Null-controllability of parabolic-hyperbolic systems

Armand Koenig

UNIVERSITÉ CÔTE D'AZUR

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VIII Partial differential equations, optimal design and nmerics

Parabolic-transport systems

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Equation we are interested in:

$$\partial_t y(t,x) + A\partial_x y(t,x) - B\partial_{xx} y(t,x) = f(t,x)1_\omega, \quad (t,x) \in [0,+\infty) \times \mathbb{T}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},\ D + D^* \text{ definite-positive}; \quad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ A' = A'^*.$$

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Question

Are these systems null-controllable (equivalently: observable) in $\omega \subset \mathbb{T}$?

$$\text{if } u = 0,\ |y(T,\cdot)|_{L^2(\mathbb{T})} \leq C|y|_{L^2([0,T]\times\omega)} \text{ ?}$$

The Theorem

Theorem (Beauchard-K-Le Balc'h 2019)
ω an open interval of $\mathbb{T}$.

$$T^* = \frac{2\pi - \mathsf{length}(\omega)}{\min_{\mu \in \mathsf{Sp}(A')} |\mu|}$$

Then

1. *the system is not null-controllable on ω in time $T < T^*$,*
2. *the system is null-controllable on ω in any time $T > T^*$.*

Parabolic frequencies, Hyperbolic frequencies

Fourier components, well-posedness

Fourier components
If $y(t,x) = \sum y_n(t)e^{inx}$

$$\partial_t y_n(t) + n^2 \left(B + \frac{i}{n} A \right) y_n(t) = 0$$

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Perturbation theory

λ_{nk} eigenvalues of $B + \frac{i}{n} A$. Perturbation of B: $\lambda_{nk} \to \lambda_k \in \mathrm{Sp}(B)$

- If $\lambda_k \neq 0$, $\lambda_{nk} \underset{n \to +\infty}{\sim} \lambda_k$: parabolic frequencies
- If $\lambda_k = 0$, $\lambda_{nk} \underset{n \to +\infty}{\sim} i\mu_k / n$: hyperbolic frequencies

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- If $\lambda_k = 0$, $\lambda_{nk} \underset{n \to +\infty}{\sim} i\mu_k / n$: hyperbolic frequencies
- Well posed if all $\Re(\lambda_k) > 0$ and $\mu_k \in \mathbb{R}$

$$\begin{aligned} y(t,x) &= \sum_{n,k} a_{nk} e^{inx - n^2 \lambda_{nk} t} y_{nk} \\ &\simeq \underbrace{\sum a_{nk} e^{inx - n^2 \lambda_k t} y_k}_{\text{parabolic frequencies}} + \underbrace{\sum a_{nk} e^{in(x - \mu_k t)} y_k}_{\text{hyperbolic frequencies}} \end{aligned}$$

Lack of null-controllability in small time

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Transport-like solutions
If $\lambda_{nk} \sim i\mu_k/n$, and y_{nk} is an associated eigenvector

$$y(t,x) = \sum_n a_n e^{inx - n^2 \lambda_{nk} t} y_{nk} \simeq \sum_n a_n e^{in(x - \mu_k t)} y_k$$

Not observable in time $T < \dfrac{2\pi - \mathrm{length}(\omega)}{|\mu_k|}$.

Minimal time = minimal time for transport equation
In the case

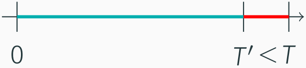
$$\partial_t y_h + A' \partial_x y_h = f_h 1_\omega$$

Solutions = sum of solutions moving at speed $\mu_k \in \mathrm{Sp}(A')$.

Null-controllability in large time

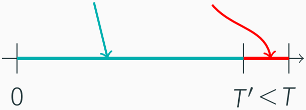
The control strategy

Decoupling the system and controlling

- 

The control strategy

Decoupling the system and controlling

- Given f_h, find f_p that steers parabolic frequencies to 0 in time T
- 

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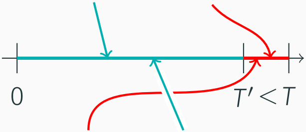
- 

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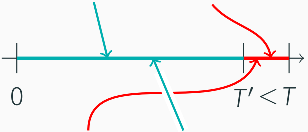
- 

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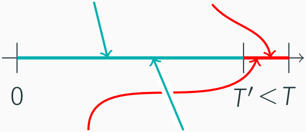
- 

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- First step: parabolic null-controllability problem in time $T - T' > 0$
- Second step: hyperbolic exact controllability problem in time T'. Ok if $T' > T^*$

What is new in our work

Dealing with a system of arbitrary size

- Previous strategy: Lebeau-Zuazua (1998) for linear thermolasticity systems (coupled wave-heat system)
- What we did: generalize to systems of arbitrary size

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Dealing with a system of arbitrary size

- Previous strategy: Lebeau-Zuazua (1998) for linear thermolasticity systems (coupled wave-heat system)
- What we did: generalize to systems of arbitrary size
- Issue: eigenvalues and eigenvectors not nice as $n \to +\infty$
- Solution: we don't need either of these
- We only need *total eigenprojections*: sums of eigenprojections on eigenvalues close to each other (Kato's perturbation theory...)

$$-\frac{1}{2i\pi}\oint_{\Gamma}(M-z)^{-1}\,\mathrm{d}z = \text{Projection on eigenspaces associated with eigenvalues of } M \text{ inside } \Gamma$$

What we (don't) know

Open problems

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- Unique continuation in small time ?
- Less controls than equations ? (partial results)
- Higher dimensions ?
- Non-constant coefficients ?

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That's all folks!