank Cao, Maxim Olshanii, **Traces of Integrability in Relaxation of One-Dimensional Two-Mass Mixtures**, *J. Stat. Phys.* **161**, 467 (2015)

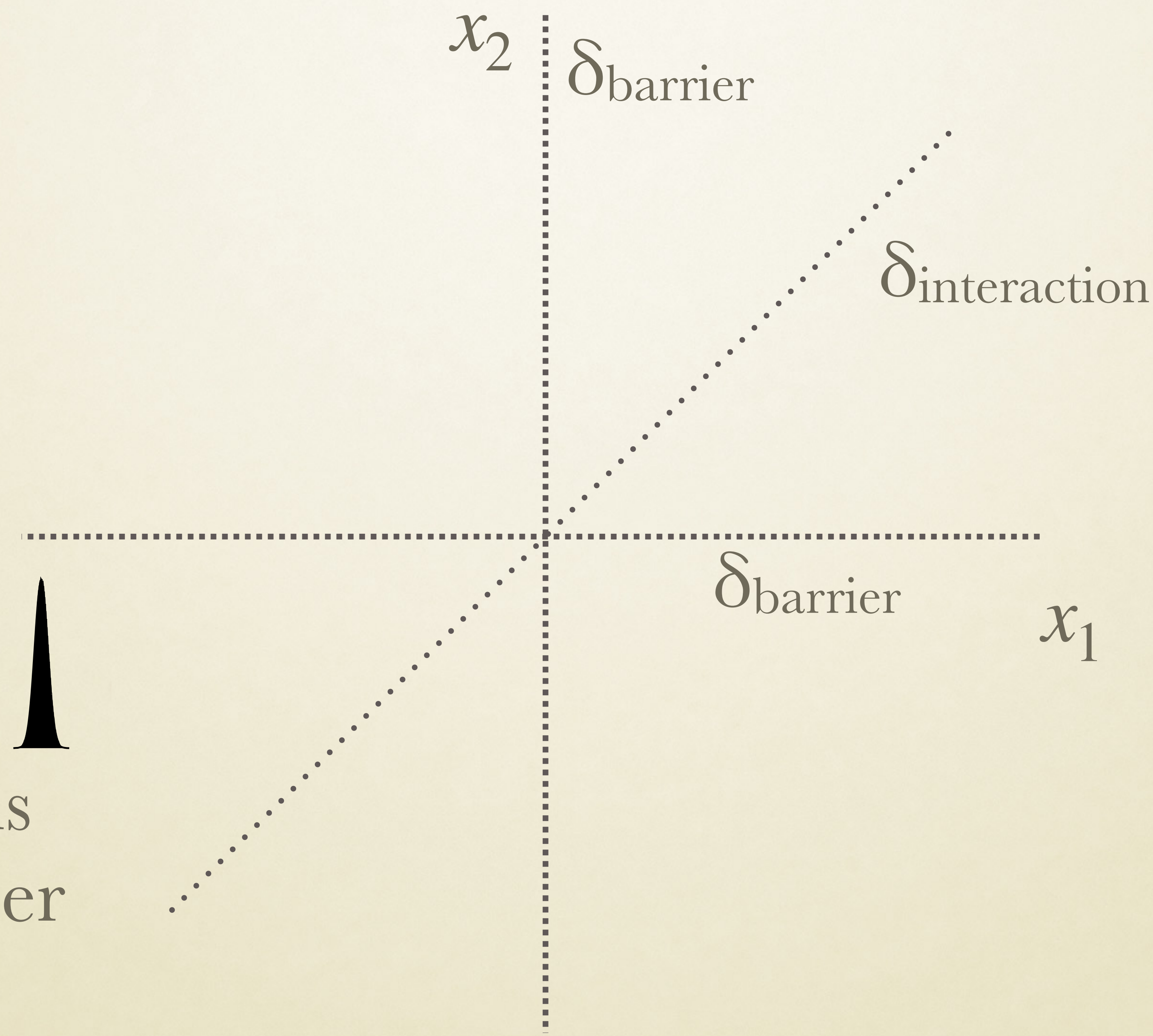
Ideas with δ-interacting particles: suppressing mirrors with nodal surfaces

Integrability-induced prohibition of dissociation of dimers on a barrier: a spatially compact readout for chip interferometers

- $I_2(4)$

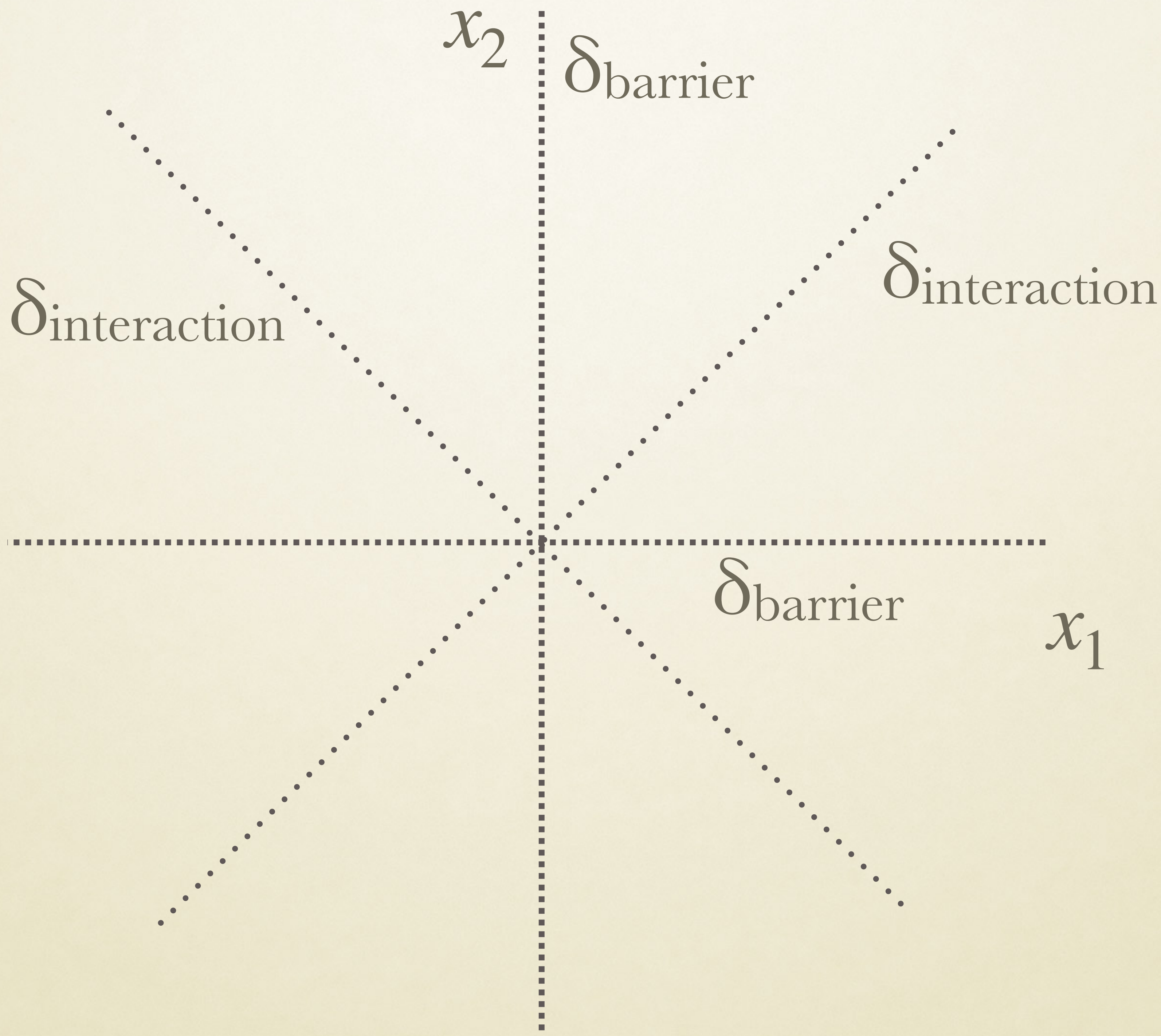
Real world, particles:





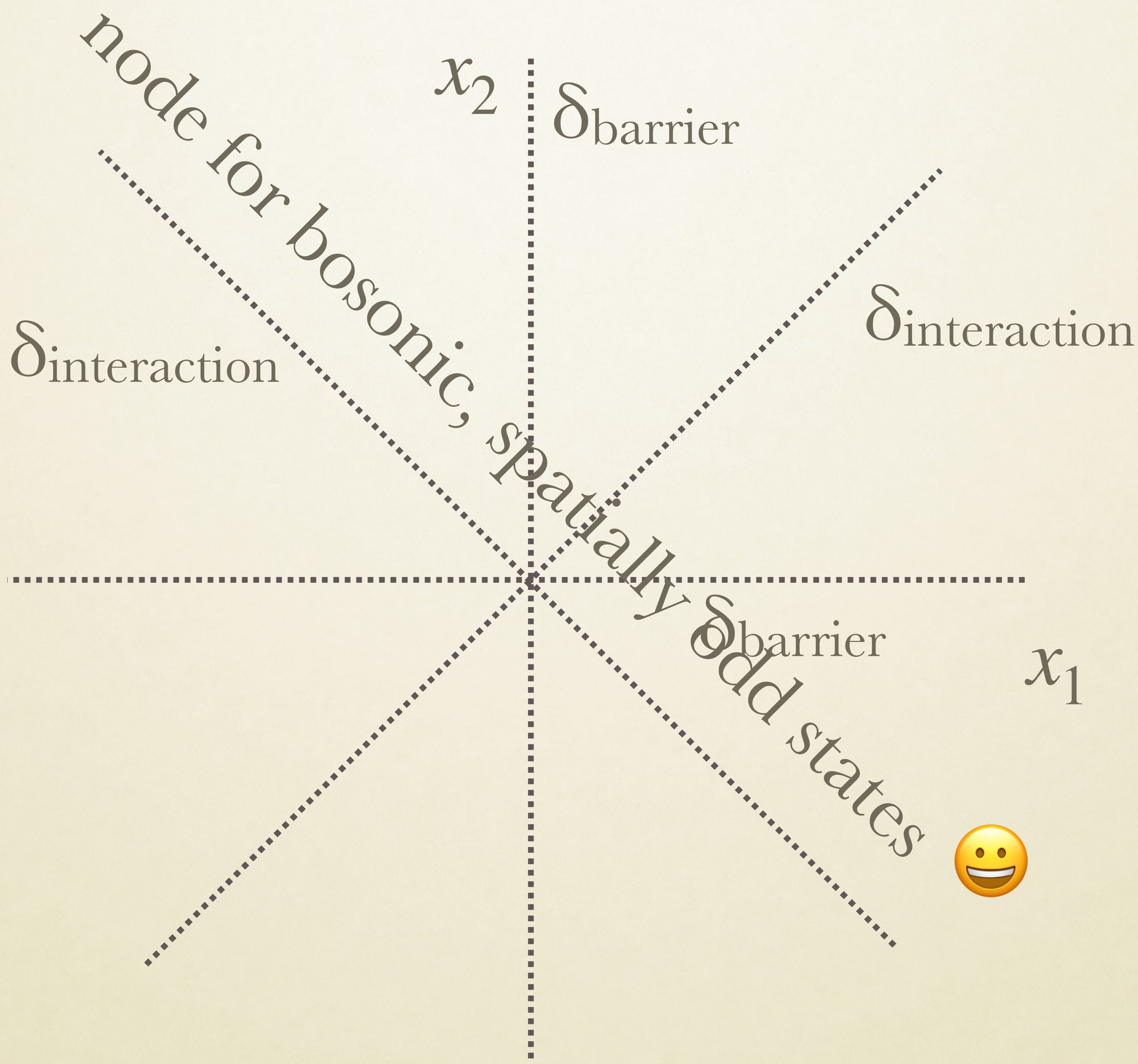
two bosons
and a barrier

Ideal world, solvable :



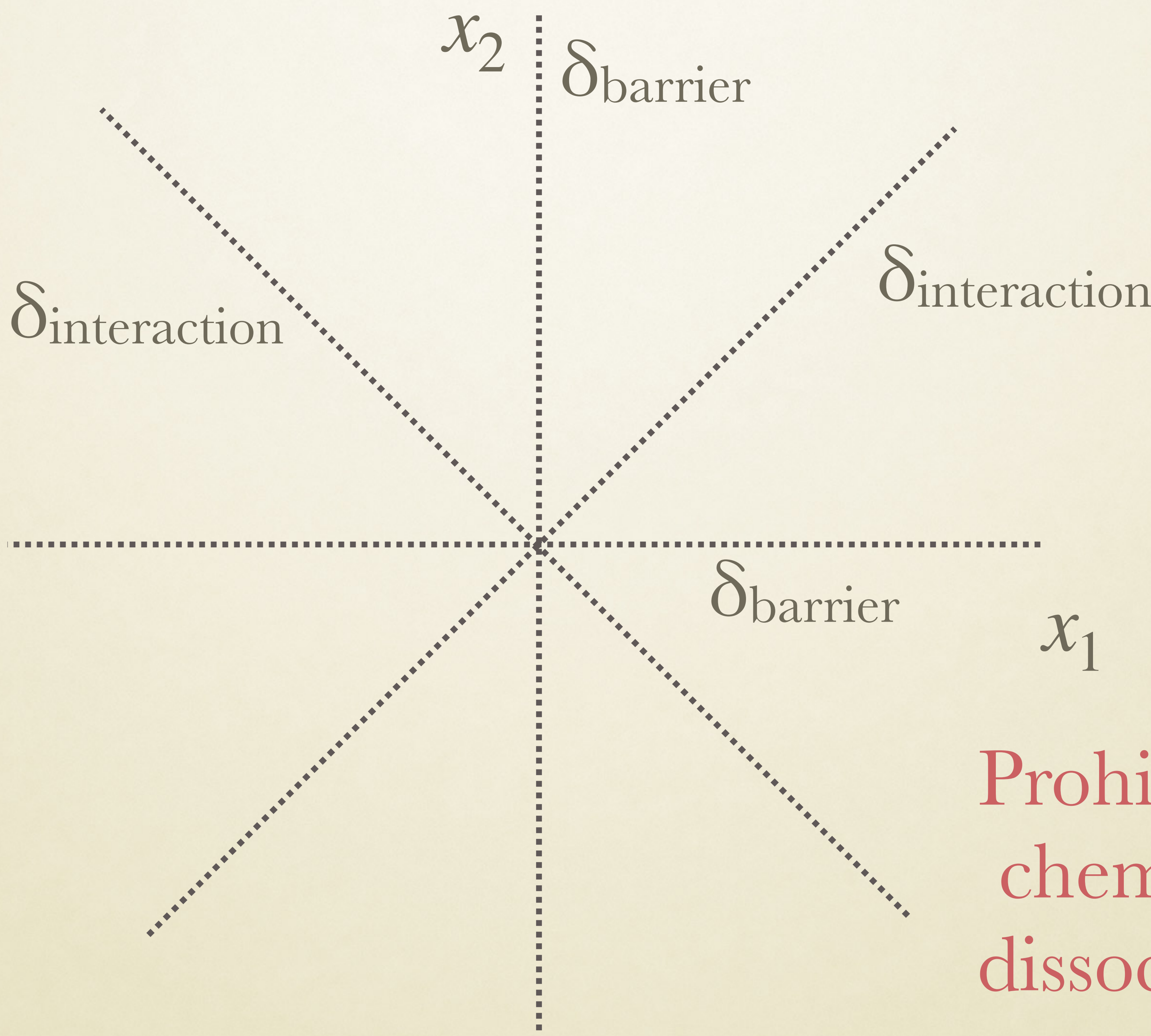
C_2

Ideal & real:

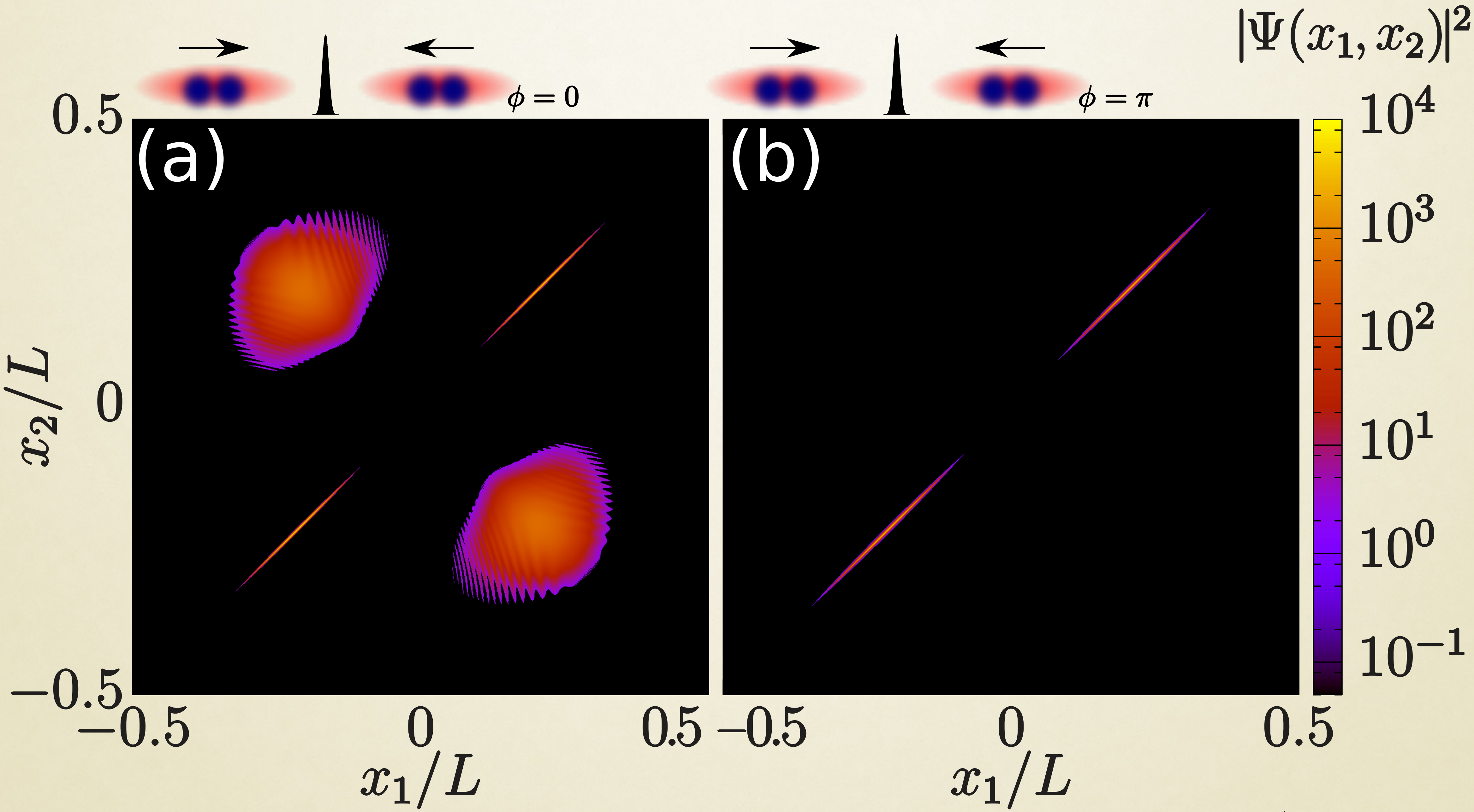


x_2
$\delta_{barrier}$
node for bosonic, spatially odd states
$\delta_{interaction}$
$\delta_{interaction}$
$\delta_{barrier}$
x_1

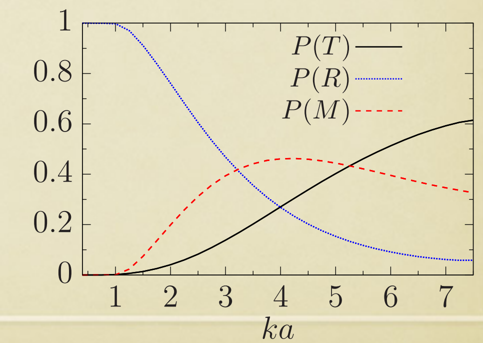
Ideal & real:



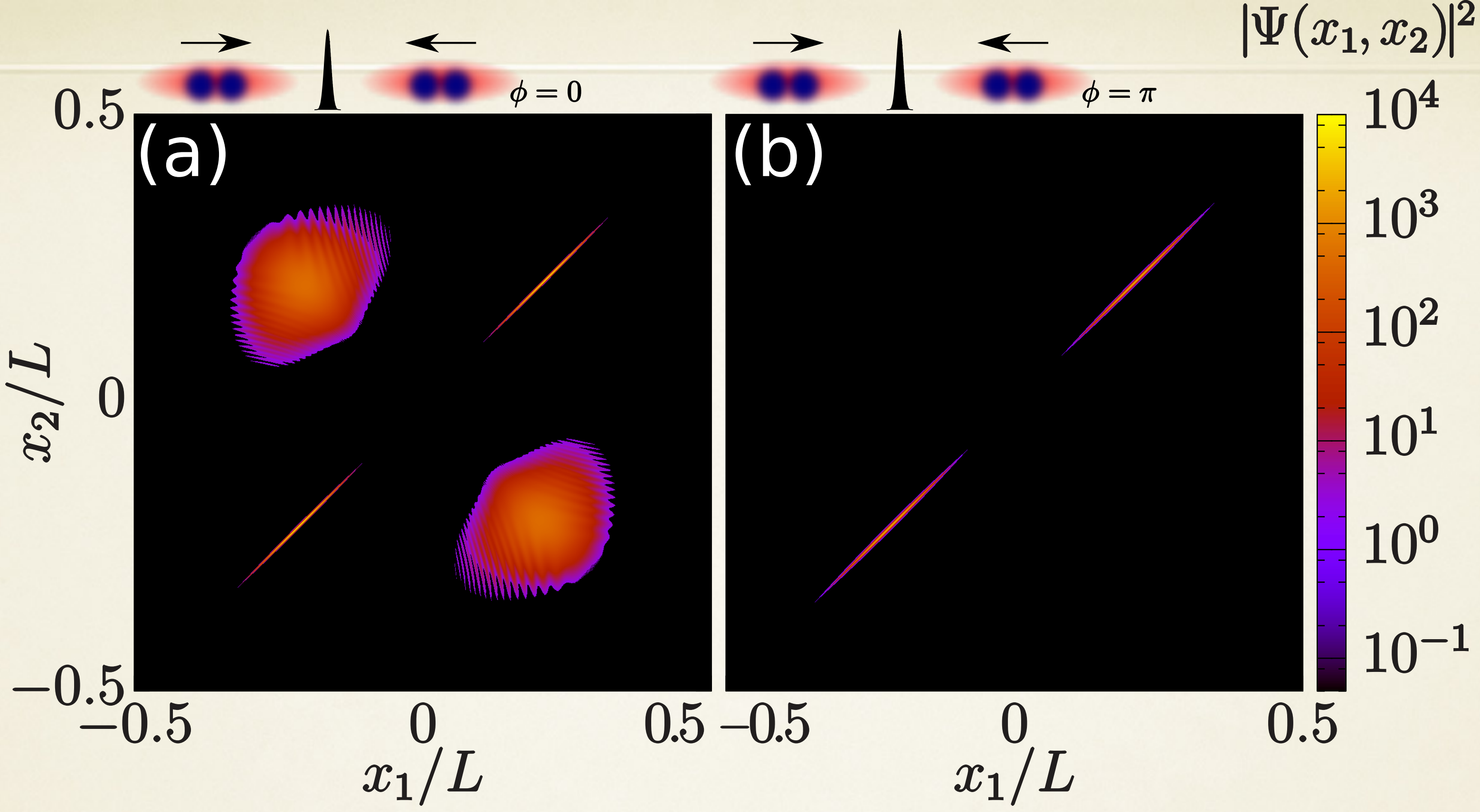
Prohibition of chemistry, of dissociation in particular



$|\Psi(x_1, x_2)|^2$
$\phi = 0$
$\phi = \pi$
(a)
(b)
0.5
0
−0.5
x_2/L
x_1/L
10^4
10^3
10^2
10^1
10^0
10^{-1}



$P(T)$
$P(R)$
$P(M)$
ka



Suggested application: compact readout in chip-based interferometers.

Can use the monomer production as a measure of relative phase between the interferometer arms, no need for spatial separation between the output beams after recombination.

[1] Juan Polo Gomez, Anna Minguzzi, Maxim Olshanii, **Traces of integrability in scattering of one-dimensional dimers on a barrier**, New J. Phys. 21, 023008 (2019)

All this is very recent, and it suspected to be a pair of a bigger whole.

Work in progress: “asymmetric Bethe ansatz”

Inspired by this work and [Yanxia Liu, Fan Qi, Yunbo Zhang, and Shu Chen, arXiv:1903.08449] (integrability of two hard cores with a 3:1 mass ratio, with δ-interactions, in a box)

Can keep unphysical interactions if the wavefunction has a node at their location

Integrability = prohibition of chemistry

Summary and outlook

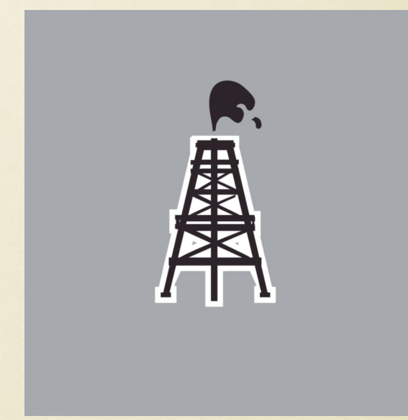
Summary

Reflection groups

- entanglement amplifier

- “integrability peaks” in mass mixtures

- interferometer output readout via monomer production



“Mathematics catalogues everything that is not self-contradictory; within that vast inventory, physics is an island of structures rich enough to contain their own beholders.”
— **Greg Egan, Oceanic**

In discussions with:

Marvin Girardeau (U Arizona),
Alfred G. Noël (UMB Mathematics),
Bala Sundaram (UMB), Adolfo del Campo (UMB),
Felix Werner (ENS), Jean-Sébastien Caux (U Amsterdam),
Dominik Schneble (Stony Brook), Randy Hulet (Rice),
Helene Perrin (Paris-Nord), Romain Dubessy (Paris-Nord)
Discovery Museums (Acton, MA)

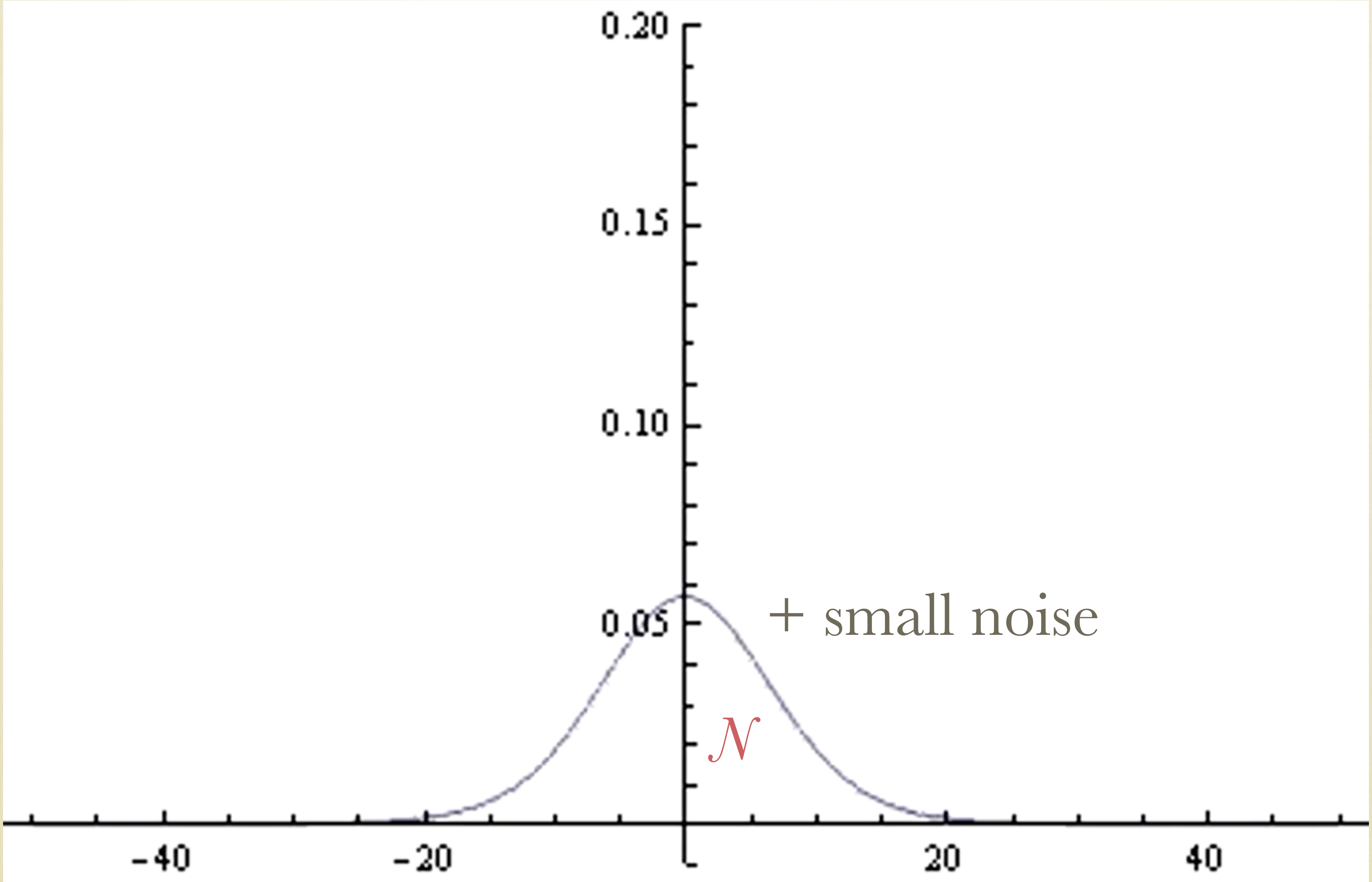
Cheap Macroscopic Quantum Coherence with the GPE Breathers

Maxim Olshanii (Olchanyi) UMass Boston
Oleksandr V. Marchukov (Tel Aviv U), Boris A. Malomed (Tel Aviv U), Vanja Dunjko (UMass Boston), Randall G. Hulet (Rice U), Vladimir A. Yurovsky (Tel Aviv U)

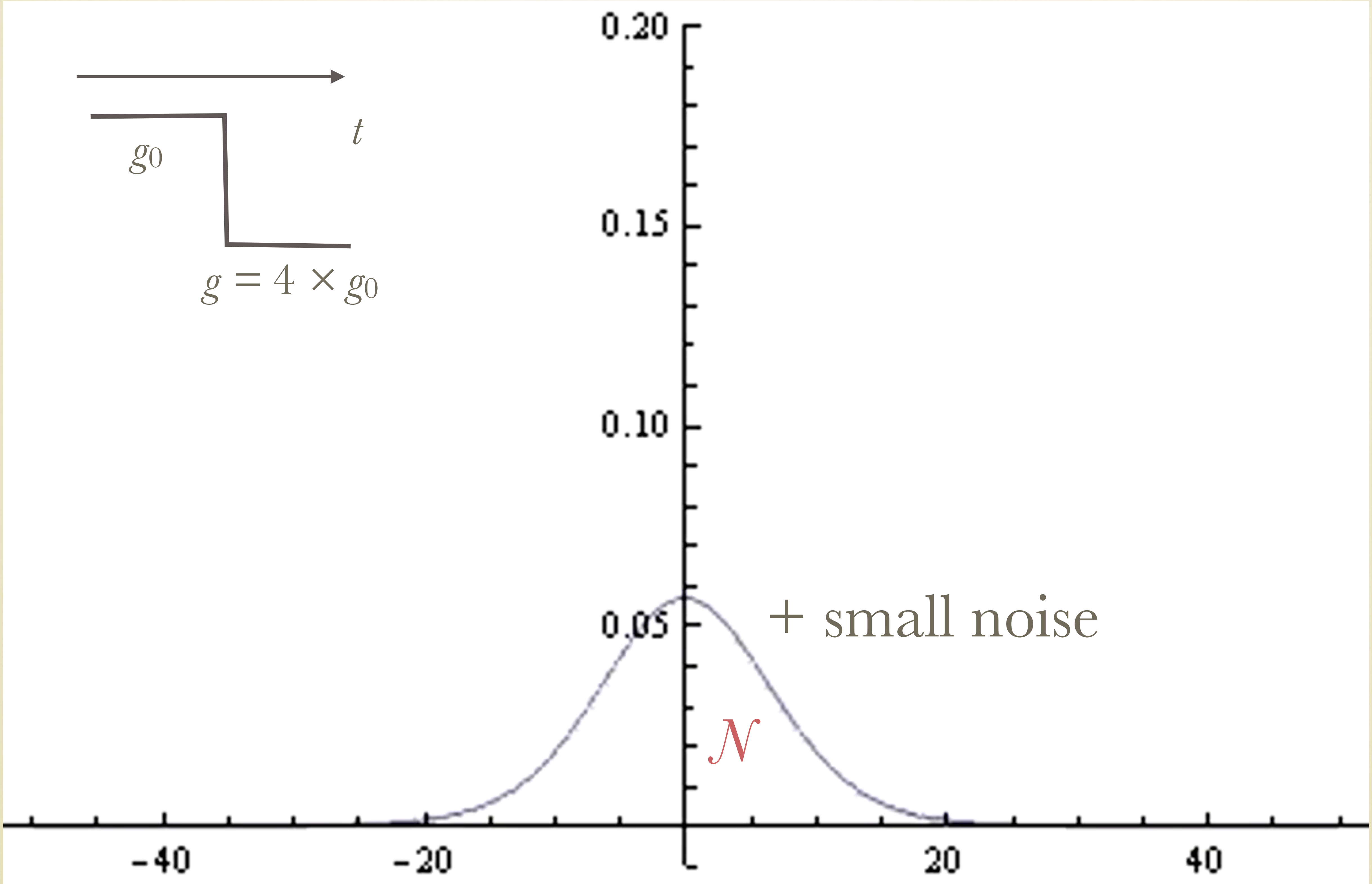
Atomtronics
Benasque
May 7, 2019

Small field fluctuations feeding the soliton relative distance fluctuations

Start from a single BEC soliton, at a coupling g_0



Quench the coupling 4-fold, $g_0 \twoheadrightarrow g = 4\times g_0$



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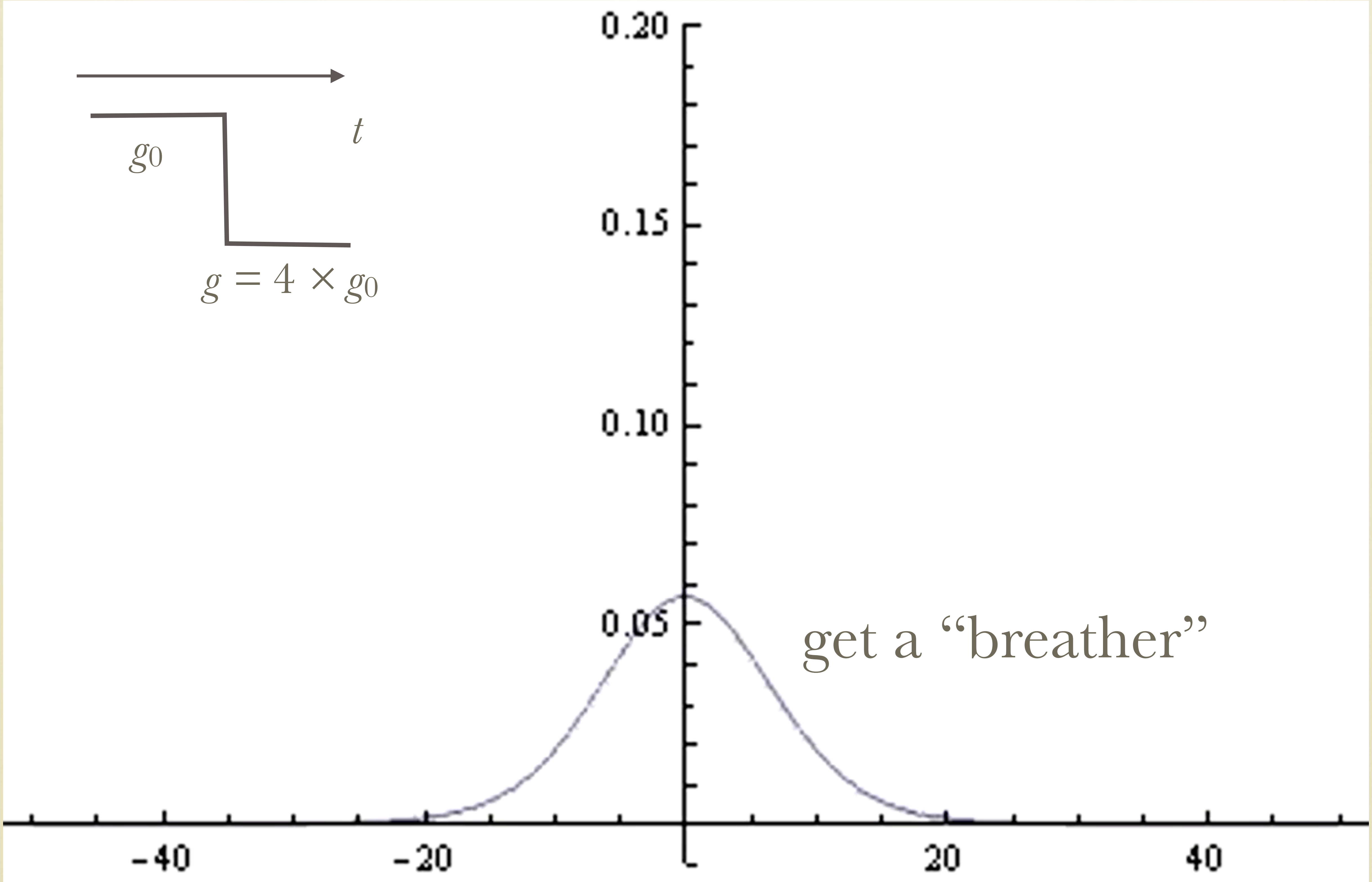

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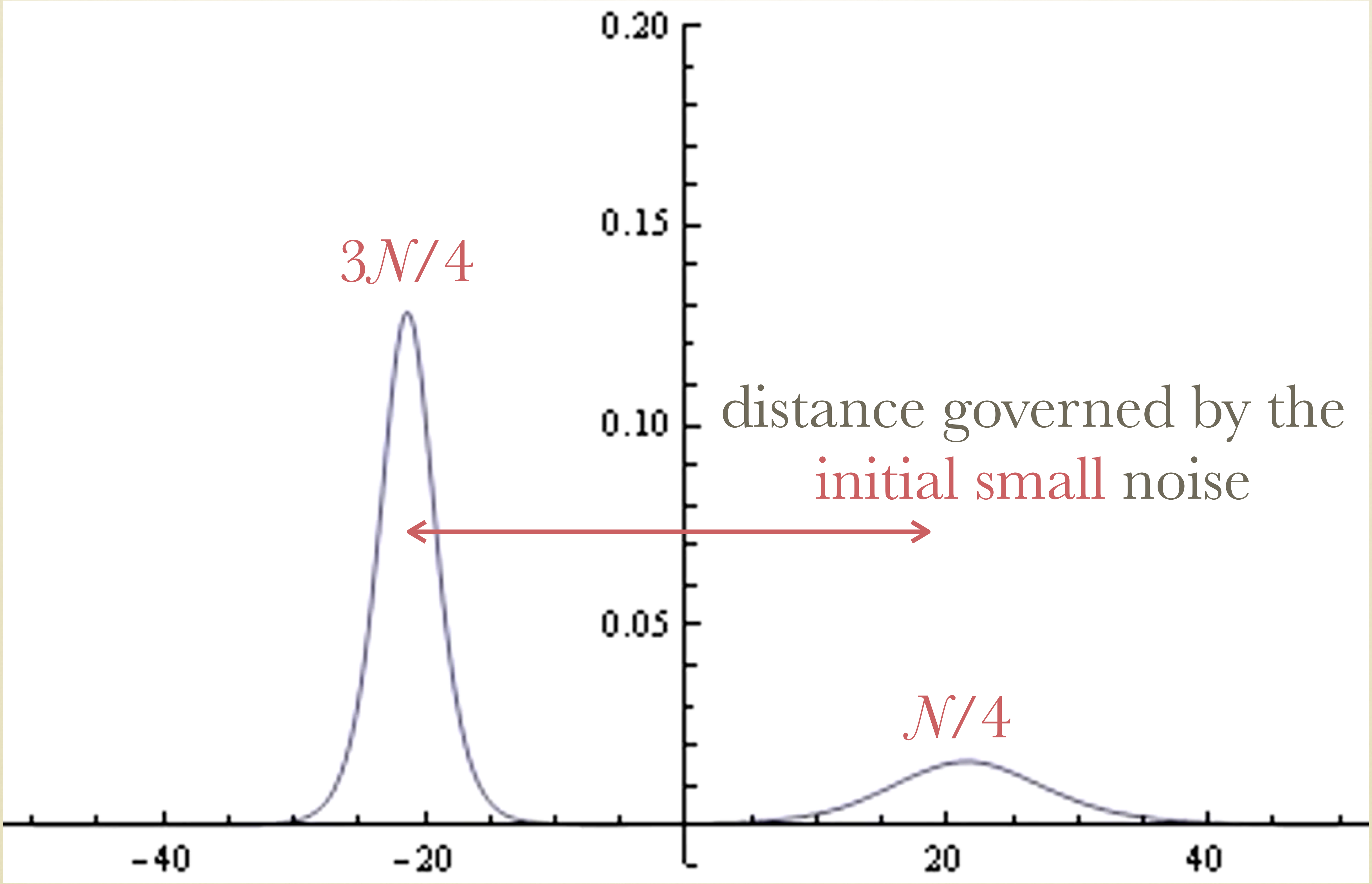


Quantum fluctuations feeding the soliton relative distance quantum fluctuations

$$T \lesssim \mu$$

Exact separable action-angle Hamiltonian, through the Inverse Scattering Transform

atoms in soliton

one-atom momentum in soliton

momentum density of the unbound atoms

$$\hat{H} = \frac{N_1 p_1^2}{2m} - \frac{m g_0^2}{24\hbar^2} N_1^3 + \int_{-\infty}^{+\infty} dp \, \frac{n(p) p^2}{2m}$$

Canonical pairs:

$$(N_1, \Phi_1);\ \ (p_1, q_1 \equiv x_{1,\mathrm{com}} N_1);\ \ (n(p), \phi(p))$$

Faddeev & Takhtadjan,
Hamilton Approach to the Soliton Theory

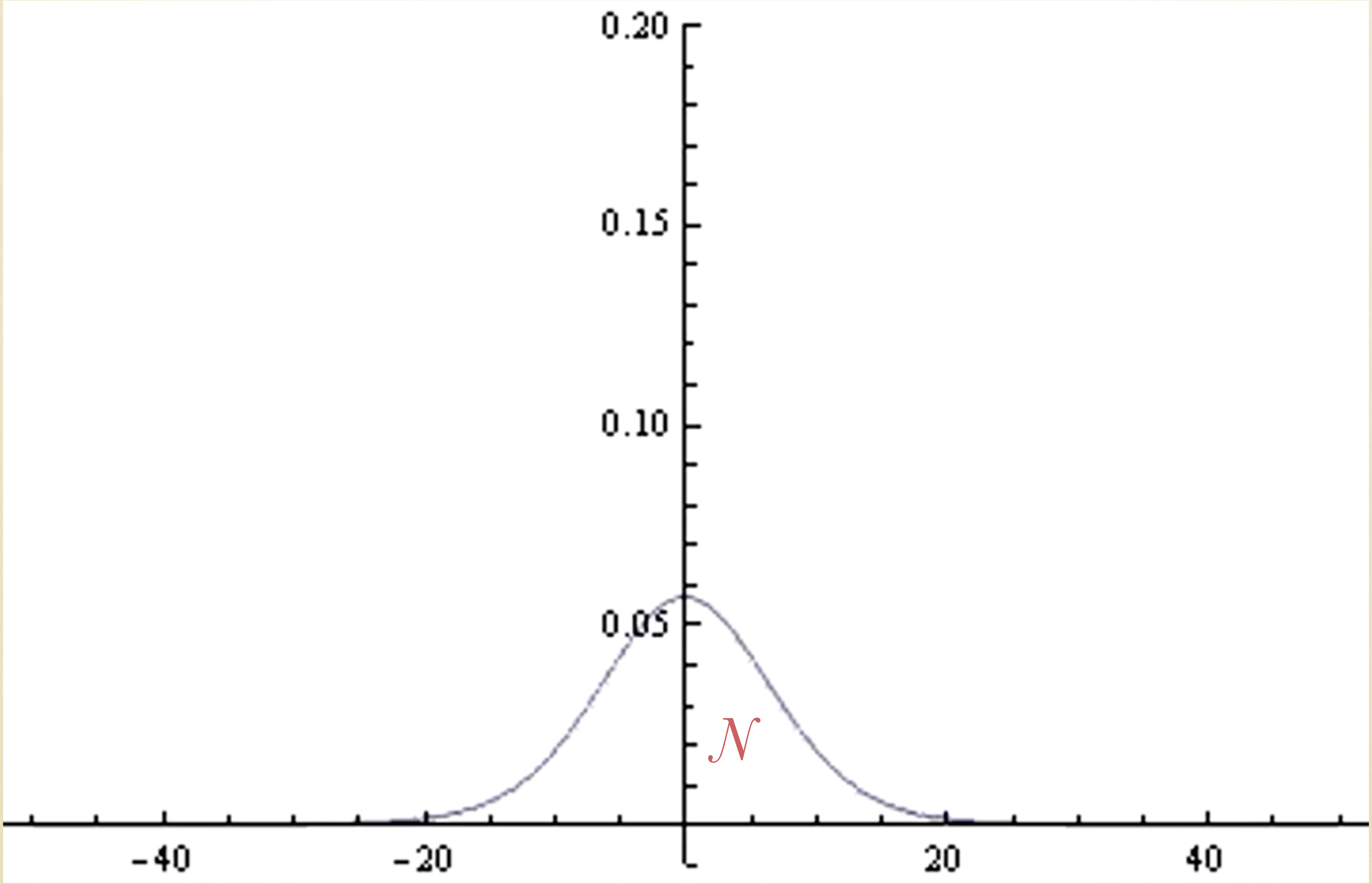
Lower part of the spectrum

soliton + one free atom

$$\mu \equiv \frac{m g_0^2}{8\hbar^2} N_1^2$$

pure soliton

Back to a single soliton, g_0; now assume $T \lesssim \mu$



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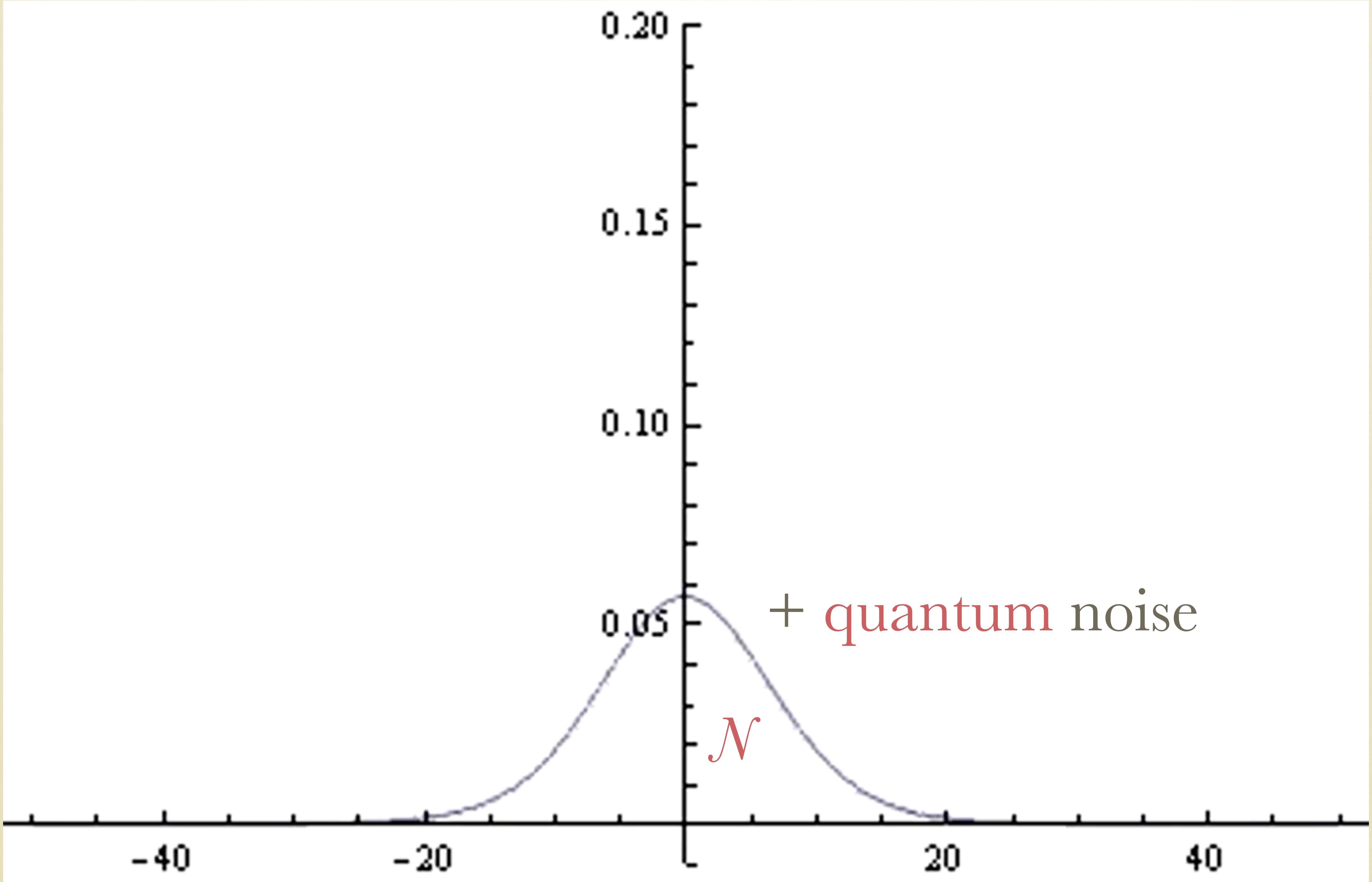
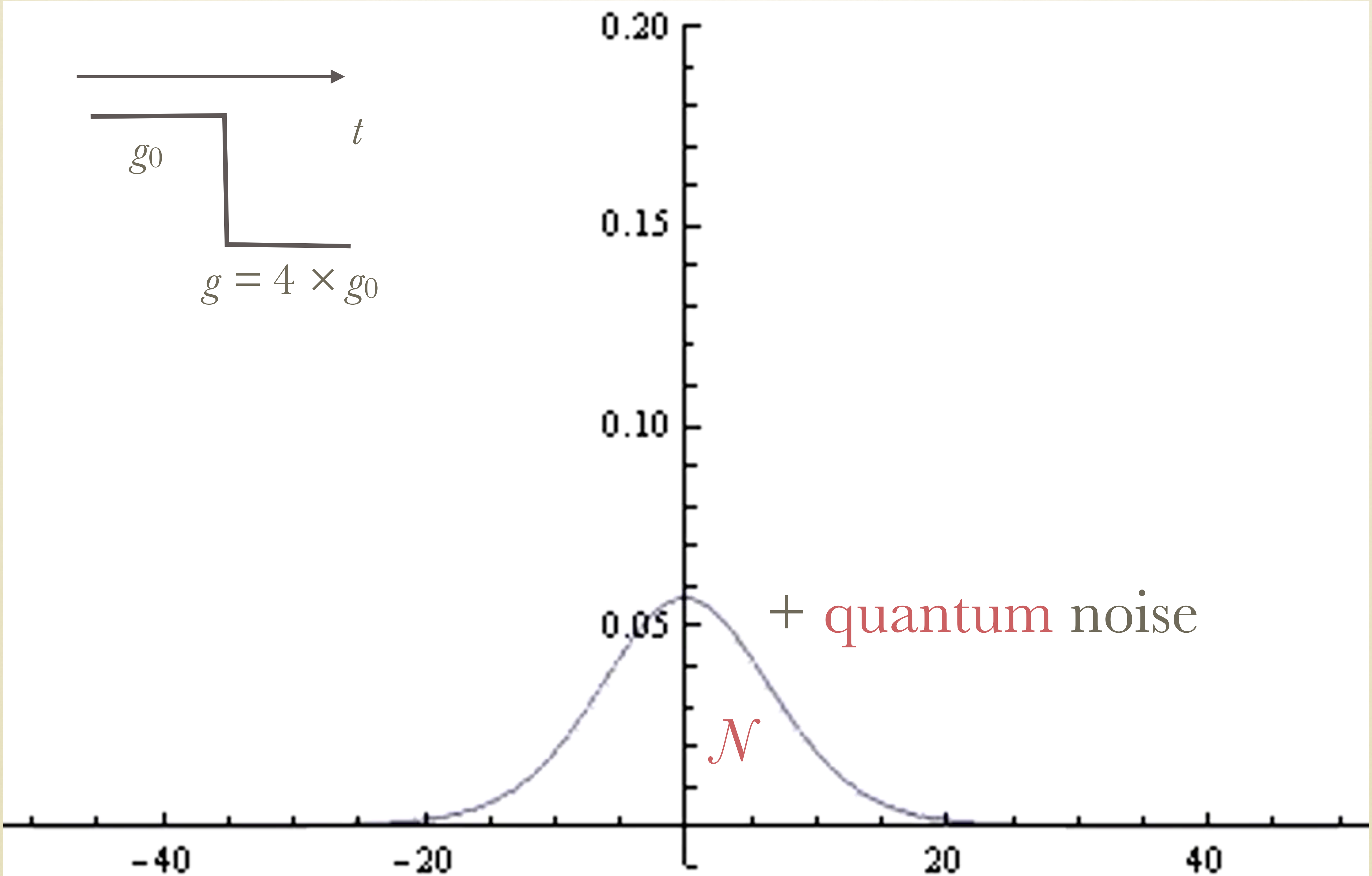


Quench the coupling 4-fold, $g_0 \twoheadrightarrow g = 4\times g_0$



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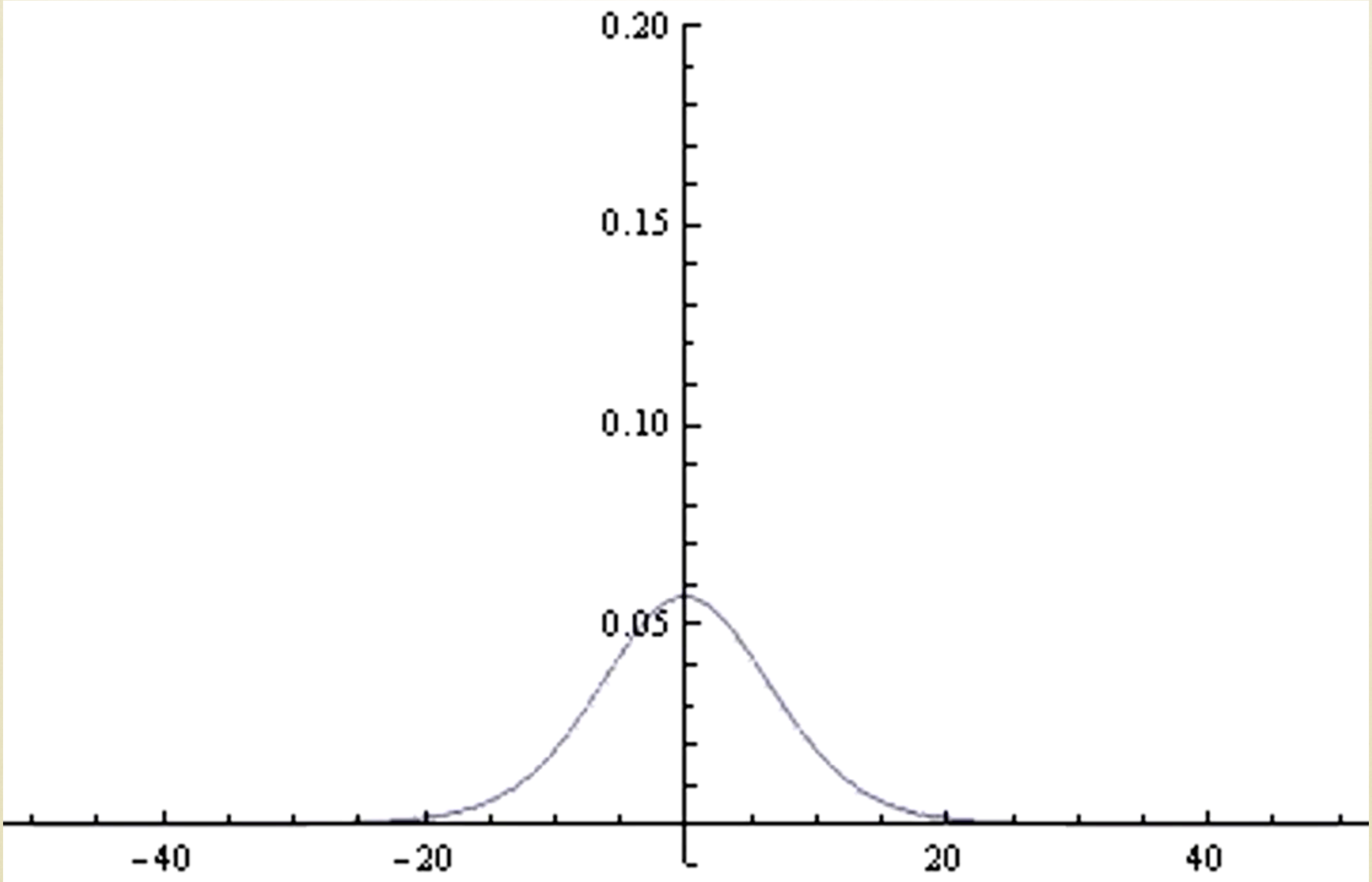

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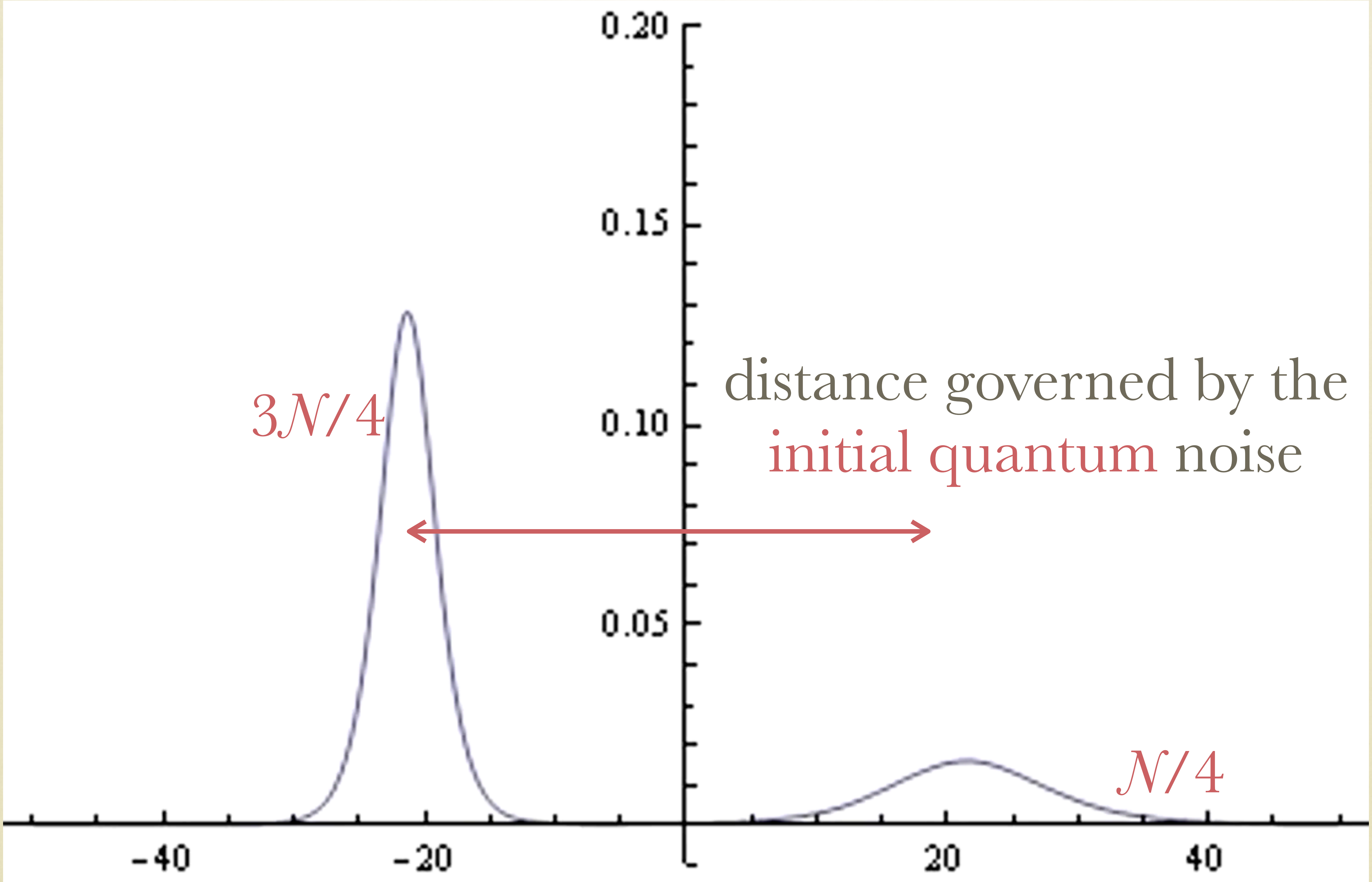

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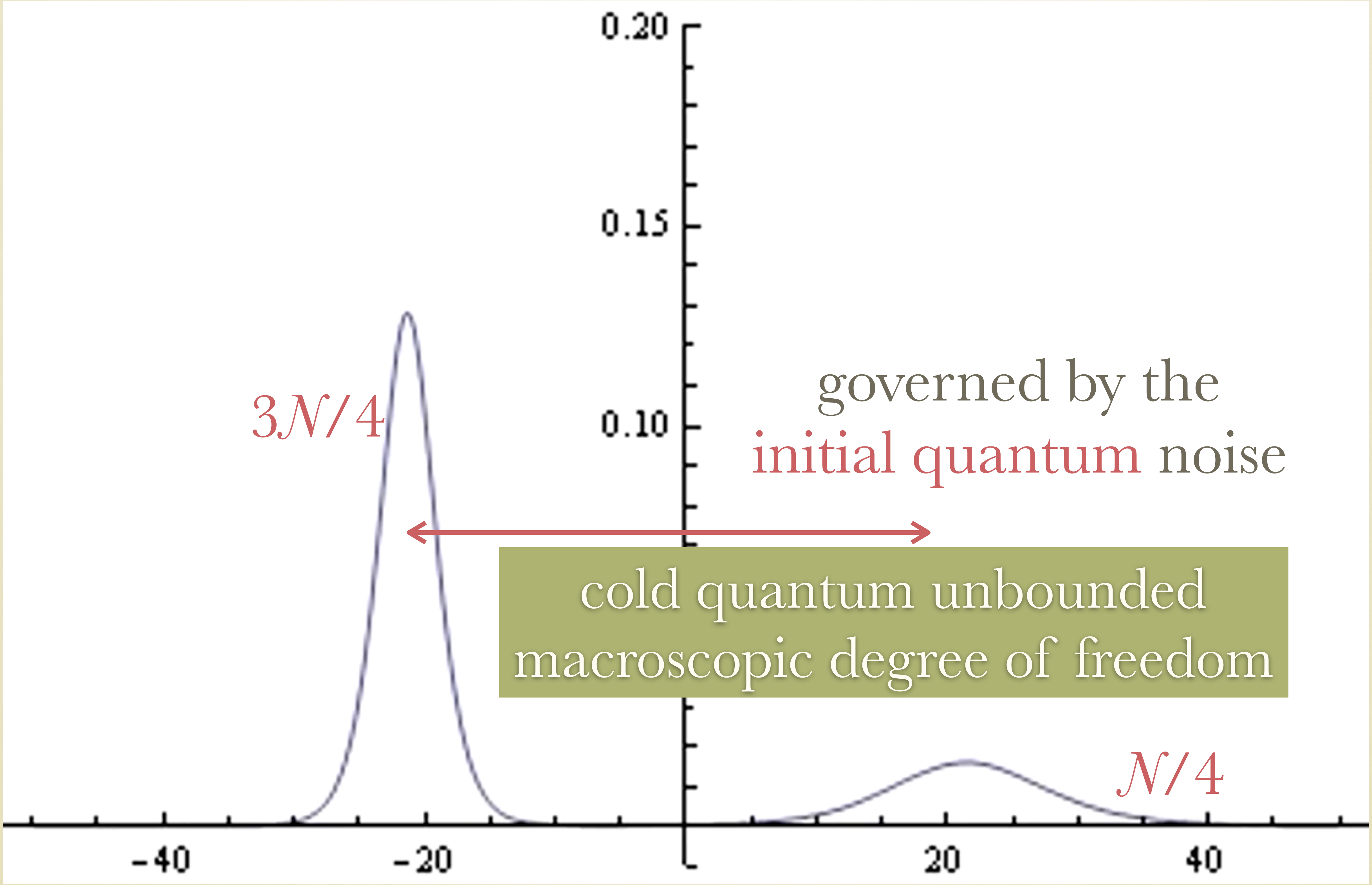


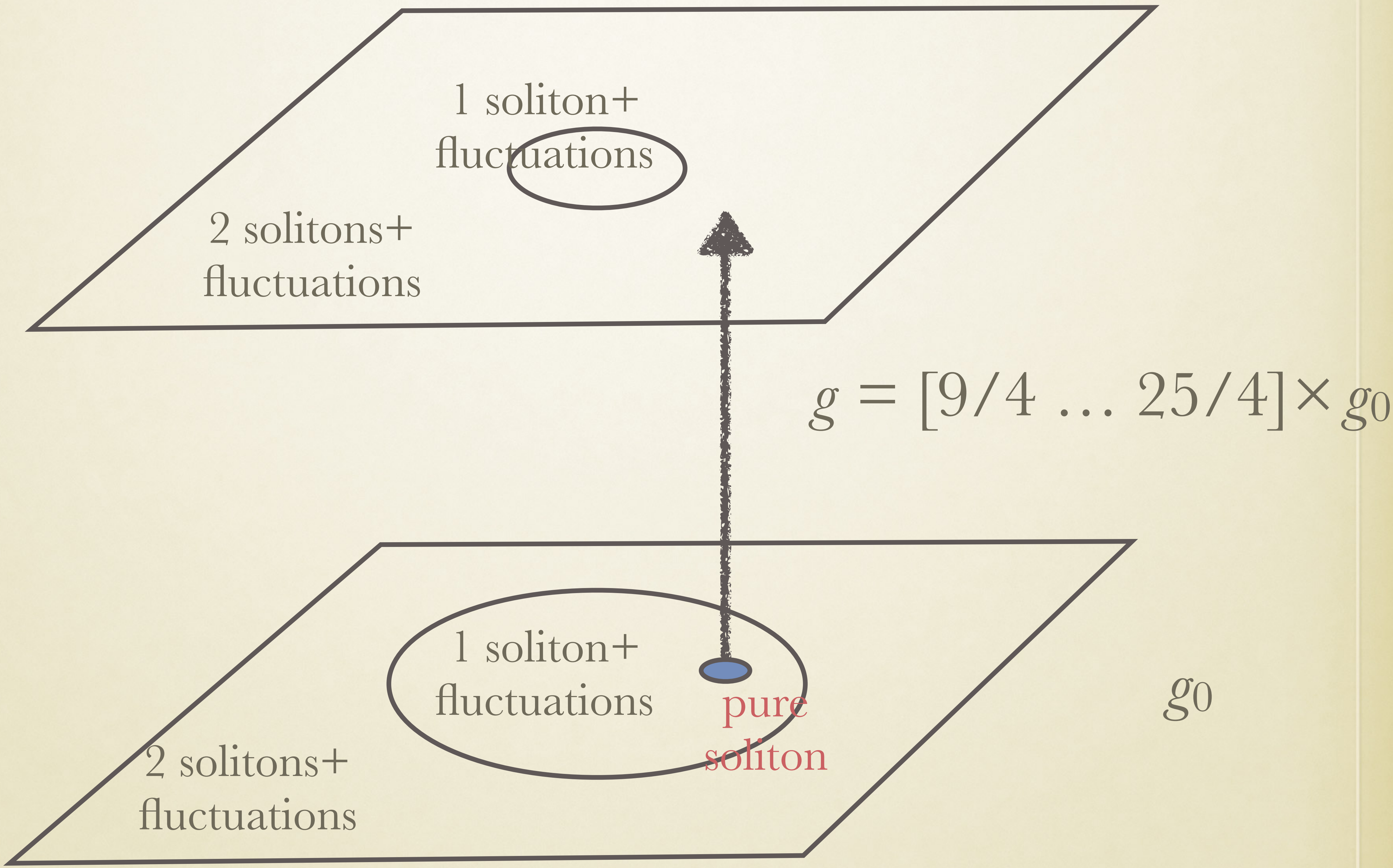
What enables the effect?

Q: What enables the effect?

A: Existence of solitons

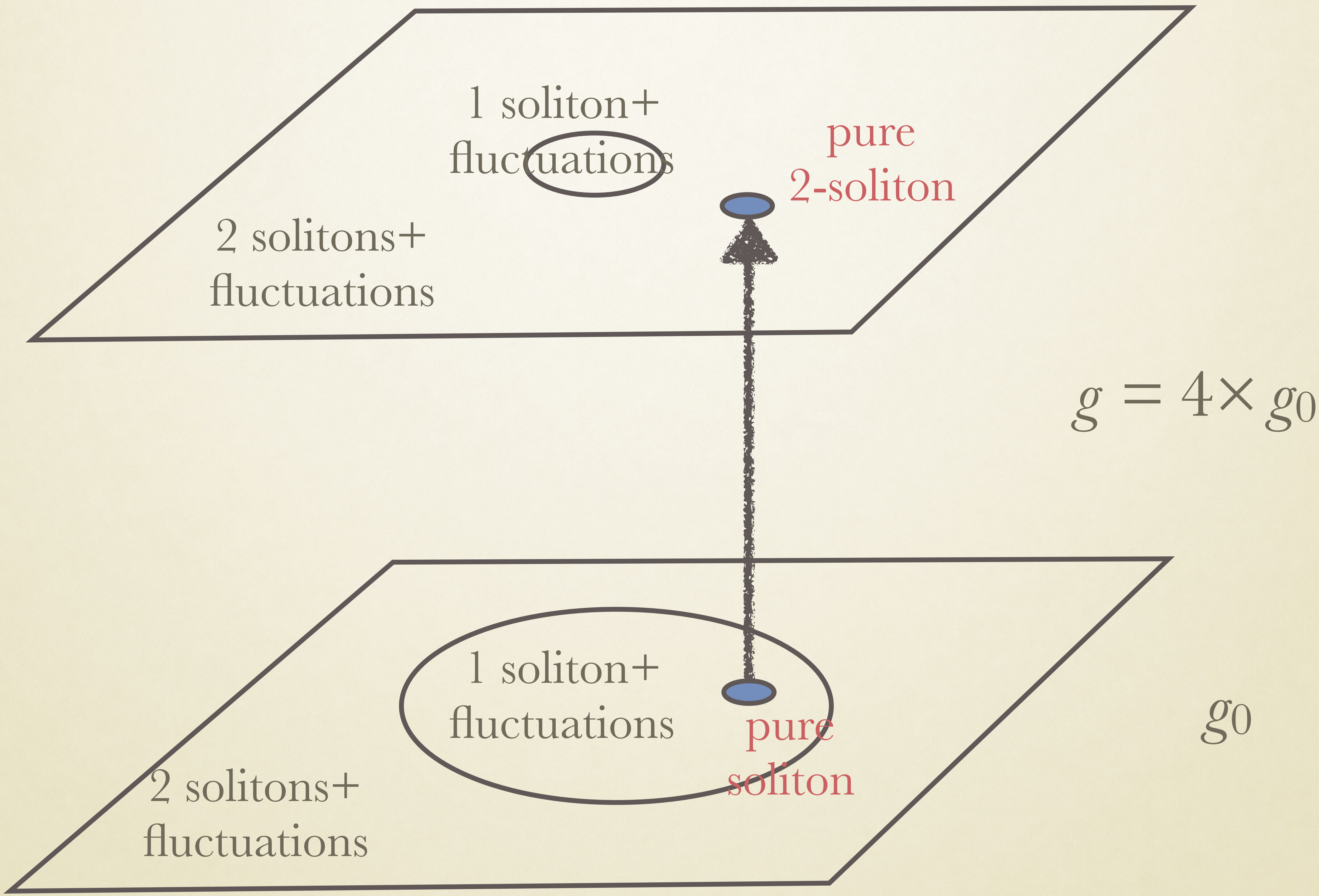
From 1-soliton to 2-soliton sheet

Satsuma & Yajima, Supplement of the Progress of Theoretical Physics, No. 55, 1974



Remark: CoM motion is completely decoupled

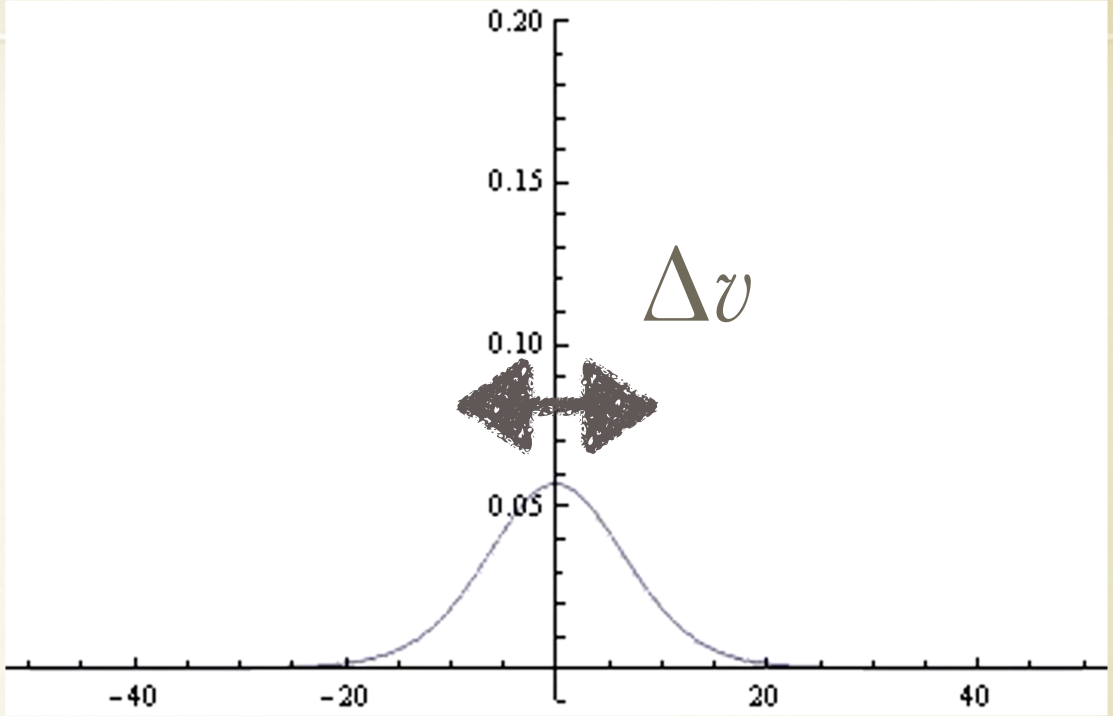
Satsuma & Yajima, Supplement of the Progress of Theoretical Physics, No. 55, 1974



Remark: CoM motion is completely decoupled

microscopic quantum fluctuations at g_0 become
macroscopic quantum fluctuations at $g = 4 \times g_0$

Predictions



^{7}Li

$N = 3 \times 10^3$

$\omega_\perp = 2\pi \times 254\text{Hz}$

$a_{\text{scatt}} = -4\, a_{\text{Bohr}}$

Bethe Ansatz for $N \lesssim 20$: $\langle (\Delta v)^2 \rangle = 0.072 \left(\frac{|g|}{\hbar}\right)^2 N$

Bogoliubov propagation starting from a white quantum noise on the mother soliton:

$$\langle (\Delta v)^2 \rangle = \frac{23}{1680} \left(\frac{|g|}{\hbar}\right)^2 N = 0.0136905 \left(\frac{|g|}{\hbar}\right)^2 N$$

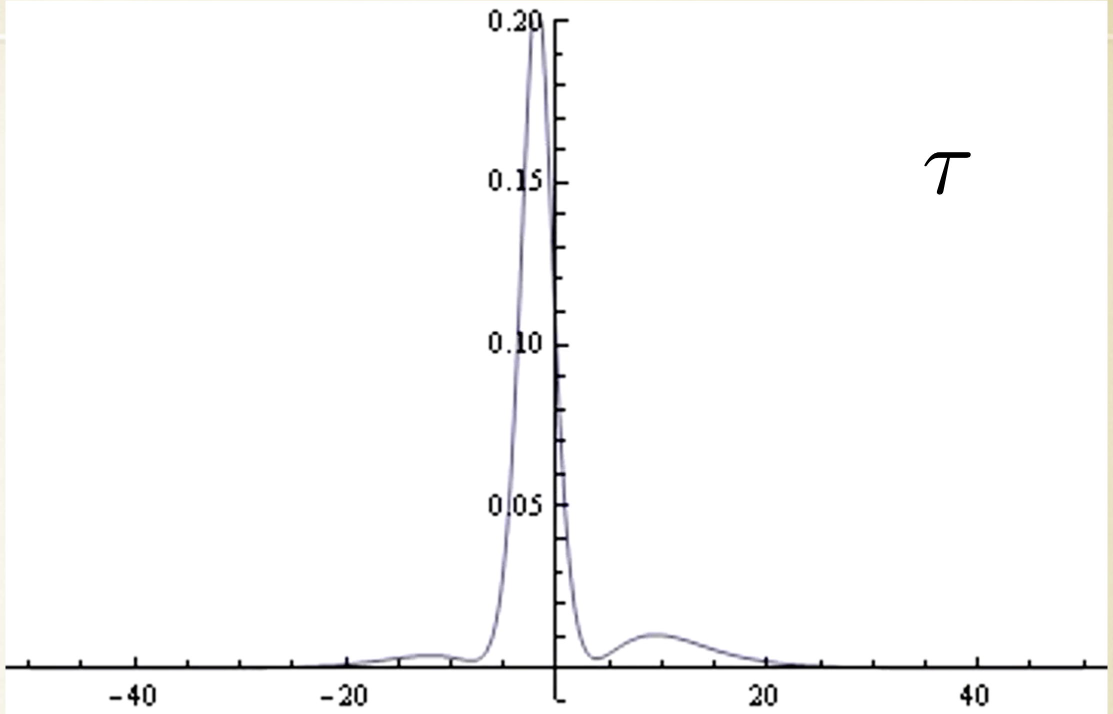
Bogoliubov propagation starting from the Bogoliubov-correlated noise on the mother soliton: $\langle (\Delta v)^2 \rangle = 0.0058 \left(\frac{|g|}{\hbar}\right)^2 N$

$^{7}\mathrm{Li}$

$N = 3 \times 10^3$

$\omega_\perp = 2\pi \times 254\mathrm{Hz}$

$a_{\mathrm{scatt}} = -4\, a_{\mathrm{Bohr}}$



Extrapolation from Bethe Ansatz for $N = 20$:

$\tau = 3.6\,\mathrm{s}$

Bogoliubov propagation starting from a white quantum noise: $\tau = 8.3\,\mathrm{s}$

Bogoliubov propagation starting from the Bogoliubov noise on mother soliton: $\tau = 12.8\,\mathrm{s}$

[2] Oleksandr V. Marchukov, Boris A. Malomed, Maxim Olshanii, Vanja Dunjko, Randall G. Hulet, and Vladimir A. Yurovsky, **Quantum fluctuations of the center-of-mass and relative parameters of NLS breather**, in preparation.

[1] Vladimir A. Yurovsky, Boris A. Malomed, Randall G. Hulet, Maxim Olshanii, **Dissociation of one-dimensional matter-wave breathers due to quantum many-body effects**, *Phys. Rev. Lett.* **119**, 220401 (2017).

Summary

A factor of 4 coupling quench from a soliton

⬇

Quantum macroscopic unbounded degree of freedom: the relative distance between daughter solitons in a breather

Observable in ambient mean-filed conditions

Support by:

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Science & Technology

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Thank you!

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Thank you!

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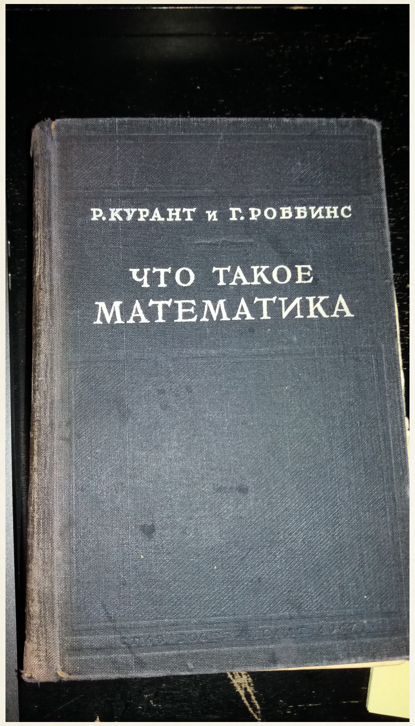
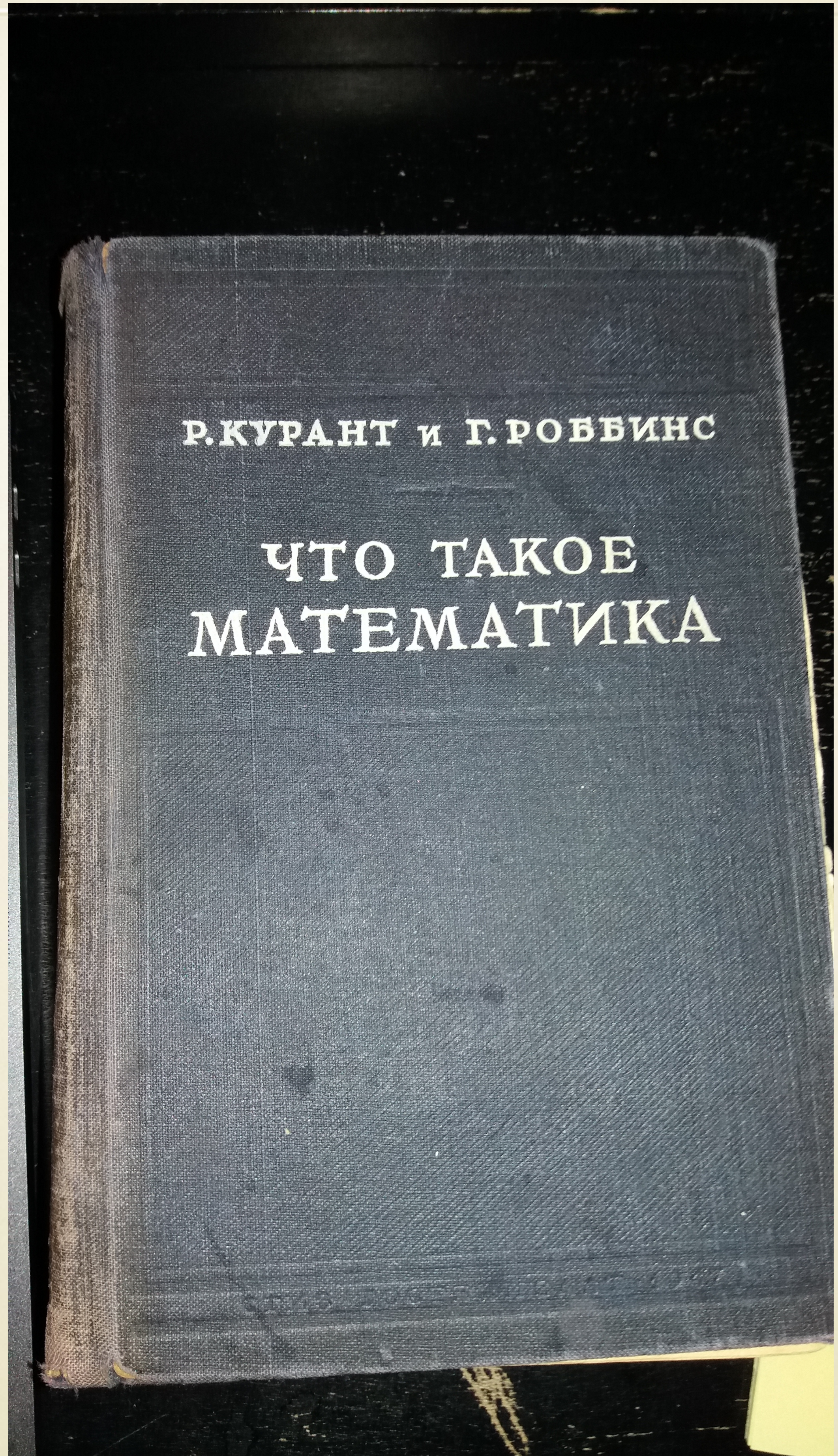
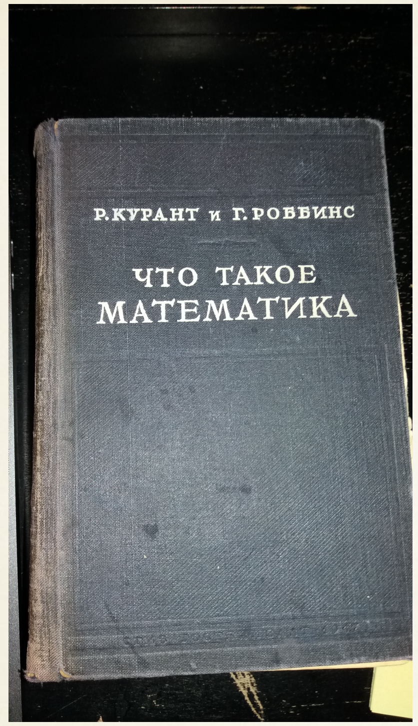
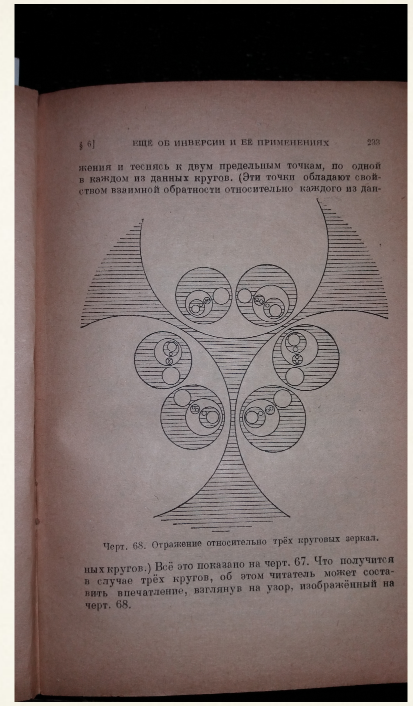
Recent developments with sphere-inversion and electrostatics: loving mirrors, changing the question



What Is Mathematics? Richard Courant & Herbert Robbins, edited by Ian Stewart


Р.КУРАНТ и Г.РОББИНС
ЧТО ТАКОЕ
МАТЕМАТИКА

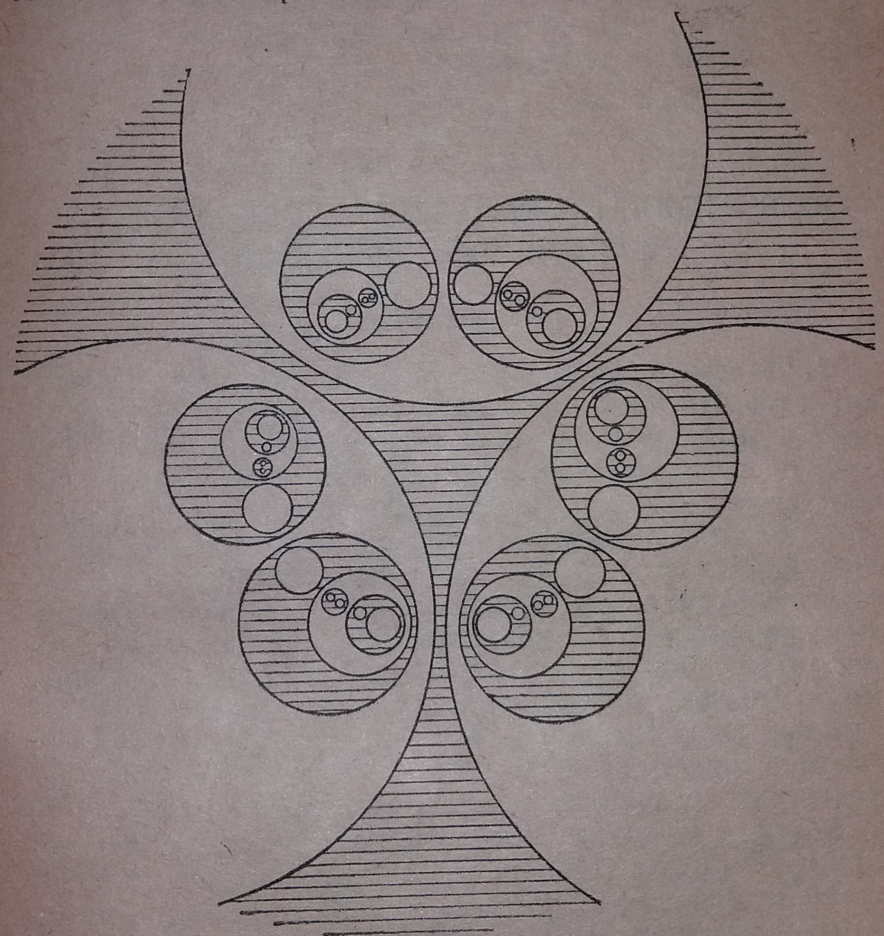



§ 6] ЕЩЁ ОБ ИНВЕРСИИ И ЕЁ ПРИМЕНЕНИЯХ 233

жения и теснясь к двум предельным точкам, по одной в каждом из данных кругов. (Эти точки обладают свойством взаимной обратности относительно каждого из данных кругов.) Всё это показано на черт. 67. Что получится в случае трёх кругов, об этом читатель может составить впечатление, взглянув на узор, изображённый на черт. 68.

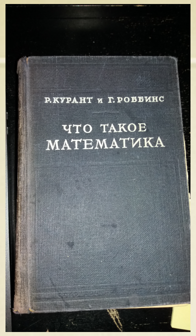
Черт. 68. Отражение относительно трёх круговых зеркал.

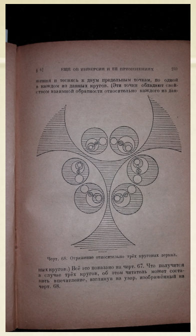
жения и теснясь к двум предельным точкам, по одной в каждом из данных кругов. (Эти точки обладают свойством взаимной обратности относительно каждого из дан-



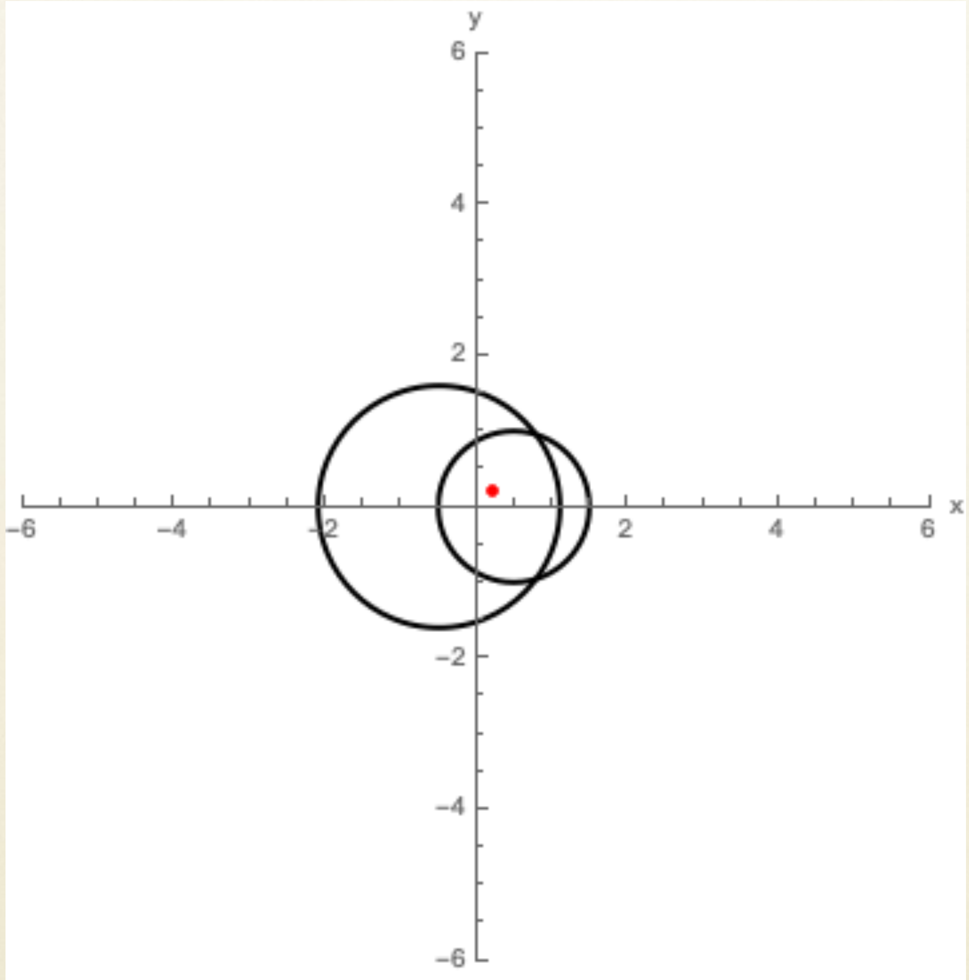
Черт. 68. Отражение относительно трёх круговых зеркал.

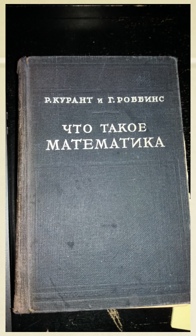
ных кругов.) Всё это показано на черт. 67. Что получится в случае трёх кругов, об этом читатель может составить впечатление, взглянув на узор, изображённый на черт. 68.

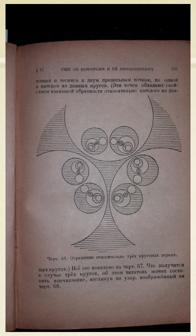




Dmitry Yampolsky, discovered numerically that sequential circle inversions lie on another circle. Yuri Styrkas (Princeton HS), proved this property

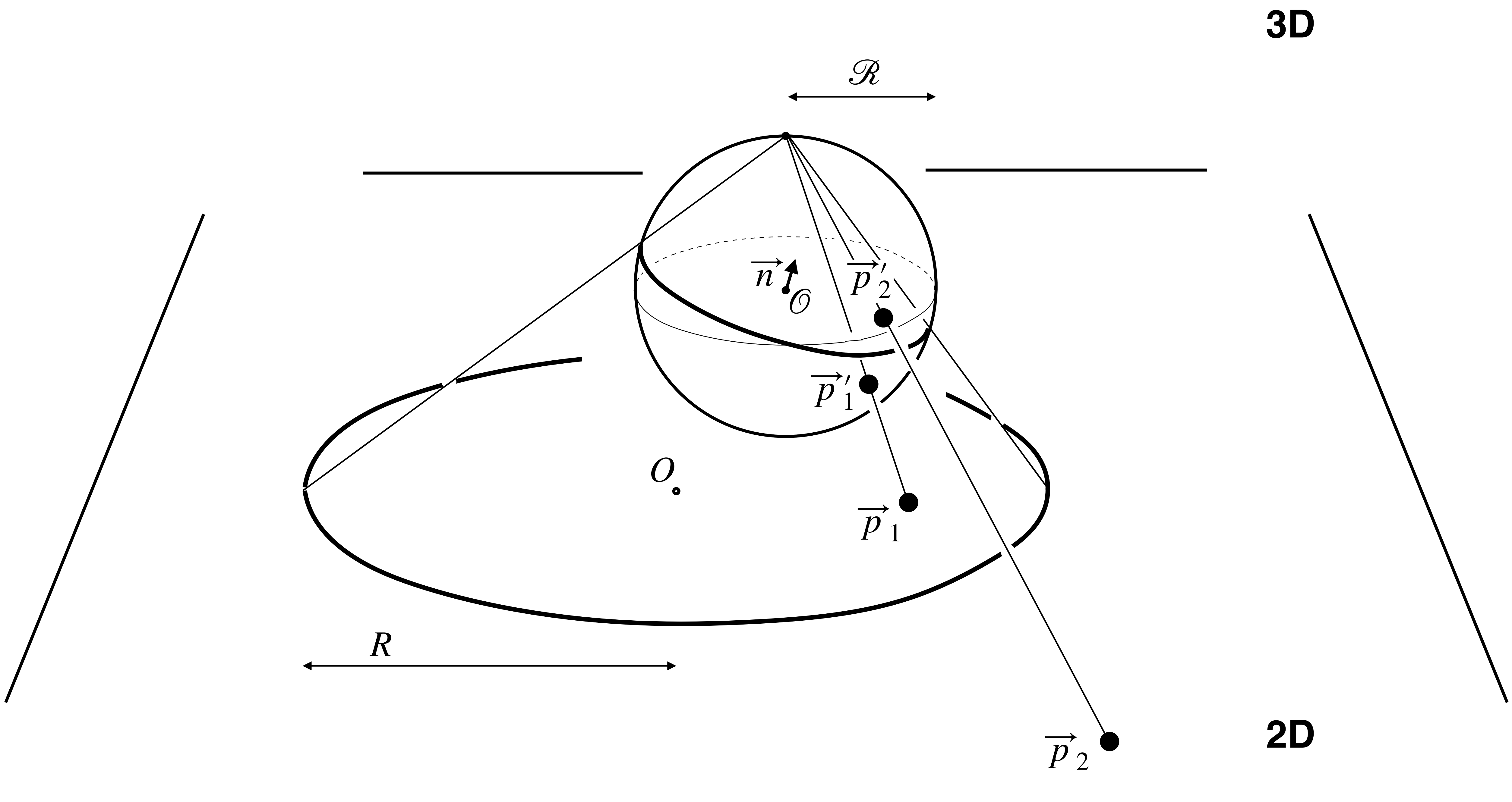

y
6
4
2
−2
−4
−6
x
−6
−4
−2
2
4
6



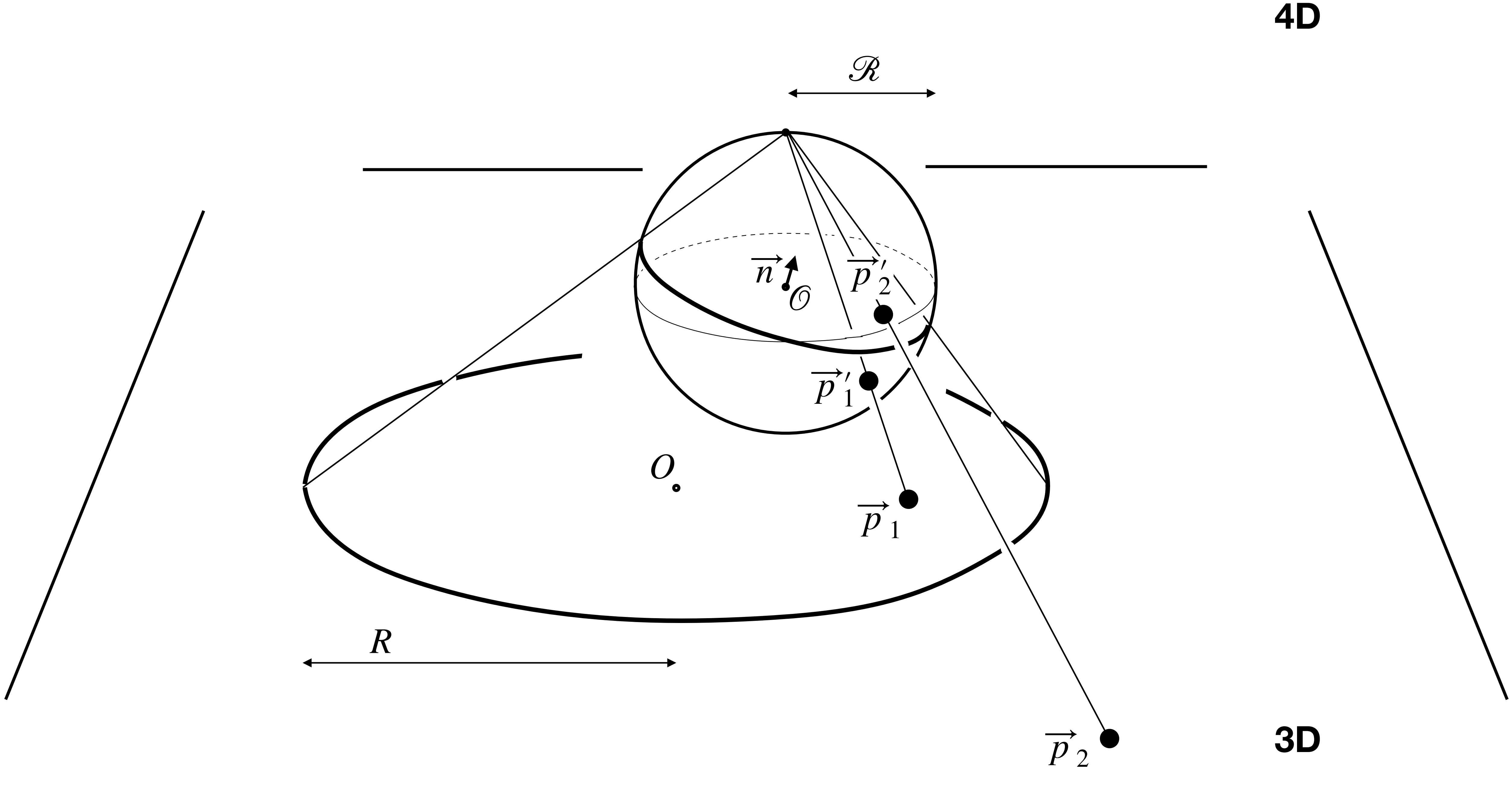


Dmitry Yampolsky, discovered numerically that sequential circle inversions lie on another circle. Yuri Styrkas, proved this property

Steven Jackson explained it in 5 min: "2D circle inversions are stereographic projections of 3D reflections". The rest follows, including lifting to 4D



3D
$\mathscr{R}$
$\vec{n}$
$\mathcal{O}$
$\vec{p}'_2$
$\vec{p}'_1$
O
$\vec{p}_1$
R
$\vec{p}_2$
2D



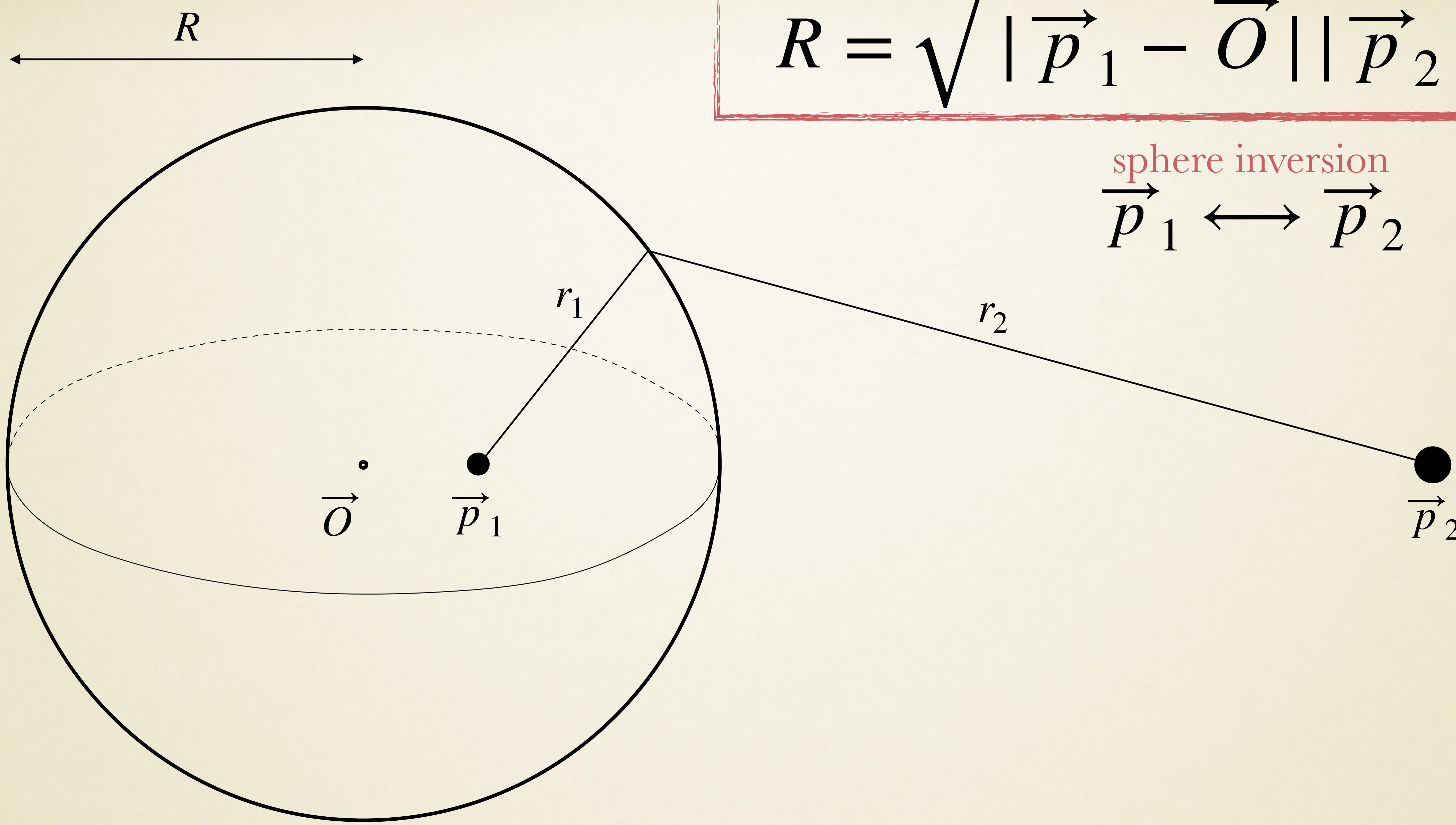
4D
$\mathscr{R}$
$\vec{n}$
$\mathcal{O}$
$\vec{p}'_2$
$\vec{p}'_1$
O
$\vec{p}_1$
R
$\vec{p}_2$
3D

Building solvable electrostatic problems

$$R = \sqrt{|\vec{p}_1 - \vec{O}||\vec{p}_2 - \vec{O}|}$$

sphere inversion

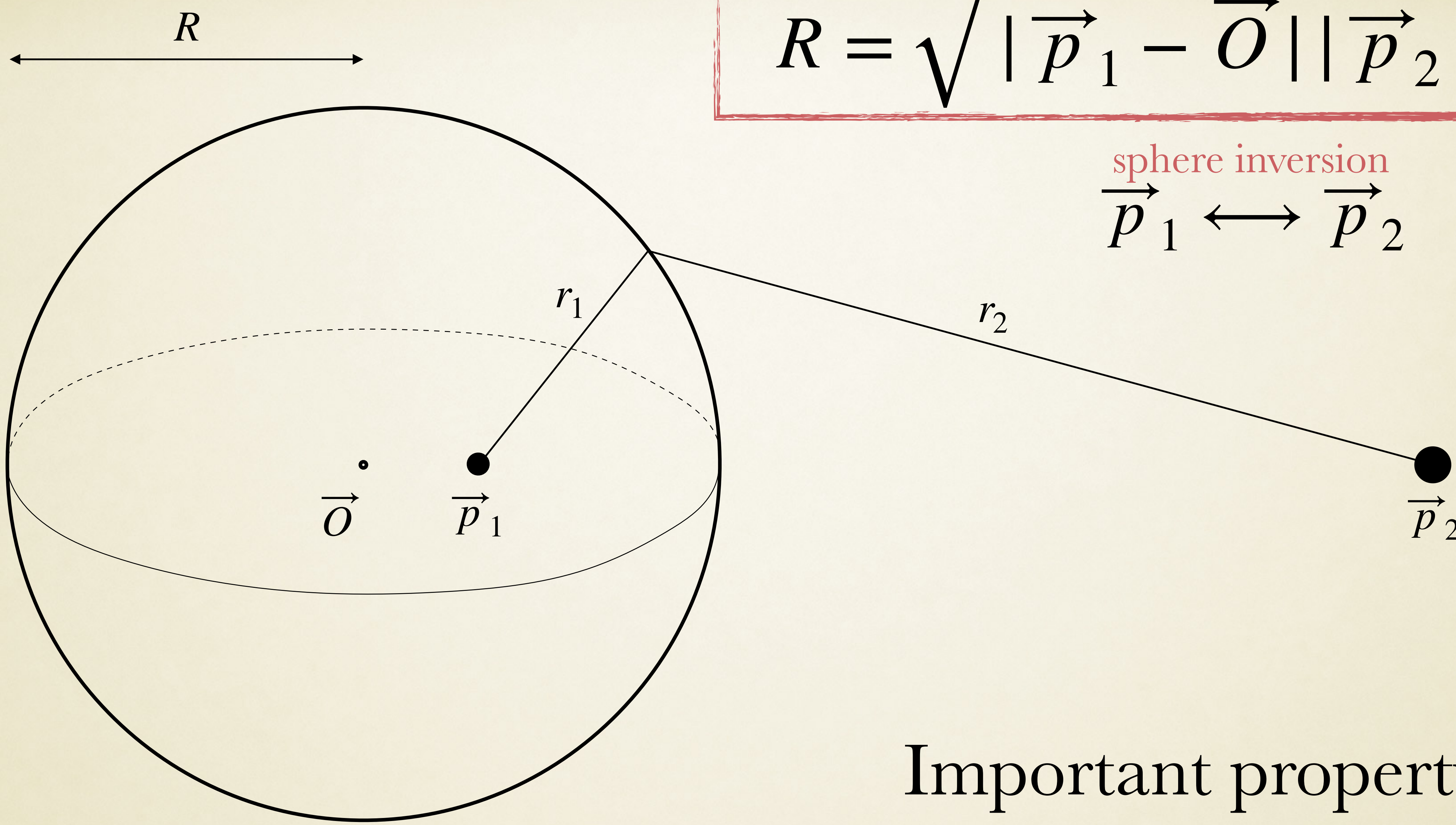
$\vec{p}_1 \longleftrightarrow \vec{p}_2$



$$R = \sqrt{|\vec{p}_1 - \vec{O}|\,|\vec{p}_2 - \vec{O}|}$$

sphere inversion

$$\vec{p}_1 \longleftrightarrow \vec{p}_2$$



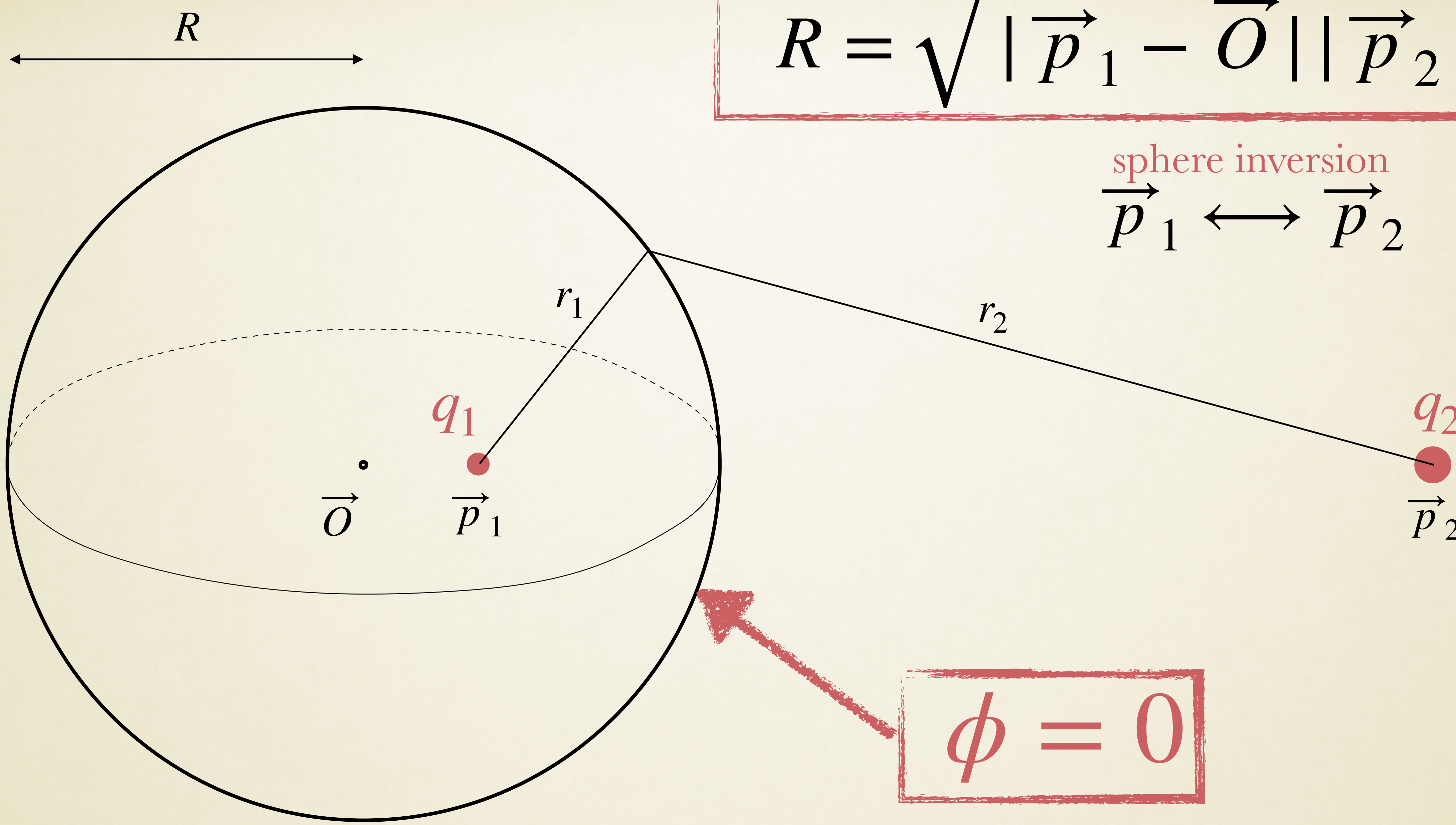
Important property:

$$\frac{r_1^2}{|\vec{p}_1 - \vec{O}|} = \frac{r_2^2}{|\vec{p}_2 - \vec{O}|}$$

$$R = \sqrt{|\vec{p}_1 - \vec{O}||\vec{p}_2 - \vec{O}|}$$

sphere inversion

$$\vec{p}_1 \longleftrightarrow \vec{p}_2$$

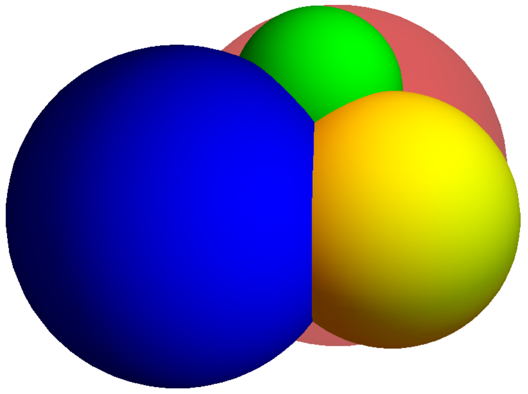


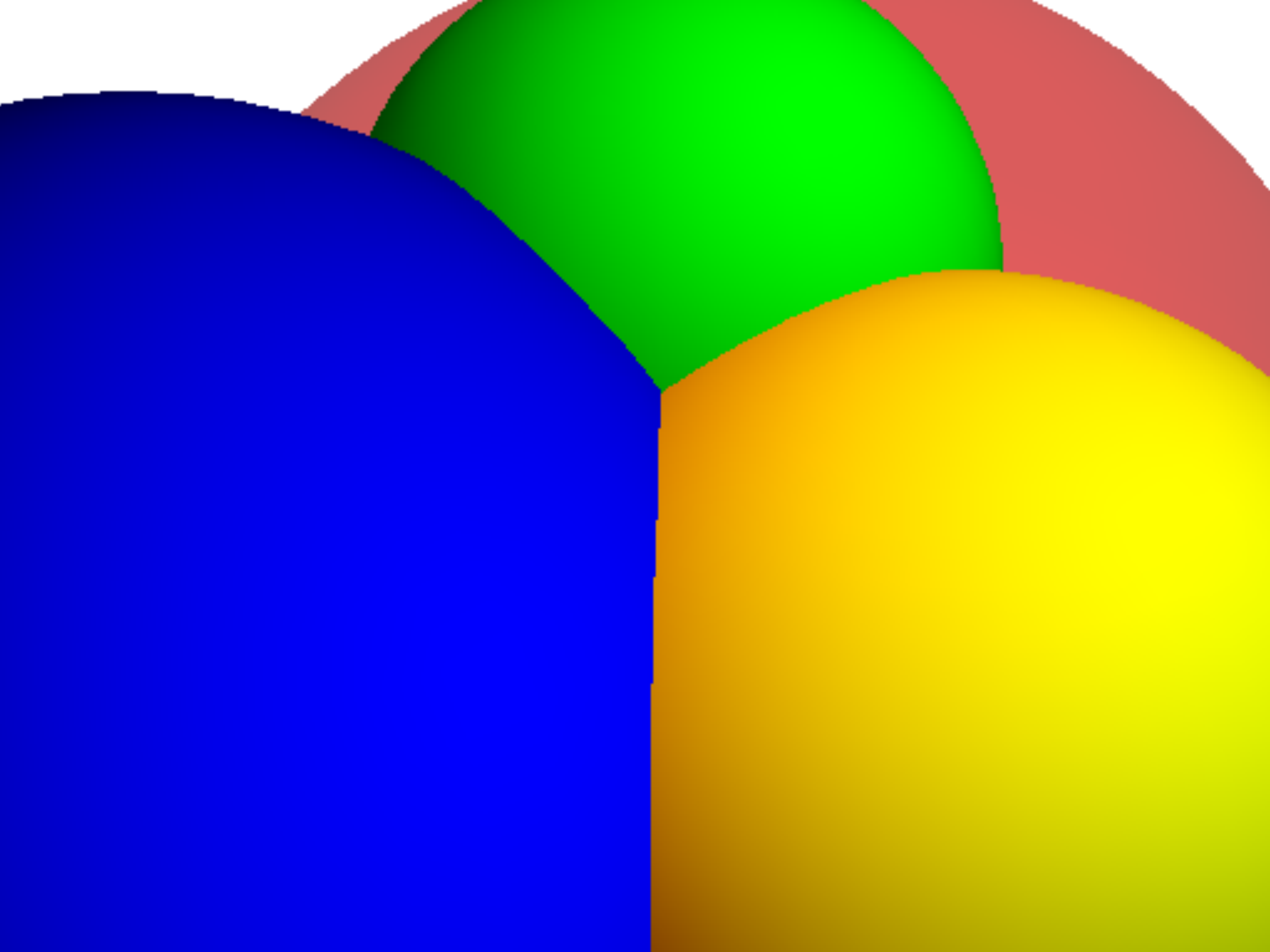
$$q_2 = -\sqrt{\frac{|\vec{p}_2 - \vec{O}|}{|\vec{p}_1 - \vec{O}|}}\, q_1$$

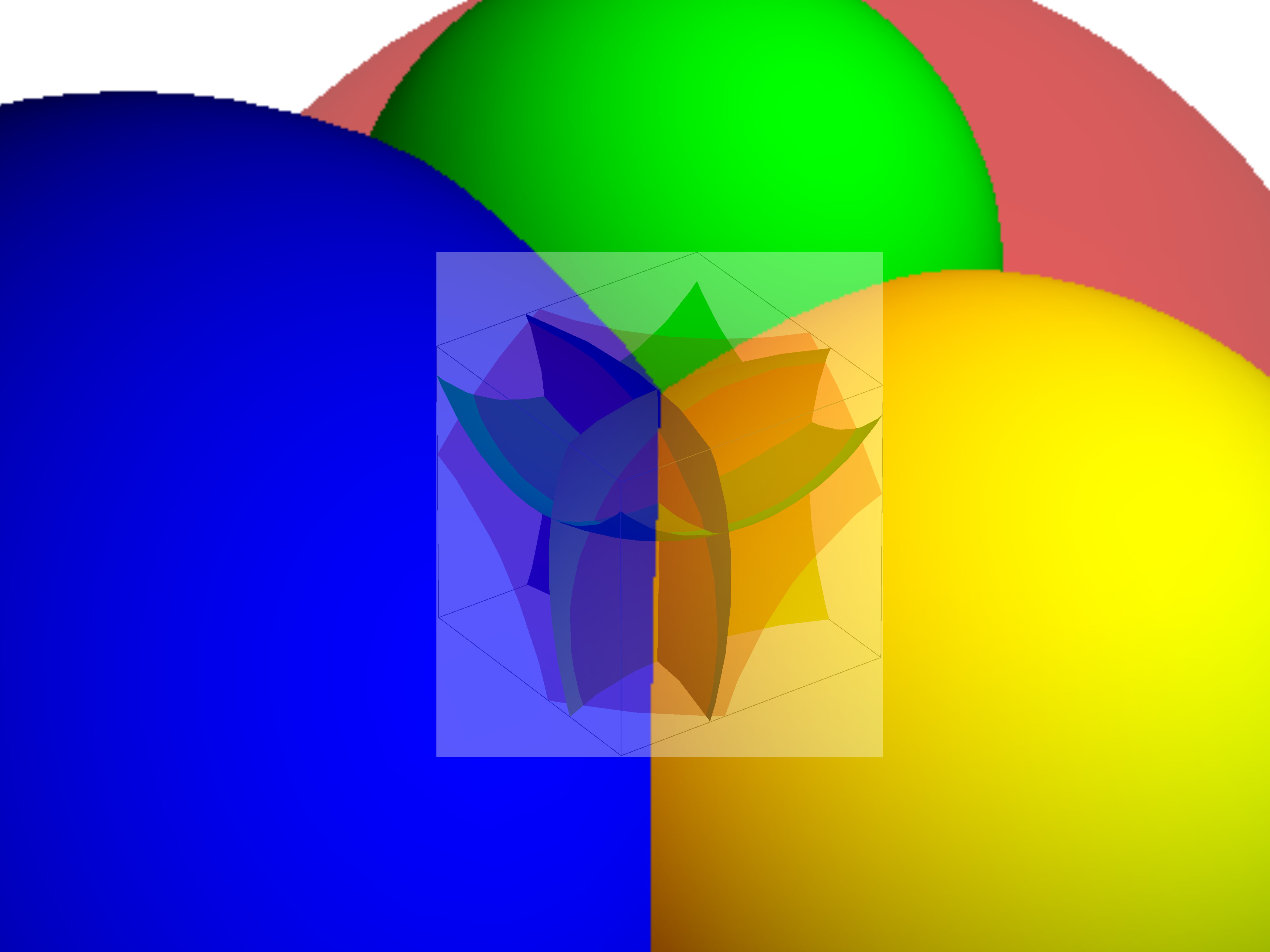
Imagine an empty cavity surrounded by a grounded conductor.

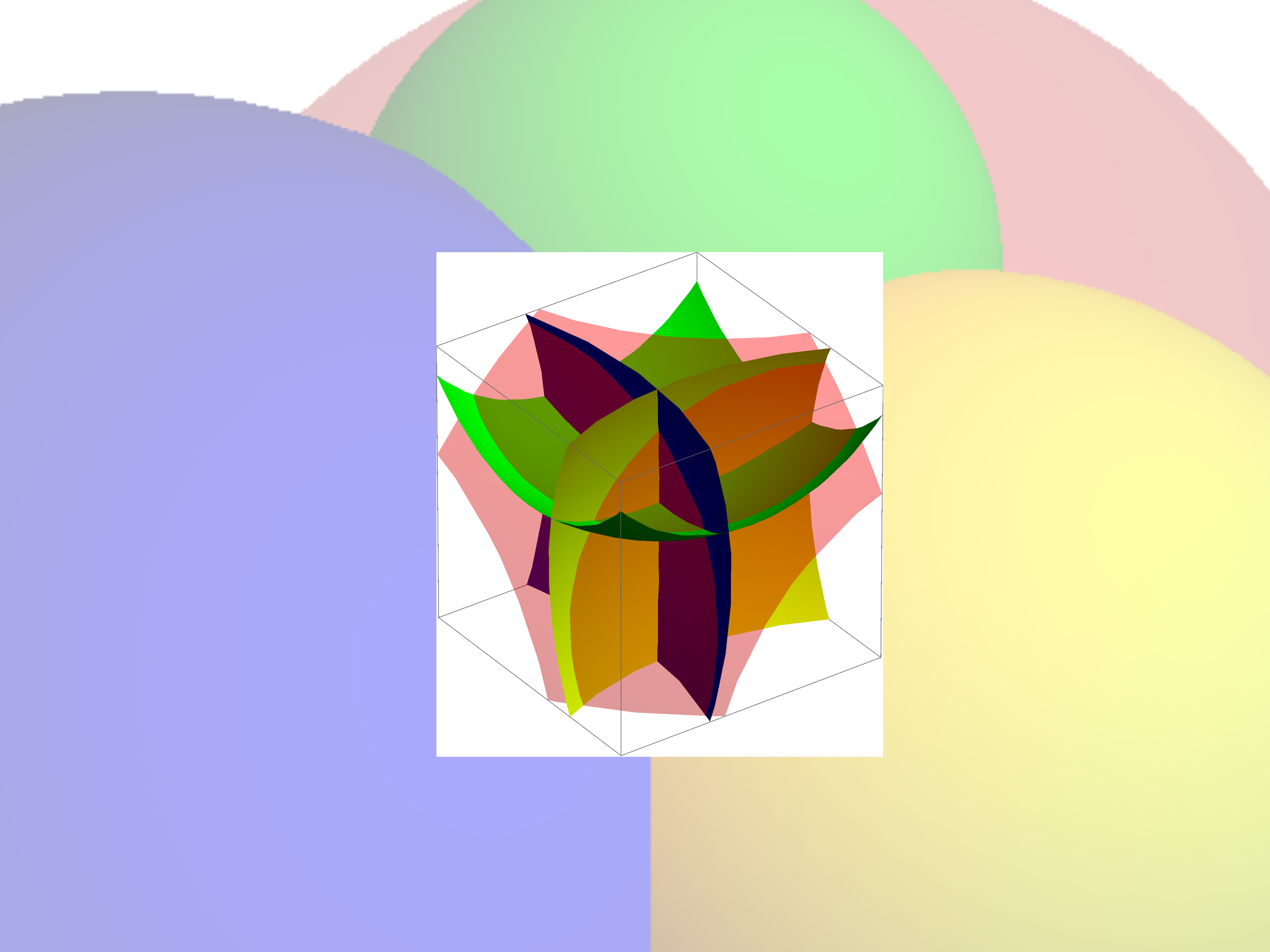
Imagine an empty cavity surrounded by a grounded conductor.

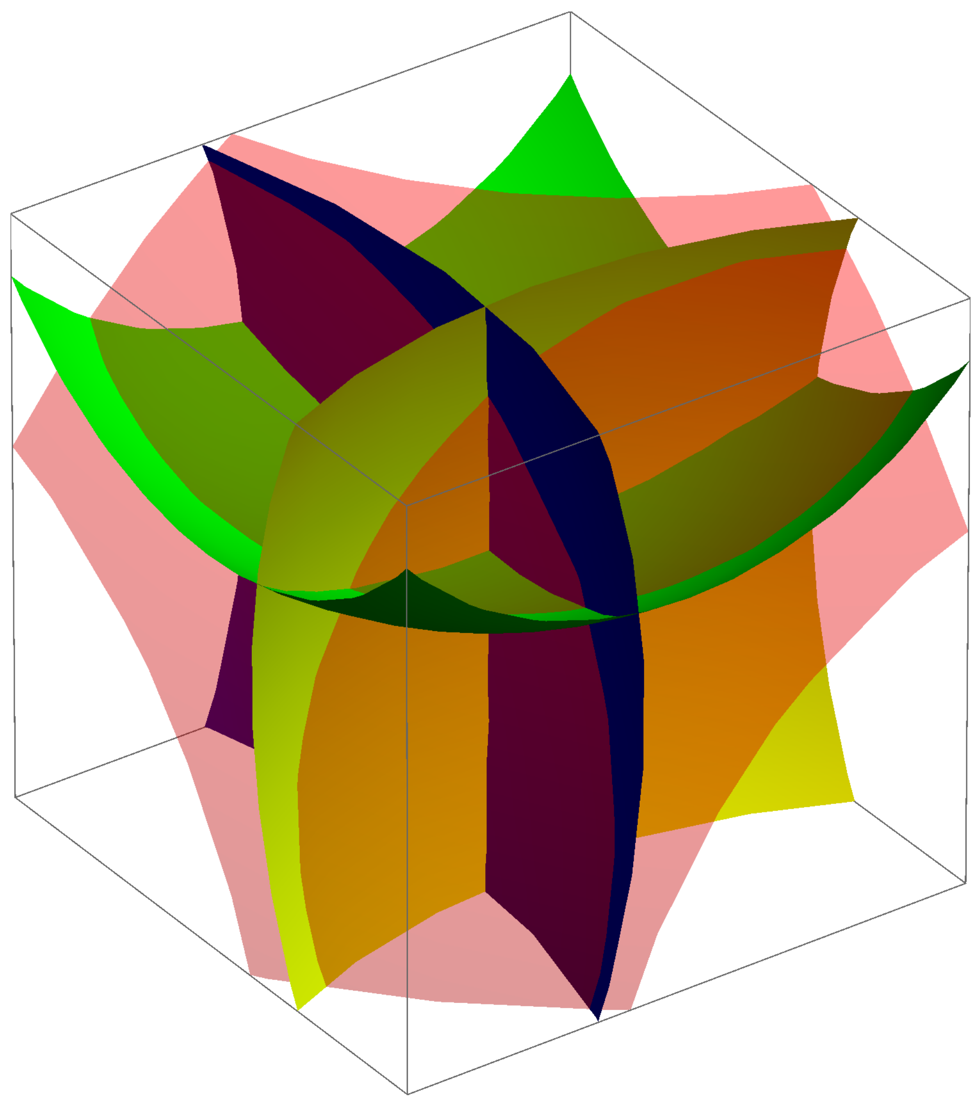
Assume that its walls are formed by segments of spherical surfaces.

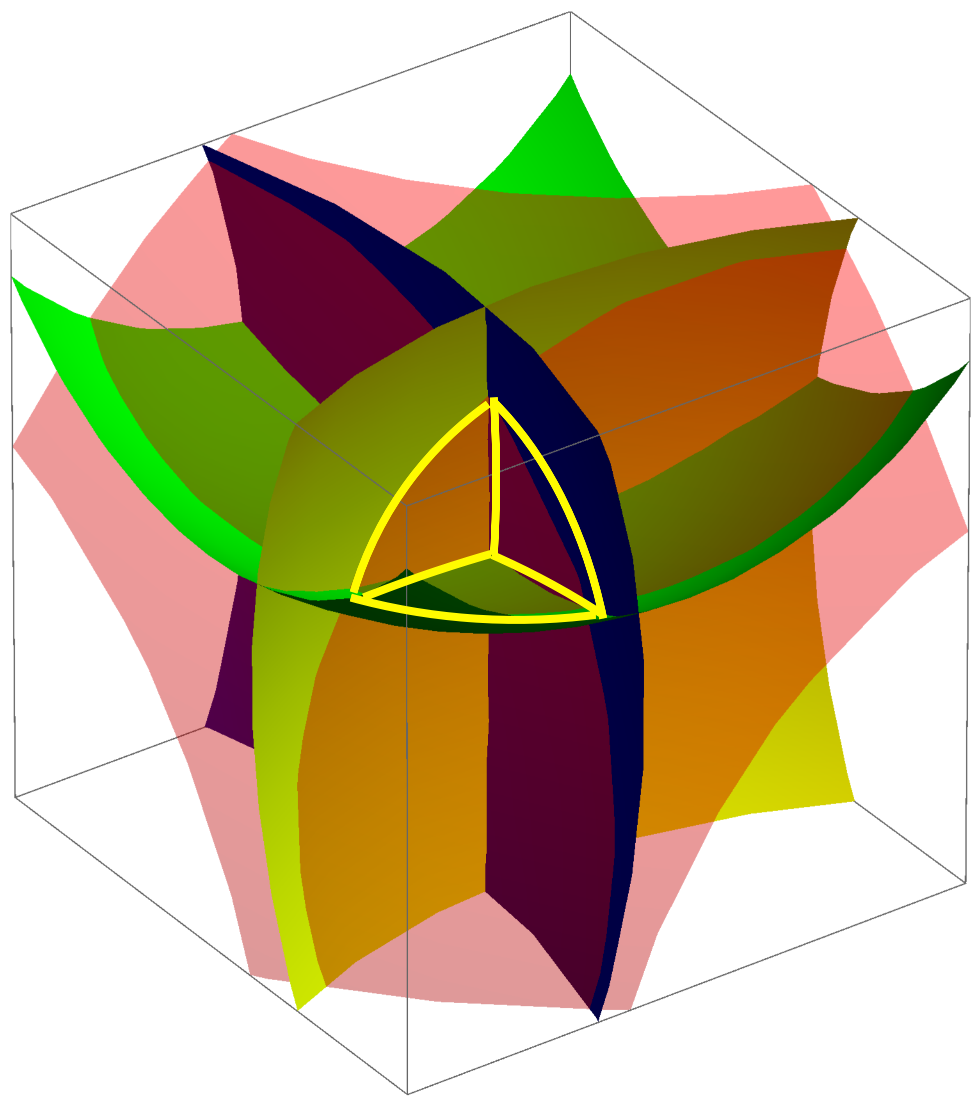


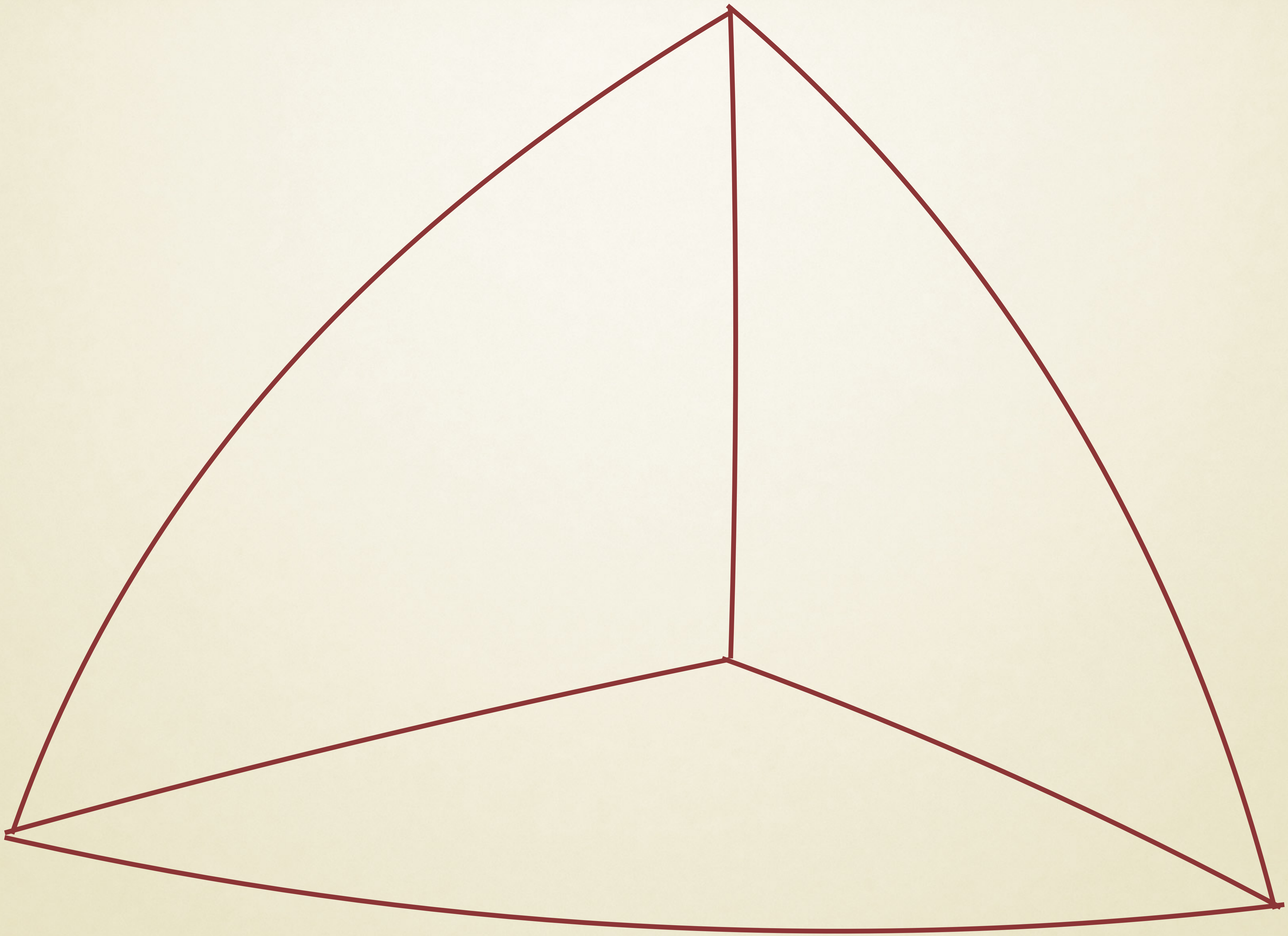












Imagine an empty cavity surrounded by a grounded conductor.

Assume that its walls are formed by segments of spherical surfaces.

Imagine an empty cavity surrounded by a grounded conductor.

Assume that its walls are formed by segments of spherical surfaces.

The field induced by a point
charge placed inside the cavity can be constructed using a method of images if the following three conditions are satisfied:

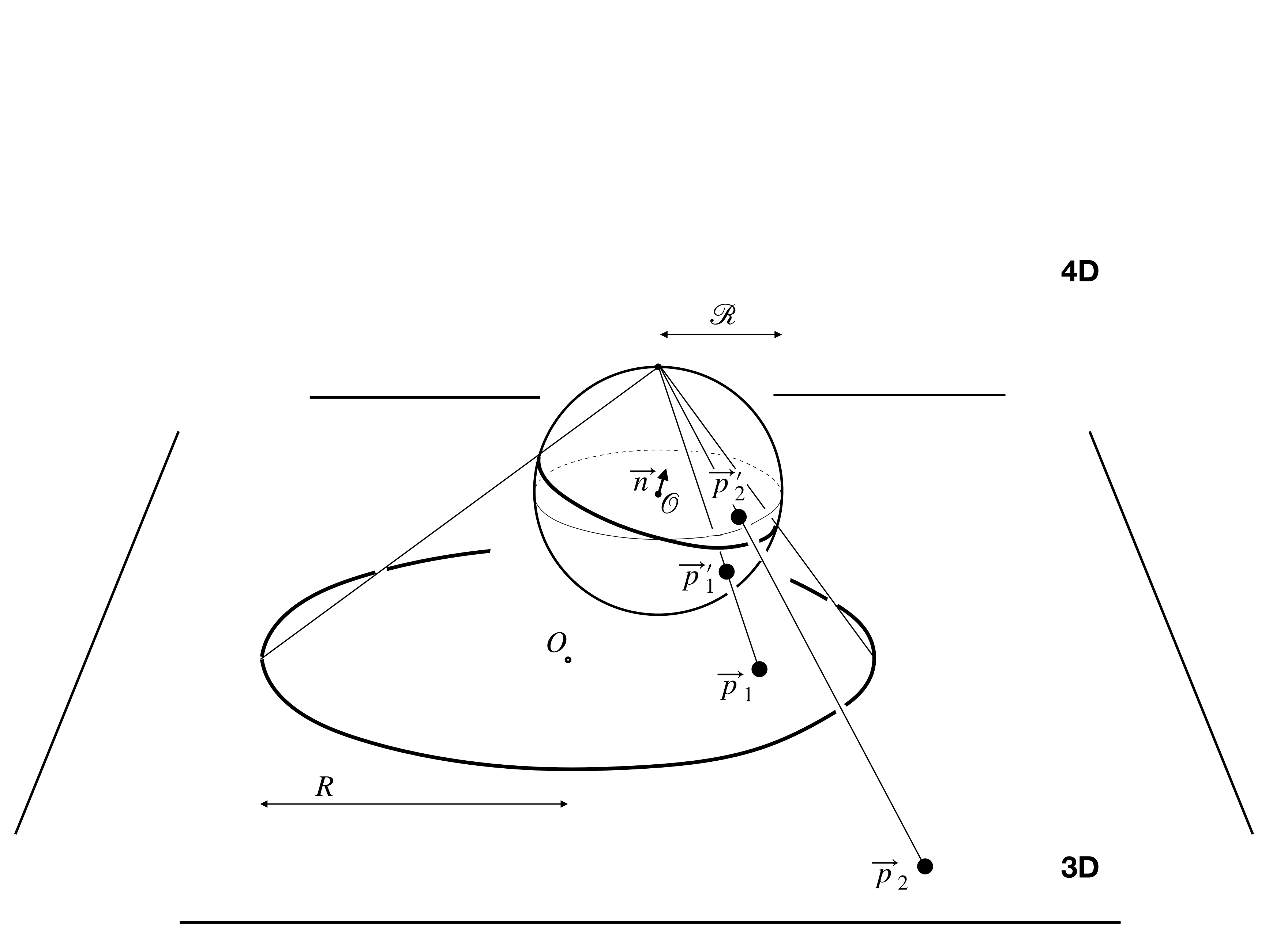
The field induced by a point
charge placed inside the cavity can be
constructed using a method of images if the
following three conditions are satisfied:

I. The set of image charge locations produced via sequential application of the inversions with respect to any of the spheres involved is finite;

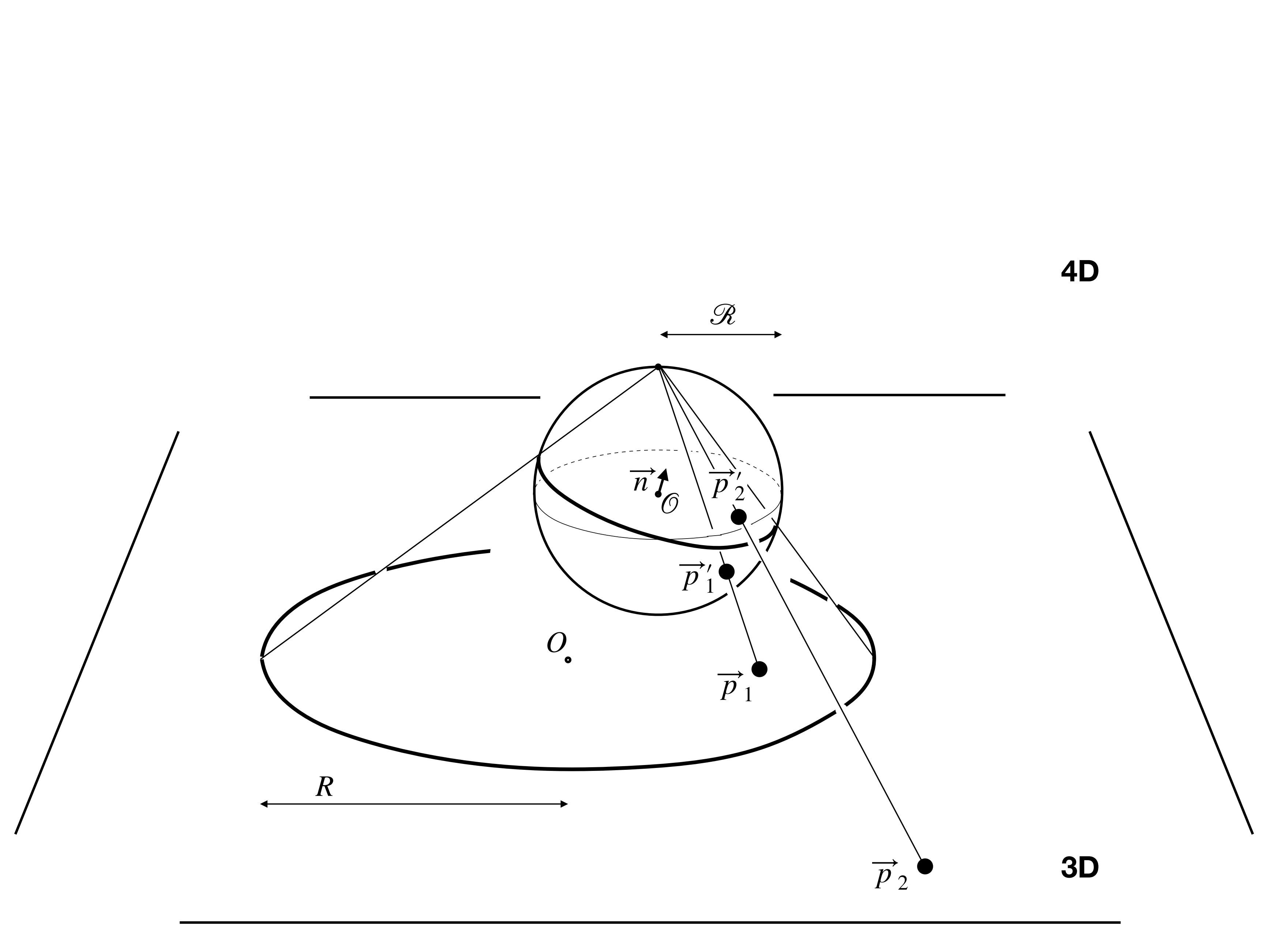
II. The values of the image charges can be unambiguously assigned;

III. No image charges are produced inside the cavity.

Consider a set of spheres each of which being a 4D stereographic projection of a grand hyper-circle on a surface of a 4D hypersphere

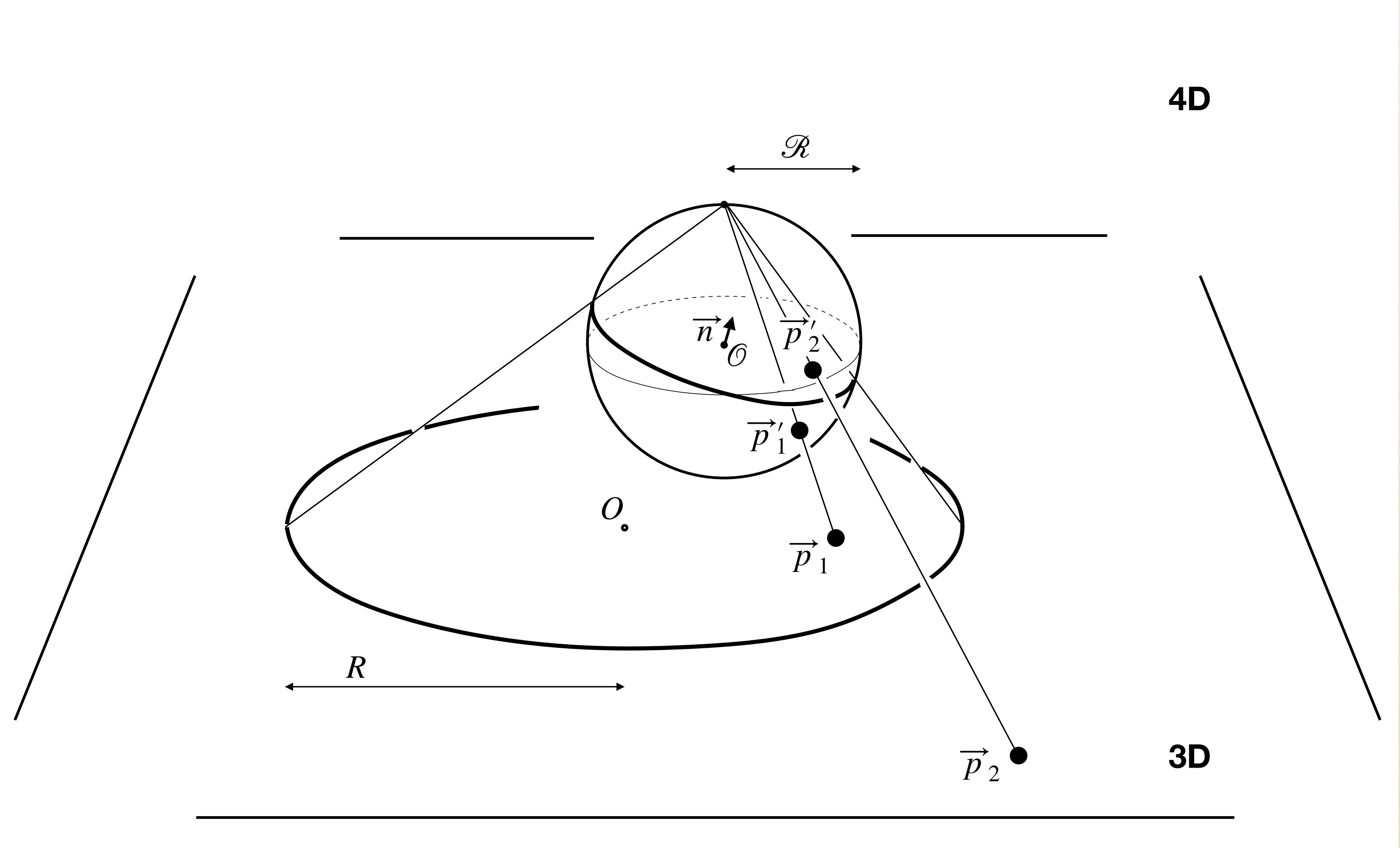


It can be explicitly shown that if two points on the hypersphere are related by a reflection via a 4D mirror, then their stereographic images will be related by a sphere inversion

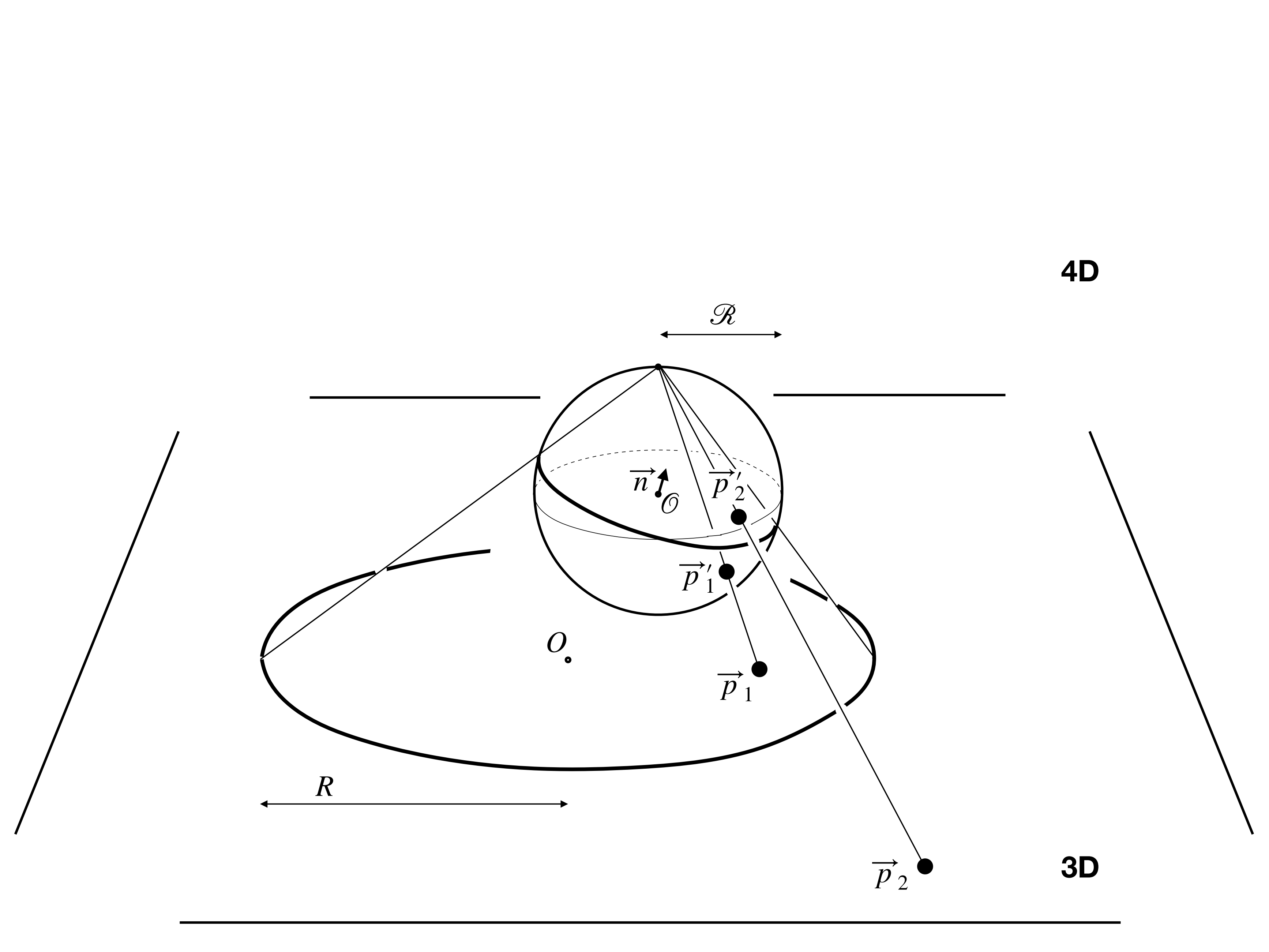


Consider several mirrors; assume they generate a finite reflection group.

Conducting cavity of interest = stereographic image of the cross-section of the principle chamber of the group and the stereographic sphere.



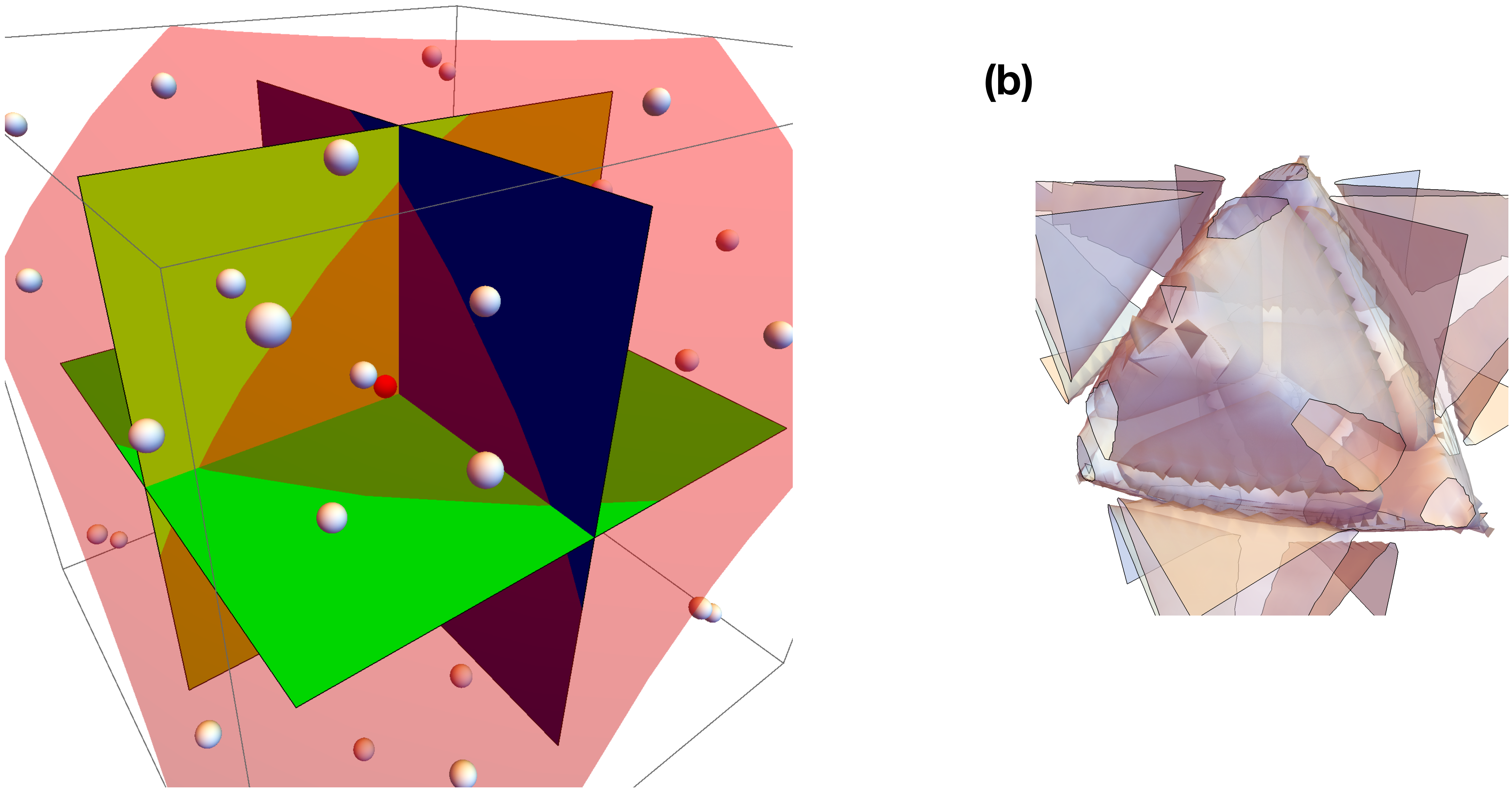
It can be shown (see my talk last Thursday) that in this case the conditions I-II-III are met



A worked example, D_4

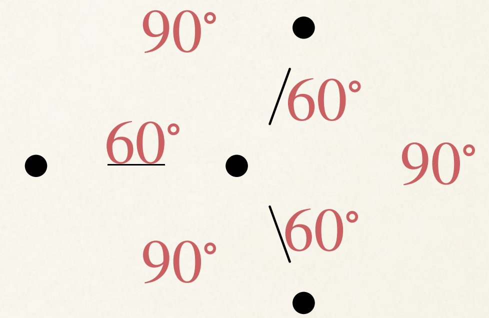
(a)

(b)



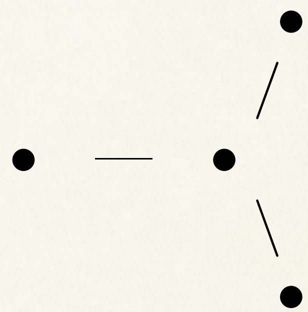
In general

A_4	$\bullet \;—\; \bullet \;—\; \bullet \;—\; \bullet$
$(B_4 = C_4)$	$\bullet \;—\; \bullet \;—\; \bullet \;\overset{4}{—}\; \bullet$
D_4	$\qquad\quad\bullet$ $\qquad\;\;/$ $\bullet \;—\; \bullet$ $\qquad\;\;\backslash$ $\qquad\quad\bullet$
F_4	$\bullet \;—\; \bullet \;\overset{4}{—}\; \bullet \;—\; \bullet$
H_4	$\bullet \;—\; \bullet \;—\; \bullet \;\overset{5}{—}\; \bullet$
$A_3 \times A_1$	$\bullet \;—\; \bullet \;—\; \bullet \qquad \bullet$
$(B_3 = C_3) \times A_1$	$\bullet \;—\; \bullet \;\overset{4}{—}\; \bullet \qquad \bullet$
$H_3 \times A_1$	$\bullet \;—\; \bullet \;\overset{5}{—}\; \bullet \qquad \bullet$
$I_2(m_1) \times I_2(m_2)$	$\bullet \;\overset{m_1}{—}\; \bullet \qquad \bullet \;\overset{m_2}{—}\; \bullet$
$I_2(m) \times A_1 \times A_1$	$\bullet \;\overset{m}{—}\; \bullet \qquad \bullet \qquad \bullet$
$A_1 \times A_1 \times A_1 \times A_1$	$\bullet \qquad \bullet \qquad \bullet \qquad \bullet$
A_3	$\bullet \;—\; \bullet \;—\; \bullet$
$(B_3 = C_3)$	$\bullet \;—\; \bullet \;\overset{4}{—}\; \bullet$
H_3	$\bullet \;—\; \bullet \;\overset{5}{—}\; \bullet$
$I_2(m) \times A_1$	$\bullet \;\overset{m}{—}\; \bullet \qquad \bullet$
$A_1 \times A_1 \times A_1$	$\bullet \qquad \bullet \qquad \bullet$
$I_2(m)$	$\bullet \;\overset{m}{—}\; \bullet$
$A_1 \times A_1$	$\bullet \qquad \bullet$
A_1	$\bullet \qquad .$

A_4	$\bullet - \bullet - \bullet - \bullet$
$(B_4 = C_4)$	$\bullet - \bullet - \bullet \overset{4}{-} \bullet$
D_4	
F_4	$\bullet - \bullet \overset{4}{-} \bullet - \bullet$
H_4	$\bullet - \bullet - \bullet \overset{5}{-} \bullet$
$A_3 \times A_1$	$\bullet - \bullet - \bullet \quad \bullet$
$(B_3 = C_3) \times A_1$	$\bullet - \bullet \overset{4}{-} \bullet \quad \bullet$
$H_3 \times A_1$	$\bullet - \bullet \overset{5}{-} \bullet \quad \bullet$
$I_2(m_1) \times I_2(m_2)$	$\bullet \overset{m_1}{-} \bullet \quad \bullet \overset{m_2}{-} \bullet$
$I_2(m) \times A_1 \times A_1$	$\bullet \overset{m}{-} \bullet \quad \bullet \quad \bullet$
$A_1 \times A_1 \times A_1 \times A_1$	$\bullet \quad \bullet \quad \bullet \quad \bullet$
A_3	$\bullet - \bullet - \bullet$
$(B_3 = C_3)$	$\bullet - \bullet \overset{4}{-} \bullet$
H_3	$\bullet - \bullet \overset{5}{-} \bullet$
$I_2(m) \times A_1$	$\bullet \overset{m}{-} \bullet \quad \bullet$
$A_1 \times A_1 \times A_1$	$\bullet \quad \bullet \quad \bullet$
$I_2(m)$	$\bullet \overset{m}{-} \bullet$
$A_1 \times A_1$	$\bullet \quad \bullet$
A_1	$\bullet$.

A_4	$\bullet \;-\; \bullet \;-\; \bullet \;-\; \bullet$
$(B_4 = C_4)$	$\bullet \;-\; \bullet \;-\; \bullet \;\overset{4}{-}\; \bullet$
D_4	$\bullet \;-\; \bullet \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array}$
F_4	$\bullet \;-\; \bullet \;\overset{4}{-}\; \bullet \;-\; \bullet$
H_4	$\bullet \;-\; \bullet \;-\; \bullet \;\overset{5}{-}\; \bullet$
$A_3 \times A_1$	$\bullet \;-\; \bullet \;-\; \bullet \quad \bullet$
$(B_3 = C_3) \times A_1$	$\bullet \;-\; \bullet \;\overset{4}{-}\; \bullet \quad \bullet$
$H_3 \times A_1$	$\bullet \;-\; \bullet \;\overset{5}{-}\; \bullet \quad \bullet$
$I_2(m_1) \times I_2(m_2)$	$\bullet \;\overset{m_1}{-}\; \bullet \quad \bullet \;\overset{m_2}{-}\; \bullet$
$I_2(m) \times A_1 \times A_1$	$\bullet \;\overset{m}{-}\; \bullet \quad \bullet \quad \bullet$
$A_1 \times A_1 \times A_1 \times A_1$	$\bullet \quad \bullet \quad \bullet \quad \bullet$
A_3	$\bullet \;-\; \bullet \;-\; \bullet$
$(B_3 = C_3)$	$\bullet \;-\; \bullet \;\overset{4}{-}\; \bullet$
H_3	$\bullet \;-\; \bullet \;\overset{5}{-}\; \bullet$
$I_2(m) \times A_1$	$\bullet \;\overset{m}{-}\; \bullet \quad \bullet$
$A_1 \times A_1 \times A_1$	$\bullet \quad \bullet \quad \bullet$
$I_2(m)$	$\bullet \;\overset{m}{-}\; \bullet$
$A_1 \times A_1$	$\bullet \quad \bullet$
A_1	$\bullet \quad .$

19 three-parametric families

A_4	$\bullet - \bullet - \bullet - \bullet$
$(B_4 = C_4)$	$\bullet - \bullet - \bullet \overset{4}{-} \bullet$
D_4	
F_4	$\bullet - \bullet \overset{4}{-} \bullet - \bullet$
H_4	$\bullet - \bullet - \bullet \overset{5}{-} \bullet$
$A_3 \times A_1$	$\bullet - \bullet - \bullet \quad \bullet$
$(B_3 =$	
$H_3 \times$	
$I_2(m$	
$I_2(m) \times A_1 \times A_1$	$\bullet \overset{m}{-} \bullet \quad \bullet \quad \bullet$
$A_1 \times A_1 \times A_1 \times A_1$	$\bullet \quad \bullet \quad \bullet \quad \bullet$
A_3	$\bullet - \bullet - \bullet$
$(B_3 = C_3)$	$\bullet - \bullet \overset{4}{-} \bullet$
H_3	$\bullet - \bullet \overset{5}{-} \bullet$
$I_2(m) \times A_1$	$\bullet \overset{m}{-} \bullet \quad \bullet$
$A_1 \times A_1 \times A_1$	$\bullet \quad \bullet \quad \bullet$
$I_2(m)$	$\bullet \overset{m}{-} \bullet$
$A_1 \times A_1$	$\bullet \quad \bullet$
A_1	$\bullet$.