Open problems on a geometric condition for controllability of conservation laws

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This talk presents open problems on a geometric condition for controllability of conservation laws.

Initial Data Identification in Conservation Laws and Hamilton-Jacobi Equations

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Abstract

In the scalar 1D case, conservation laws and Hamilton–Jacobi equations are deeply related. For both, we characterize those profiles that can be attained as solutions at a given positive time corresponding to at least one initial datum. Then, for each of the two equations, we precisely identify all those initial data yielding a solution that coincide with a given profile at that positive time. Various topological and geometrical properties of the set of these initial data are then proved.

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1 Introduction

Under suitable conditions on the flow $f\colon \mathbb{R} \to \mathbb{R}$ and on the initial datum, solutions to a scalar conservation law in 1 space dimension, namely to

$$\partial_t u + \partial_x f(u) = 0\,, \tag{1.1}$$

are known to be obtained through $u = \partial_x U$ from solutions to the Hamilton–Jacobi equation

$$\partial_t U + f(\partial_x U) = 0\,. \tag{1.2}$$

A peculiar feature of these equations is their irreversibility. In particular, in the case of (1.1), inexorable shock formations cause an unavoidable loss of information, so that different initial data may well evolve into the same profile. Usual identification techniques, often based on linearizations or fixed point arguments, have no chances to be effective when dealing with (1.1) or (1.2).

Below, we provide a full characterization of the set of the initial data for (1.1), respectively (1.2), that evolve into a given profile. Geometric and topological properties of this set are also obtained. To this aim, a refinement of the results in [22], see also [12, 23], on the relation between (1.1) and (1.2) had to be obtained.

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For any suitable initial datum u_o, we denote by $(t,x) \to S_t^{CL} u_o(x)$ the weak entropy solution to (1.1). Symmetrically, we denote by $(t,x) \to S_t^{HJ} U_o(x)$ the viscosity solution to (1.2). Below, we consider the case of a uniformly convex $\mathbf{C}^2$ flux f and we obtain complete characterizations of both sets

$$\begin{array}{rcll} I_T^{CL}(w) & := & \left\{u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})\colon S_T^{CL} u_o = w\right\} & \text{and} \\ I_T^{HJ}(W) & := & \left\{U_o \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})\colon S_T^{HJ} U_o = W\right\} & \end{array} \tag{1.3}$$

and we use the notation I_T whenever we refer to both sets in (1.3).

First, we identify those profiles such that the corresponding set I_T is non empty. This proof is constructive, in the sense that an initial datum in I_T is explicitly constructed, see Theorem 3.1. Here, we consider in detail the case of the conservation law (1.1). A key role is played by the decay of rarefaction waves, a phenomenon typically described through Oleinik decay estimates that goes back to [28], was recently improved in [18], extended to systems of conservation laws in [9] and of balance laws in [11], see also the reference texts [8, Chapter 6, Ex.5] and [15, Theorem 11.2.1]. For related problems dealing with the reachable set of (1.1), also in the case of the initial – boundary value problem, we refer to [6, 21] and [1].

Once I_T is ensured to be non empty, in its characterization as well as in establishing its properties a key role is played by two sets, say X_i and X_{ii}, whose precise definitions are in (2.4). For x varying in the former one, X_i, the value attained at x by any initial datum in I_T is essentially uniquely determined. On the contrary, for x varying in the latter one, X_{ii}, the value attained at x by any initial datum in I_T is subject to rather loose constraints. Moreover, coherently with the finite propagation speed typical of (1.1) and (1.2), the values attained at x by any initial datum in I_T on each of the different connected components of X_i and X_{ii} are entirely independent from each other.

Instrumental in these proofs is the ability to go back and forth between solutions to (1.1) and solutions to (1.2). To this aim, we needed to complete the results in [22] that deal with the connection from (1.2) to (1.1). Indeed, Proposition 2.5 details how to pass from solutions to (1.1) to solutions to (1.2).

On the basis of the obtained characterizations, several properties of $I_T^{CL}(w)$ are then proved. First, we re-obtain its convexity, which was already stated in [19]. Then, the unique extreme point of $I_T^{CL}(w)$ is fully characterized and we prove that, remarkably, this set is a cone admitting no finite dimensional extremal faces.

The characterization below directly shows that as soon as I_T^{CL} is non empty, then also $I_T^{CL} \cap \mathbf{BV}(\mathbb{R};\mathbb{R})$ is non empty, meaning that any profile reached by an initial datum with unbounded total variation can also be reached by a (different) initial datum in $\mathbf{BV}$. Moreover, we prove that I_t^{CL} always contains one sided Lipschitz continuous functions but more regular initial data may also be available. The initial datum constructed *"prolonging backwards all shocks"* yields a solution whose interaction potential [8, Formula (10.10)] is *constant* on $]0,T[$.

Further motivations for the present study are provided by parameter identification or inverse problems based on (1.1) or (1.2). In particular, we defer to the related paper [19] that motivates the present problem through applications to the study of sonic booms in (1.1), also providing several illustrative examples and visualizations. In the case of (1.2), U is typically the value function associated to a time reversed control problem, f being related to the dynamics and to the running cost, with U_o playing the role of the terminal cost. Here, the present result amounts to characterizing the terminal cost corresponding to given initial cost, see [16, Section 10.3] for further connections to optimal control problems. The present analytic results can also help in numerical investigations such as those in [2, 10, 26, 27].

Sections 2 to 5 collect the analytic results, while all proofs are deferred to sections 6 to 9

2 Notation and Preliminary Results

Throughout, T is fixed and strictly positive. Below, we mostly refer to [5, § 3.2] for results about **BV** functions. In Section 6 we briefly recall the definition and the min properties of $\mathbf{SBV}(\mathbb{R};\mathbb{R})$, refer to [4] or [15, § 1.7] for more details. As usual, we also use functions u in $\mathbf{BV_{loc}}(\mathbb{R};\mathbb{R})$, respectively in $\mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$, meaning that the restriction $u_{|I}$ of u to any bounded real interval I is in $\mathbf{BV}(I;\mathbb{R})$, respectively in $\mathbf{SBV}(I;\mathbb{R})$. If $u \in \mathbf{BV_{loc}}(\mathbb{R};\mathbb{R})$, then we set $u(x\pm) = \lim_{\xi\to x\pm} u(\xi)$ and we convene that we choose as u the left continuous representative of its class, so that $u(x) = u(x-)$.

We assume the following condition on the function defining (1.1) or (1.2), where c is a suitable positive constant:

$$\textbf{(F):} \qquad f \in \mathbf{C}^2(\mathbb{R};\mathbb{R}) \text{ is such that } f'' \geq c > 0 \text{ and } f(0) = \min_{\mathbb{R}} f = 0. \tag{2.1}$$

Clearly, the latter part of the above condition is not restrictive, since it can be achieved through *ad hoc* translations of the u or $\partial_x U$ variable and of the flux f.

As general references on the theory of scalar conservation laws we use [8, Chapter 6], [15, Chapter XI] or [30, Vol. 1, Chapter 2]. As specified in Definition 2.1 below, by *solution* to (1.1), we always mean a weak entropy solution in the sense of [8, § 4.4], see also [15, Definition 6.2.1] or [30, Definition 2.3.3].

Definition 2.1. *A map* $u \in \mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}) \cap \mathbf{C}^0([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};\mathbb{R}))$ *is a* weak entropy solution *to* (1.1) *if*

$$\iint_{\mathbb{R}^2} \Big(|u-k|\,\partial_t\varphi + \operatorname{sgn}(u-k)\,\big(f(u)-f(k)\big)\,\partial_x\varphi\Big)\,\mathrm{d}t\,\mathrm{d}x \geq 0 \qquad \begin{array}{l}\forall \varphi \in \mathbf{C}^1_c(\mathbb{R}^+\times\mathbb{R};\mathbb{R}^+)\,,\\ \forall k \in \mathbb{R}\,.\end{array}$$

Consider the Cauchy problem for (1.1) with an $\mathbf{L}^\infty$ initial datum assigned at time $t = 0$. Then, by [31, Theorem 16.1], condition (2.1) ensures that as soon as a weak entropy solution exists, then it has locally bounded total variation in space at any positive time.

Concerning (1.2), we use the standard definition of viscosity solution based on super-solutions and sub-solutions, see [16, Chapter 10], [13, Definition I.1] or [22, Section 1].

Definition 2.2. *A map* $U \in \mathbf{W}^{1,\infty}([0,T]\times\mathbb{R};\mathbb{R}) \cap \mathbf{C}^0([0,T];\mathbf{W}^{1,\infty}_{\mathbf{loc}}(\mathbb{R};\mathbb{R}))$ *is a* viscosity solution *to* (1.2) *if for all* $\varphi \in \mathbf{C}^\infty(]0,T[\times\mathbb{R};\mathbb{R})$ *and all* $(t_o,x_o) \in\,]0,T[\times\mathbb{R}$

- *if* $U-\varphi$ *has a local maximum at* (t_o,x_o)*, then* $\partial_t\varphi(t_o,x_o) + f\big(\partial_x\varphi(t_o,x_o)\big) \leq 0$*;*
- *if* $U-\varphi$ *has a local minimum at* (t_o,x_o)*, then* $\partial_t\varphi(t_o,x_o) + f\big(\partial_x\varphi(t_o,x_o)\big) \geq 0$.

For the existence of a semigroup generated by (1.2) yielding solutions in the sense of Definition 2.2, we refer for instance to the classical result [13, Theorem VI.2].

The space derivation, i.e., the map $U \to u = \partial_x U$, shows the equivalence between solutions to (1.2) in the sense of Definition 2.2 and solutions to (1.1) in the sense of Definition 2.1, see [22, Theorem 1.1] and the references therein.

Throughout this paper, the following function plays a key role.

Notation 2.3. *For a fixed $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, we denote*

$$\begin{array}{rccl} p & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & x - T\, f'\big(w(x)\big)\,. \end{array} \tag{2.2}$$

In the case of (1.1), as soon as $I_T^{CL}(w) \neq \emptyset$, p assigns to each $x \in \mathbb{R}$ the intersection of the minimal backward characteristic for (1.1) through (T,x), see [15, Chapter X], with the axis $t=0$. In the case of (1.2), we clearly set $p(x) := x - T\, f'\left(\partial_x W(x)\right)$.

The choice of the *left* continuous representative of w is here crucial to obtain *minimal* backward characteristics.

Oleinik condition on the decay of positive waves [9, 18, 28], see also [8, Chapter 6, Ex.5] and [15, Theorem 11.2.1], is equivalent to require that p, defined in (2.2), be weakly increasing:

$$\textbf{(O):} \quad \text{for all } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^+ \setminus \{0\} \quad \begin{array}{l} p(x) \leq p(x+y)\,, \text{ equivalently} \\ f'\left(w(x+y)\right) - f'\left(w(x)\right) \leq y/T\,. \end{array} \tag{2.3}$$

On the basis of Oleinik Condition (2.3), we partition $\mathbb{R}$ into two sets X_i and X_{ii} that play a key role in the sequel.

Proposition 2.4. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that (2.3) holds and $p \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$. Introduce the sets*

$$\begin{array}{rcl} X_i & := & p\left(\left\{x \in \mathbb{R} \colon p \text{ is differentiable at } x \text{ and } p'(x) \neq 0\right\}\right)\,, \\ X_{ii} & := & \bigcup_{x\in\mathbb{R}} \left]p(x-), p(x+)\right[\,. \end{array} \tag{2.4}$$

Then, $\mathbb{R} \setminus (X_i \cup X_{ii})$ has Lebesgue measure 0.

We now investigate the equivalence between the Conservation Law (1.1) and the Hamilton–Jacobi equation (1.2). A key result is [22, Theorem 1.1], to which we provide here a completion, in the sense explained through the following diagrams:

$$\begin{array}{ccc} & [22, \text{Theorem } 1.1] & \\ U_o & \longrightarrow & S_t^{HJ} U_o \\ \partial_x \big\downarrow & & \big\downarrow \partial_x \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array} \qquad\qquad \begin{array}{ccc} & \text{Proposition } 2.5 & \\ U_o & \longrightarrow & S_t^{HJ} U_o \\ \int^x \big\uparrow & & \big\uparrow \text{Formula (2.5)} \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array}$$

Proposition 2.5. *Let f satisfy (2.1). Fix $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ and call u the solution in the sense of Definition 2.1 to the Cauchy problem for the Conservation Law (1.1) with datum u_o at time $t=0$. For a path $\gamma \in \mathbf{W}^{1,\infty}([0,T];\mathbb{R})$ and a constant $c \in \mathbb{R}$, define*

$$U(t,x) = \int_{\gamma(t)}^x u(t,\xi)\,\mathrm{d}\xi + \int_0^t \Big(\dot\gamma(\tau)\, u\left(\tau,\gamma(\tau)\right) - f\left(u\left(\tau,\gamma(\tau)\right)\right)\Big)\,\mathrm{d}\tau + c\,. \tag{2.5}$$

Then, U solves Hamilton–Jacobi equation (1.2) with datum $U_o(x) = \int_{\gamma(0)}^x u_o(\xi)\,\mathrm{d}\xi + c$ in the sense of Definition 2.2.

3 Construction of a Remarkable Element of $I_t^{CL}(w)$

We now prove that Oleinik Condition (2.3) characterizes those profiles w such that $I_T^{CL}(w) \neq \emptyset$. Indeed, if a profile w satisfies Oleinik condition (2.3), then the conservation law (1.1) can be integrated backwards in time, taking w as final datum at time T and yielding a **BV** initial profile at time 0. Technically, we reverse the space variable, rather than reversing time, and we explicitly construct an element of $I_T^{CL}(w)$ that will play a key role in the sequel.

Theorem 3.1. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that (2.3) holds. Then, there exists a unique function $u_o^* \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ characterized by each one of the following two equivalent conditions:*

(i)* $u_o^*(x) = \tilde u(T,-x)$, *where* $\tilde u$ *is a solution to* $\left\{\begin{array}{l}\partial_t \tilde u + \partial_x f(\tilde u) = 0\,,\\ \tilde u(0,x) = w(-x)\,.\end{array}\right.$

(ii)* u_o^* *is such that*

(ii.i) for all $x \in \mathbb{R}$ where p is differentiable and $p'(x) \neq 0$,*

$$\lim_{y\to x} \frac{1}{p(y)-p(x)} \int_{p(x)}^{p(y)} u_o^*(\xi)\,\mathrm{d}\xi = w(x)\,;$$

(ii.ii) for all $x \in \mathbb{R}$ where $w(x-) \neq w(x+)$, for all $v \in [w(x+), w(x-)]$,*

$$\begin{aligned}\frac{1}{T}\int_{x-Tf'(v)}^{x-Tf'(w(x+))} u_o^*(\xi)\,\mathrm{d}\xi &= \big(v\,f'(v) - f(v)\big) - \Big(w(x+)\,f'\left(w(x+)\right) - f\left(w(x+)\right)\Big)\,,\\ \frac{1}{T}\int_{x-Tf'(w(x-))}^{x-Tf'(v)} u_o^*(\xi)\,\mathrm{d}\xi &= \Big(w(x-)\,f'\left(w(x-)\right) - f\left(w(x-)\right)\Big) - \big(v\,f'(v) - f(v)\big)\,.\end{aligned}$$

Moreover, u_o^* enjoys the following properties:

(1)* $u_o^* \in I_T^{CL}(w)$;

(2)* u_o^* *is one sided Lipschitz and* $u_o^* \in \mathbf{BV}(\mathbb{R};\mathbb{R})$;

(3) for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}^+$,* $f'\left(u_o^*(x-y)\right) - f'\left(u_o^*(x)\right) \leq \dfrac{y}{T}$.

We underline that condition *(ii*)* naturally determines two subsets of $\mathbb{R}$, related to X_i and X_{ii}. Indeed, in *(ii*)* we explicitly specify the exact sets of those points x where the two conditions *(ii*.i)* and *(ii*.ii)* have to be satisfied by w at time $t = T$. Essentially, the results above show that if $u_o \in I_T^{CL}(w)$, then the restriction $u_{o|X_i}$ yields the continuous part of w, while $u_{o|X_{ii}}$ yields the shocks.

As a first consequence of Theorem 3.1 we obtain the following characterization of those profiles w such that $I_T^{CL}(w)$ is non empty.

Corollary 3.2. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. With the notations (1.3) and (2.2), the following statements are equivalent:*

(a) $I_T^{CL}(w) \neq \emptyset$;

(b) *a suitable representative of w satisfies Oleinik Condition (2.3).*

Note that condition (b), and hence the requirement $I_T^{CL}(w) \neq \emptyset$, also ensures that $w \in \mathbf{BV_{loc}}(\mathbb{R};\mathbb{R})$. Hence, w admits a left continuous representative satisfying (b) for all $x \in \mathbb{R}$.

The translation of the results above to the case of Hamilton–Jacobi equation (1.2) essentially relies on Proposition 2.5 and is here omitted. We only recall that the conditions *(ii*.i)* and *(ii*.ii)* above are restated in terms of a primitive U_o^* of u_o^* in $(I^\flat)$ and $(II^\flat)$ in Lemma 7.3.

4 Characterizations of $I_T^{CL}(w)$ and $I_T^{HJ}(W)$

In view of [7, Theorem 1.1], for any initial datum u_o and for all but countably many times T, the map $w = S_T u_o$ leads to a function p in $\mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$, see also [4, Theorem 1.2] and [15, Theorem 11.3.5]. Therefore, we restrict our analysis below to functions w such that $p \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$.

We proceed with our main result, in the version referring to the Conservation Law (1.1)

Theorem 4.1. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$ and $p \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$. Then, a map $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ is in $I_T^{CL}(w)$ if and only if the following two conditions hold:*

(i) for all $x \in \mathbb{R}$ such that p is differentiable at x and $p'(x) \neq 0$,

$$\lim_{y \to x} \frac{1}{p(y) - p(x)} \int_{p(x)}^{p(y)} u_o(\xi)\, \mathrm{d}\xi = w(x)\,; \tag{4.1}$$

(ii) for all $x \in \mathbb{R}$ such that $w(x-) \neq w(x+)$, for all $v \in [w(x+), w(x-)]$,

$$\begin{aligned}
\frac{1}{T}\int_{x-Tf'(v)}^{x-Tf'(w(x+))} u_o(\xi)\,\mathrm{d}\xi \;&\leq\; \left(v\, f'(v) - f(v)\right) - \Big(w(x+)\, f'\left(w(x+)\right) - f\left(w(x+)\right)\Big)\,,\\
\frac{1}{T}\int_{x-Tf'(w(x-))}^{x-Tf'(v)} u_o(\xi)\,\mathrm{d}\xi \;&\geq\; \Big(w(x-)\, f'\left(w(x-)\right) - f\left(w(x-)\right)\Big) - \left(v\, f'(v) - f(v)\right)\,.
\end{aligned}$$

Note that *(i)* holds, in particular, whenever $p(x)$ is a Lebesgue point of u_o. Moreover, at *(i)*, we mean both that the limit in the left hand side of (4.1) exists and that its value is $w(x)$. With reference to Proposition 2.4, X_i in (2.4) is the set where the values of u_o are constrained by *(i)* and, similarly, X_{ii} is the set where the values of u_o are constrained by *(ii)*.

As a side remark note that, as is to be expected, if the flow f is varied by any additive constant, both conditions *(i)* and *(ii)* remain unchanged.

Towards a restatement of Theorem 4.1 in the case of Hamilton–Jacobi equation we provide the following Theorem, whose proof is instrumental in the characterization of $I_T^{CL}(w)$. Therein, we use the Legendre transform f^* of f, see Proposition 7.1 for the precise definition.

Theorem 4.2. *Let (2.1) hold, T be positive an p be as in (2.2). Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$ and $w \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$. Then, a map $U \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})$ is such that $\partial_x U_o \in I_T^{CL}(w)$ if and only if the following two conditions hold:*

(I) for all $x \in \mathbb{R}$ such that p is differentiable at x and $p'(x) \neq 0$,

$$\lim_{y \to x} \frac{U_o\left(p(y)\right) - U_o\left(p(x)\right)}{p(y) - p(x)} = \partial_x W(x)\,; \tag{4.2}$$

(II) *for all* $x \in \mathbb{R}$ *such that* $\partial_x W(x-) \neq \partial_x W(x+)$, *for all* $y \in \left]p(x-), p(x+)\right[$,

$$\frac{U_o\left(p(x+)\right) - U_o(y)}{T} \leq f^*\left(\frac{x-y}{T}\right) - f^*\left(\frac{x-p(x+)}{T}\right),$$
$$\frac{U_o(y) - U_o\left(p(x-)\right)}{T} \geq f^*\left(\frac{x-p(x-)}{T}\right) - f^*\left(\frac{x-y}{T}\right).$$

Here we remark that the conditions in Theorem 4.1 and those in Theorem 4.2 are equivalent.

Lemma 4.3. *Under the assumptions and notations of Theorem 4.1 and Theorem 4.2, if* $U_o \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})$ *and* $u_o = \partial_x U_o$, *then*

$$\begin{array}{lcl} u_o \text{ satisfies (i) in Theorem 4.1} & \iff & U_o \text{ satisfies (I) in Theorem 4.2;} \\ u_o \text{ satisfies (ii) in Theorem 4.1} & \iff & U_o \text{ satisfies (II) in Theorem 4.2.} \end{array}$$

However, the former CL–formulation leads to an easier proof of the necessity condition, while the latter integral formulation leads to a simpler verification of the sufficiency part. Therefore, in the proofs of theorems 4.1 and 4.2 we follow this scheme:

$$\begin{array}{ccccc} & u_o \in I_T^{CL}(w) & \iff & \partial_x U_o \in I_T^{CL}(w) & \\ \text{Theorem 4.1} & \Downarrow & & \Uparrow & \text{Theorem 4.2} \\ & u_o \text{ satisfies (i)} & \iff & U_o \text{ satisfies (I)} & \\ & u_o \text{ satisfies (ii)} & \iff & U_o \text{ satisfies (II)} & \\ & & \text{Lemma 4.3} & & \end{array}$$

On the basis of Theorem 4.2, we now deal with the Hamilton–Jacobi equation (1.2) and state the characterization of $I_T^{HJ}(W)$.

Theorem 4.4. *Let* (2.1) *hold and* T *be positive. Fix* $W \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})$ *such that* $I_T^{HJ}(W) \neq \emptyset$ *and* $\partial_x W \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$. *Let* p *be as in* (2.2), *with* $w = \partial_x W$. *Then,* $U_o \in I_T^{HJ}(W)$ *if and only if the following two conditions hold:*

(I^{HJ}) *for all* $x \in \mathbb{R}$ *such that* p *is differentiable at* x *and* $p'(x) \neq 0$,

$$\lim_{y \to x} \frac{U_o\left(p(y)\right) - U_o\left(p(x)\right)}{p(y) - p(x)} = \partial_x W(x);$$

(II^{HJ}) *for all* $x \in \mathbb{R}$ *such that* $\partial_x W(x-) \neq \partial_x W(x+)$,

$$\begin{array}{lrcl} \forall\, y \in \left]p(x-), p(x+)\right[& U_o(y) + T\, f^*\left(\frac{x-y}{T}\right) & \geq & W(x), \\ \forall\, y \in \left\{p(x-), p(x+)\right\} & U_o(y) + T\, f^*\left(\frac{x-y}{T}\right) & = & W(x). \end{array}$$

5 Geometric Properties of $I_T^{CL}(w)$

On the basis of the characterization provided by Theorem 4.1 and Theorem 4.2, we obtain the following information on topological and geometrical properties of the set $I_T^{CL}(w)$.

Proposition 5.1. *Let* (2.1) *hold and* T *be positive. Fix* $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ *such that* $I_T^{CL}(w) \neq \emptyset$ *and* $w \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$. *Then, with respect to the* $\mathbf{L}^1_{\mathbf{loc}}$ *topology, the set* $I_T^{CL}(w)$ *is:*

(T1) closed;

(T2) with empty interior.

Proposition 5.2. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$ and $w \in \mathbf{SBV}_{\mathbf{loc}}(\mathbb{R};\mathbb{R})$. Then,*

(G1) the set $I_T^{CL}(w)$ reduces to a singleton if and only if $w \in \mathbf{C}^0(\mathbb{R};\mathbb{R})$;

(G2) the set $I_T^{CL}(w)$ is a convex cone having as unique extremal point at its vertex the map u_o^* defined in Theorem 3.1.

(G3) if $u_o \in I_T^{CL}(w)$ and $u_o \neq u_o^*$, then for any $n \in \mathbb{N} \setminus \{0\}$ there exist $v_0, v_1, \ldots, v_N \in I_T^{CL}(w)$ such that

$$u_o = \frac{1}{N+1} \sum_{i=0}^{N} v_i \tag{5.1}$$

and $v_1 - v_0, v_2 - v_0, \ldots, v_N - v_0$ are linearly independent.

Above, by *singleton* we mean up to equality a.e. or, equivalently, that the *precise representative* is unique. By precise representative of $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ we mean that

$$u_o(x) = \begin{cases} \lim_{r\to 0} \frac{1}{r} \int_x^{x+r} u_o(\xi)\, \mathrm{d}\xi & \text{whenever this limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

6 Proofs Related to § 2

The Lebesgue measure in $\mathbb{R}$ is denoted by $\mathcal{L}$. Given $u \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R};\mathbb{R})$, we define the set $\mathfrak{Leb}\,(u)$ of its Lebesgue points as the set of those $x \in \mathbb{R}$ such that $\lim_{r\to 0} \frac{1}{r} \int_x^{x+r} \left|u(\xi) - u(x)\right| \mathrm{d}\xi = 0$. By [17, Corollary 2, Chapter 1, § 7]), $\mathcal{L}\left(\mathbb{R} \setminus \mathfrak{Leb}\,(u)\right) = 0$.

Below, we often use the decomposition $u = u_{ac} + u_j + u_c$ of a $\mathbf{BV}$ function u into its absolutely continuous part u_{ac}, its jump part u_j, which is a possibly infinite sum of Heaviside functions, and its Cantor part u_c. Whenever $u_c = 0$, we say that $u \in \mathbf{SBV}(\mathbb{R};\mathbb{R})$. Recall that if $u \in \mathbf{BV}(\mathbb{R};\mathbb{R})$, then its weak derivative Du is a Radon measure [17, § 1.1] that admits the decomposition $Du = (Du)_{ac} + (Du)_j + (Du)_c$, $(Du)_{ac}$ being absolutely continuous with respect to $\mathcal{L}$, $(Du)_j$ is a, possibly infinite, sum of Dirac deltas and $(Du)_c$ is the Cantor part of Du. As is well known [5, Corollary 3.33], up to sets of Lebesgue measure 0,

$$(Du)_{ac} = D(u_{ac})\,, \quad (Du)_j = D(u_j) \quad \text{and} \quad (Du)_c = D(u_c)\,.$$

We also denote by u' the density of $(Du)_{ac}$ with respect to the Lebesgue measure, so that $(Du)_{ac} = u'\,\mathcal{L}$ and $u' = (u_{ac})'$. By [5, Theorem 3.28] for a.e. $x \in \mathbb{R}$, $u'(x)$ coincides with the limit of the incremental ratio of u or u_{ac} at x.

For later use, we need the following variation of the Area Formula, see e.g. [5, § 2.10].

Lemma 6.1. *Let $\varphi \in \mathbf{SBV}(\mathbb{R};\mathbb{R})$ be weakly increasing. Then, for any measurable set E,*

$$\int_E \varphi'_{ac}(\xi)\, \mathrm{d}\xi = \int_{\mathbb{R}} \mathrm{card}\left(E \cap \varphi^{-1}(\xi)\right) \mathrm{d}\xi$$

with φ'_{ac} being the absolutely continuous part of φ'.

Proof. Throughout this proof, A^c is the complement of the set A in $\mathbb{R}$. Denote by H_ξ the Heaviside function centered at ξ, i.e., $H_\xi(x) = 1$ for $x \geq \xi$ and $H_\xi(x) = 0$ for $x < \xi$.

Define $\varphi_s = \sum_n \alpha_n H_{\xi_n}$, with $\alpha_n = \varphi(\xi_n+) - \varphi(\xi_n-)$ and $\{\xi_n : n \in \mathbb{N}\}$ being the set of points of jump in φ. Since $\varphi \in \mathbf{SBV}(\mathbb{R};\mathbb{R})$, by [29, Chapter 2, Section 25], the function $\varphi_{ac} = \varphi - \varphi_s$ is in $\mathbf{AC}(\mathbb{R};\mathbb{R})$.

For any measurable set E, define $\mu(E) = \int_E \varphi'_{ac}(\xi)\,\mathrm{d}\xi$. By construction, μ is a measure and for any $a, b \in \mathbb{R}$ with $a \leq b$, we have that $\mu([a,b]) = \varphi_{ac}(b) - \varphi_{ac}(a)$. Hence, choosing a and b among the continuity points of φ, we have that

$$\mu([a,b]) = \varphi(b) - \varphi(a) - \sum_{n\colon \xi_n \in]a,b[} \alpha_n\,. \tag{6.1}$$

Define, for any measurable set E, also $\nu(E) := \int_{\mathbb{R}} \operatorname{card}\left(E \cap \varphi^{-1}(\xi)\right)\mathrm{d}\xi$. The set function ν is a measure, as it follows from the Monotone Convergence Theorem, see e.g. [5, Theorem 1.19], and from the countable additivity of the counting measure.

Denote $A := \bigcup_{n\in\mathbb{N}} \left]\varphi(\xi_n-), \varphi(\xi_n+)\right[$. For any $a, b \in \mathbb{R}$, with $a \leq b$ being continuity points of φ, by the monotonicity of φ note that

$$\operatorname{card}\left([a,b] \cap \varphi^{-1}(\xi)\right) = \begin{cases} 0 & \xi < \varphi(a) \text{ or } \xi > \varphi(b) \text{ or } \xi \in A\,, \\ 1 & \varphi^{-1}(\xi) \text{ is a singleton in } [a,b] \cap A^c\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, if $[a,b] \cap \varphi^{-1}(\xi)$ contains two points, say ξ_1 and ξ_2 with $\xi_1 < \xi_2$, then $[\xi_1, \xi_2] \subseteq \left([a,b] \cap \varphi^{-1}(\xi)\right)$, so that $\operatorname{card}\left([a,b] \cap \varphi^{-1}(\xi)\right) = +\infty$ and there exist only countably many such points. As a consequence,

$$\nu([a,b]) = \int_{\mathbb{R}} \operatorname{card}\left([a,b] \cap \varphi^{-1}(\xi)\right)\mathrm{d}\xi = \int_{\mathbb{R}} \chi_{[\varphi(a),\varphi(b)]\cap A^c}(\xi)\,\mathrm{d}\xi = \mathcal{L}([\varphi(a),\varphi(b)] \cap A^c)$$

so that

$$\begin{aligned} [\varphi(a),\varphi(b)] \cap A^c &= [\varphi(a),\varphi(b)] \cap \left(\bigcup_{n\in\mathbb{N}} \left]\varphi(\xi_n-), \varphi(\xi_n+)\right[\right)^c \\ &= [\varphi(a),\varphi(b)] \setminus \bigcup_{n\colon \xi_n \in]a,b[} \left]\varphi(\xi_n-), \varphi(\xi_n+)\right[\end{aligned}$$

and the latter union in the right hand side above is contained in $[\varphi(a), \varphi(b)]$, due to our choice of a and b. Passing to the Lebesgue measure of the sets on the two sides of the latter equality,

$$\begin{aligned} \nu\left([a,b]\right) &= \mathcal{L}\left([\varphi(a),\varphi(b)] \cap A^c\right) \\ &= \mathcal{L}\left([\varphi(a),\varphi(b)] \setminus \bigcup_{n\colon \xi_n \in]a,b[} \left]\varphi(\xi_n-), \varphi(\xi_n+)\right[\right) \\ &= \mathcal{L}\left([\varphi(a),\varphi(b)]\right) - \sum_{n\colon \xi_n \in]a,b[} \mathcal{L}\left(\left]\varphi(\xi_n-), \varphi(\xi_n+)\right[\right) \\ &= \varphi(b) - \varphi(a) - \sum_{n\colon \xi_n \in]a,b[} \alpha_n \end{aligned} \tag{6.2}$$

By (6.1) and (6.2) we have that $\mu = \nu$ on all intervals $[a,b]$ with a, b continuity points of φ. The choice of φ ensures that these points are dense in $\mathbb{R}$, completing the proof. □

Proof of Proposition 2.4. Introduce the sets

$$P_1 := \{x \in \mathbb{R} \colon p \text{ or } p_{ac} \text{ is not differentiable at } x\} \qquad \text{and} \qquad P_2 := \left\{x \in \mathbb{R} \setminus P_1 \colon p'(x) = 0\right\} .$$

Remark that if $x \in P_2$, then p, p_{ac} and p_j are all differentiable at x and $p'_{ac}(x) = p'_j(x) = 0$, since p, p_{ac} and p_j are all weakly increasing by *(b)* in Theorem 3.2.

By [29, Chapter 1, Section 2], P_1 has Lebesgue measure 0 and $\int_{P_1 \cup P_2} p'_{ac}(x)\,\mathrm{d}x = 0$. Note that Lemma 6.1, which can be applied since $p \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$, ensures that $\int_{P_1\cup P_2} p'_{ac}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \operatorname{card}\left((P_1 \cup P_2) \cap p^{-1}(\xi)\right) \mathrm{d}\xi$, so that $\operatorname{card}\left((P_1 \cup P_2) \cap p^{-1}(\xi)\right) = 0$ for a.e. $\xi \in \mathbb{R}$ and, equivalently, $p(P_1 \cup P_2)$ is negligible, i.e.,

$$\mathcal{L}\left(p(P_1 \cup P_2)\right) = 0\,. \tag{6.3}$$

Observe that

$$p(\mathbb{R}) = X_i \cup p(P_1 \cup P_2)\,. \tag{6.4}$$

By (2.3), the function p is non decreasing, hence X_{ii} is an at most countable union of non empty, disjoint and open intervals. By the properties of backward characteristics, the set $\mathbb{R} \setminus \left(X_{ii} \cup p(\mathbb{R})\right)$ is at most countable, so that

$$\mathcal{L}\left(\mathbb{R} \setminus \left(X_{ii} \cup p(\mathbb{R})\right)\right) = 0\,. \tag{6.5}$$

Using the relations above,

$$\begin{array}{rcll} \mathbb{R} \setminus (X_i \cup X_{ii}) & \subseteq & \mathbb{R} \setminus \left(X_i \cup X_{ii} \cup p(P_1 \cup P_2)\right) \cup p(P_1 \cup P_2) & \text{[by (6.4)]} \\ & \subseteq & \mathbb{R} \setminus \left(X_{ii} \cup p(\mathbb{R})\right) \cup p(P_1 \cup P_2) & \\ \mathcal{L}\left(\mathbb{R} \setminus (X_i \cup X_{ii})\right) & \leq & \mathcal{L}\left(\mathbb{R} \setminus \left(X_{ii} \cup p(\mathbb{R})\right)\right) + \mathcal{L}\left(p(P_1 \cup P_2)\right) & \text{[by (6.3) and (6.5)]} \\ & \leq & 0\,, & \end{array}$$

completing the proof. □

The following classical result is of use in the subsequent proof as well as in what follows,

Proposition 6.2 ([14, Lemma 3.2]). *Let f satisfy (2.1) and let u be a weak entropy solution to (1.1). Let $a, b \in [0, T]$ with $a < b$ and choose two maps $\alpha, \beta \in \mathbf{C}^{0,1}([a, b];\mathbb{R})$ with $\alpha \leq \beta$. Then, for a.e. $t_1, t_2 \in [a, b]$ with $t_1 \leq t_2$,*

$$\begin{aligned} \int_{\alpha(t_2)}^{\beta(t_2)} u(t_2, x)\,\mathrm{d}x - \int_{\alpha(t_1)}^{\beta(t_1)} u(t_1, x)\,\mathrm{d}x \; = \; & \int_{t_1}^{t_2} \Big(f\left(u(t, \alpha(t)-)\right) - \dot\alpha(t)\, u(t, \alpha(t)-)\Big)\,\mathrm{d}t \\ & - \int_{t_1}^{t_2} \Big(f\left(u(t, \beta(t)+)\right) - \dot\beta(t)\, u(t, \beta(t)+)\Big)\,\mathrm{d}t\,. \end{aligned}$$

Remark that if γ is a Lipschitz curve, then

$$\int_{t_1}^{t_2} \Big(f\left(u(t, \gamma(t)-)\right) - \dot\gamma(t)\, u(t, \gamma(t)-)\Big)\,\mathrm{d}t = \int_{t_1}^{t_2} \Big(f\left(u(t, \gamma(t)+)\right) - \dot\gamma(t)\, u(t, \gamma(t)+)\Big)\,\mathrm{d}t\,,$$

as it follows from Proposition 6.2 in the case $\alpha = \beta = \gamma$.that

Proof of Proposition 2.5. Introduce the map $\tilde U(t,x) = (S_t^{HJ} U_o)(x)$. We prove that $\tilde U$ coincides with the map defined in (2.5). By [22, Theorem 1.1], the map $\partial_x \tilde U$ solves the Conservation Law (1.1), so that

$$\partial_x \tilde U(t,x) = u(t,x) = \partial_x \int_{\gamma(t)}^x u(t,\xi)\, \mathrm{d}\xi \,.$$

Hence, there exist a map $\Upsilon \in \mathbf{W}^{1,\infty}([0,T];\mathbb{R})$ with $\Upsilon(0) = 0$ and a $c \in \mathbb{R}$ such that

$$\tilde U(t,x) = \int_{\gamma(t)}^x u(t,\xi)\, \mathrm{d}\xi + \Upsilon(t) + c$$

and, using Proposition 6.2 with $\alpha(t) = \gamma(t)$, $\beta(t) = x$, $t_2 = t$ and $t_1 = 0$,

$$\begin{aligned}
& \tilde U(t,x) - \tilde U_o(x) \\
= & \int_{\gamma(t)}^x u(t,\xi)\, \mathrm{d}\xi - \int_0^x u(t,\xi)\, \mathrm{d}\xi + \Upsilon(t) \\
= & \int_0^t \Big(f\Big(u\left(\tau,\gamma(\tau)-\right)\Big) - \dot\gamma(\tau)\, u\left(\tau,\gamma(\tau)-\right) \Big)\, \mathrm{d}\tau - \int_0^t f\left(u(\tau,x+)\right) \mathrm{d}\tau + \Upsilon(t)
\end{aligned}$$

By the differentiability properties of $\tilde U$, see [16, Theorem 1, Section 10.1.2], we have

$$\tilde U(t,x) - \tilde U_o(x) = -\int_0^t f\left(\partial_x \tilde U(\tau,x)\right) \mathrm{d}\tau = -\int_0^t f\left(u(\tau,x)\right) \mathrm{d}\tau \,. \tag{6.6}$$

So that

$$\Upsilon(t) = \int_0^t \Big(\dot\gamma(\tau)\, u\left(\tau,\gamma(\tau)-\right) - f\Big(u\left(\tau,\gamma(\tau)-\right)\Big)\Big)\, \mathrm{d}\tau$$

completing the proof. □

7 Proofs Related to § 3

A tool used below is the following classical representation formula for the solutions to (1.1).

Proposition 7.1 ([25, Theorem 2.1]). *Let (2.1) hold and $u_o \in \mathbf{L}^1(\mathbb{R};\mathbb{R})$. The solution to*

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_o(x) \end{cases} \tag{7.1}$$

in the sense of Definition 2.1 is the map

$$u(t,x) = g\left(\frac{x - y(t,x)}{t}\right) \qquad \text{where} \qquad \begin{array}{l} y(t,x) \text{ minimizes } y \to s(t,x,y)\,, \\ s(t,x,y) = t\, f^*\left(\frac{x-y}{t}\right) + \int_0^y u_o(\xi)\, \mathrm{d}\xi \,, \\ f^*(\lambda) = \lambda\, g(\lambda) - f\left(g(\lambda)\right)\,, \\ g(\lambda) = (f')^{-1}(\lambda)\,. \end{array} \tag{7.2}$$

Formula (7.2) is a direct consequence of the classical Lax–Hopf formula, see [25, Theorem 2.1], [20, 24] or [15, § 11.4], adapted to the present assumption (2.1) on f. Here we only remark that y is uniquely defined for a.e. $x \in \mathbb{R}$.

We pass to some properties on the solution to the conservation law obtained reversing space in (1.1) and assigning w as initial datum.

Lemma 7.2. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that (2.3) holds. Call $\tilde u$ the weak entropy solution to*

$$\begin{cases} \partial_\tau \tilde u + \partial_\xi f(\tilde u) = 0 \\ \tilde u(0,\xi) = w(-\xi) \end{cases} \tag{7.3}$$

and define $u_o^*(x) := \tilde u(T,-x)$. Then:

(R1) The map $\tilde u$ is Lipschitz continuous on any compact subset of $]0,T[\times\mathbb{R}$.

(R2) The map u_o^* is one sided Lipschitz and in $\mathbf{BV}(\mathbb{R};\mathbb{R})$;

(R3) The map u_o^* satisfies $u_o^* \in I_T^{CL}(w)$.

Proof. By (2.1) and [15, Theorem 11.2.1], for any $\tau \in]0,T]$, $\xi \in \mathbb{R}$ and $h > 0$, we have

$$f'\left(\tilde u(\tau,\xi+h)\right) - f'\left(\tilde u(\tau,\xi)\right) \le \frac{h}{\tau}. \tag{7.4}$$

On the other hand, for any $\tau \in [0,T[$, $\xi \in \mathbb{R}$ and $h > 0$, introduce the values attained at $\tau = 0$ by the minimal backward characteristics originating from $(\tau,\xi+h)$ and (τ,ξ):

$$\xi_r = \xi + h - \tau f'\left(\tilde u(\tau,\xi+h)\right) \quad \text{and} \quad \xi_\ell = \xi - \tau f'\left(\tilde u(\tau,\xi)\right), \tag{7.5}$$

so that

$$\begin{array}{rll} & f'\left(\tilde u(\tau,\xi+h)\right) - f'\left(\tilde u(\tau,\xi)\right) & \\ \ge & f'\left(\tilde u_o(\xi_r-)\right) - f'\left(\tilde u_o(\xi_\ell+)\right) & \text{[15, Theorem 11.1.3]} \\ \ge & -\dfrac{\xi_r - \xi_\ell}{T} & \text{[by Oleinik condition (2.3)]} \\ = & -\dfrac{h - \tau\left(f'\left(\tilde u(\tau,\xi+h)\right) - f'\left(\tilde u(\tau,\xi)\right)\right)}{T} & \text{[by (7.5)]} \end{array}$$

hence

$$f'\left(\tilde u(\tau,\xi+h)\right) - f'\left(\tilde u(\tau,\xi)\right) \ge -\frac{h}{T-\tau}. \tag{7.6}$$

The two inequalities (7.4) and (7.6) ensure that $x \to f'\left(\tilde u(\tau,x)\right)$ is Lipschitz continuous for $\tau \in]0,T[$, a Lipschitz constant being $\max\left\{1/\tau,\ 1/(T-\tau)\right\}$. Therefore, by (7.3) and (2.1), $\partial_\tau \tilde u$ is in $\mathbf{L}^\infty$, proving *(R1)* and *(R2)*.

For any $\mathbf{C}^1$ entropy – entropy flux pair (η,q) for (7.3), see [15, Chapter 3, § 2], we have

$$\partial_\tau \eta(\tilde u) + \partial_\xi q(\tilde u) = 0 \qquad \text{a.e. in } [0,T]\times\mathbb{R}.$$

Passing from the (τ,ξ) to the (t,x) variables and setting

$$\begin{array}{rcl} t & := & T-\tau \\ x & := & -\xi \end{array} \qquad u(t,x) := \tilde u(T-t,-x), \tag{7.7}$$

we obtain that $\partial_t \eta(u) + \partial_x q(u) = 0$ in distributional sense. By [15, Chapter 6, § 2], u is a weak entropy solution to (1.1) such that $u(T) = w$ and hence $u_o^* \in I_T^{CL}(w)$, proving *(R3)*. □

Lemma 7.3. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$ and $p \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$. Then, there exists a unique $u_o^\flat \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that any of its primitives $U_o^\flat$ satisfies*

($I^\flat$) for all $x \in \mathbb{R}$ such that p is differentiable at x and $p'(x) \neq 0$,

$$\lim_{y \to x} \frac{U_o^\flat\left(p(y)\right) - U_o^\flat\left(p(x)\right)}{p(y) - p(x)} = w(x)\,;$$

($II^\flat$) for all $x \in \mathbb{R}$ such that $w(x-) \neq w(x+)$, for all $y \in \left]p(x-), p(x+)\right[$,

$$\begin{aligned}
\frac{U_o^\flat\left(p(x+)\right) - U_o^\flat(y)}{T} &= f^*\left(\frac{x-y}{T}\right) - f^*\left(\frac{x-p(x+)}{T}\right),\\
\frac{U_o^\flat(y) - U_o^\flat\left(p(x-)\right)}{T} &= f^*\left(\frac{x-p(x-)}{T}\right) - f^*\left(\frac{x-y}{T}\right).
\end{aligned}$$

Proof. We prove separately existence and uniqueness, referring to X_i and X_{ii} defined in (2.4).

Existence. Let $u_o \in I_T^{CL}(w)$ and call U_o any of its primitives. For all $y \in \mathbb{R} \setminus X_{ii}$, define $U_o^\flat(y) = U_o(y)$ so that *($I^\flat$)* holds since $p(\mathbb{R}) \subseteq \mathbb{R} \setminus X_{ii}$.

Consider now a $y \in X_{ii}$. Then, there exists a unique $x \in \mathbb{R}$ such that $y \in \left]p(x-), p(x+)\right[$. Define, for all $y \in [p(x-), p(x+)]$,

$$U_o^\flat(y) := U_o\left(p(x-)\right) + T\left(f^*\left(\frac{x-p(x-)}{T}\right) - f^*\left(\frac{x-y}{T}\right)\right). \tag{7.8}$$

Straightforward computations show that $U_o^\flat(p(x-)) = U_o\left(p(x-)\right)$ and that *($II^\flat$)* holds.

Uniqueness. Fix $\bar{y} \in X_i \cap \mathfrak{Leb}\left(u_o^\flat\right)$. There exists an $\bar{x} \in \mathbb{R}$ such that $p(\bar{x}) = \bar{y}$, p is differentiable at $\bar{x}$ and $p'(\bar{x}) > 0$, then

$$\lim_{y \to x} \frac{U_o^\flat\left(p(y)\right) - U_o^\flat\left(p(\bar{x})\right)}{p(y) - p(\bar{x})} = \begin{cases} w(\bar{x}) & \text{[by } (I^\flat)\text{]} \\ u_o^\flat(\bar{y}) & \text{[by the choice of } \bar{y}\text{]} \end{cases}$$

On the other hand, if $\bar{y} \in X_{ii}$, there exists an $\bar{x} \in \mathbb{R}$ such that $\bar{y} \in \left]p(\bar{x}-), p(\bar{x}+)\right[$ and by *($II^\flat$)*, for all $y \in \left]p(\bar{x}-), p(\bar{x}+)\right[$,

$$U_o^\flat(y) = U_o^\flat\left(p(\bar{x}-)\right) + T\left(f^*\left(\frac{\bar{x}-p(\bar{x}-)}{T}\right) - f^*\left(\frac{\bar{x}-y}{T}\right)\right)$$

so that $U_o^\flat$ is of class $\mathbf{C}^1$ on the interval $\left]p(\bar{x}-), p(\bar{x}+)\right[$, which implies that $u_o^\flat(\bar{y}) = g\left(\frac{\bar{x}-\bar{y}}{T}\right)$, with g defined as in Proposition 7.1.

Thus, Proposition 2.4 ensures that $u_o^\flat$ is a.e. uniquely defined. □

Lemma 7.4. *Let (2.1) hold and T be positive. Fix $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $p \in \mathbf{SBV_{loc}}(\mathbb{R};\mathbb{R})$ and (2.3) holds. Then, the function u_o^* defined in Lemma 7.2 and the function $u_o^\flat$ defined in Lemma 7.3 coincide.*

Proof. It is sufficient to prove that the map u_o^* defined in Lemma 7.2 satisfies *($I^\flat$)* and *($II^\flat$)* in Lemma 7.3. We refer below to $\tilde{u}$ and u as defined in Lemma 7.2 and related as in (7.7).

The proof consists of the following steps.

Generalized Characteristics: The fact that $\tilde{u}$ and u are locally Lipschitz on $]0,T[\, \times \mathbb{R}$ implies that the generalized characteristics are actually classical characteristics and only one of them goes through a point (τ,ξ) or (t,x) when $\tau, t \in \,]0,T[$.

Of course at times 0 and T it is still possible to have multiple characteristics since the semi-Lipschitz property only guarantees uniqueness in one direction. Indeed, u or $\tilde{u}$ may display rarefaction waves generated at time 0 or shock waves forming at time T.

Furthermore simple calculations show the equivalence through (7.7) of

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = f'(\tilde{u}(\tau,\xi(\tau))) \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = f'(u(t,x(t))). \tag{7.9}$$

Properties of u_0^*: Now let us consider a point $x \in \mathbb{R}$ such that $w(x^-) > w(x^+)$. Therefore, the minimal and maximal backward characteristics for u through (T,x), say γ^- and γ^+, are

$$\gamma^+(t) = x - (T-t)\, f'\left(w(x^+)\right) \quad \text{and} \quad \gamma^-(t) = x - (T-t)\, f'\left(w(x^-)\right). \tag{7.10}$$

By *(b)* in Theorem 3.2, the function p is non decreasing, hence

$$\forall z \in \mathbb{R}, \qquad p(z) = z - T\, f'\left(w(z)\right), \tag{7.11}$$

we have

$$\gamma^-(0) = p(x^-) \quad \text{and} \quad \gamma^+(0) = p(x^+). \tag{7.12}$$

Using (7.7), γ^- and γ^+ provide generalized characteristics for $\tilde{u}$ through the formulæ

$$\varphi^-(\tau) = -\gamma^-(T-\tau) \quad \text{and} \quad \varphi^+(\tau) = -\gamma^+(T-\tau). \tag{7.13}$$

Now consider $y \in \left]p(x^-), p(x^+)\right[$. We have of course

$$\varphi^-(T) > -y > \varphi^+(T). \tag{7.14}$$

Since $\varphi^-(0) = \varphi^+(0) = -x$ and the characteristics do not cross for $\tau \in \,]0,T[$, the minimal and maximal backward characteristics through $(T,-y)$ for $\tilde{u}$ are straight lines and, in fact, coincide and go through $(0,-x)$. Thus, we have

$$\tilde{u}(T,(-y)^+) = \tilde{u}(T,(-y)^-) \quad \text{and} \quad -y - f'(\tilde{u}(T,-y)) = -x. \tag{7.15}$$

From which we get that u_0^* is given between $p(x^-)$ and $p(x^+)$ by the formula

$$\forall y \in \left]p(x^-), p(x^+)\right[, \qquad u_0^*(y) = g\left(\frac{x-y}{T}\right), \tag{7.16}$$

where g is the reciprocal function of f', as in (7.2).

Condition (I♭) holds: Fix $x, y \in \mathbb{R}$ with $x < y$. Apply Proposition 6.2 with $a = 0$, $b = T$ and as α, respectively β, the minimal, respectively maximal, backward characteristic from (T, x), respectively (T, y). Note that our choice $u \in \mathbf{C}^0([0,T]; \mathbf{L}^1(\mathbb{R};\mathbb{R}))$ in Definition 2.1 allows to select $t_1 = 0$ and $t_2 = T$. Then, with the notation (2.2),

$$
\begin{aligned}
& U_o^\flat\left(p(y)\right) - U_o^\flat\left(p(x)\right) \\
= \; & \int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi \\
= \; & \int_x^y w(\xi)\,\mathrm{d}\xi - \int_0^t \Big(f\left(u(t,\alpha(t)-)\right) - \dot\alpha(t)\, u\left(t,\alpha(t)\right)\Big)\,\mathrm{d}t \\
& + \int_0^t \Big(f\left(u(t,\beta(t)-)\right) - \dot\beta(t)\, u\left(t,\beta(t)\right)\Big)\,\mathrm{d}t \\
= \; & \int_x^y w(\xi)\,\mathrm{d}\xi \qquad\qquad\qquad\qquad\qquad\qquad \text{[by [15, Theorem 11.1.3]]} \\
& -T\Big(f\left(w(x)\right) - w(x)\, f'\left(w(x)\right)\Big) + T\Big(f\left(w(y)\right) - w(y)\, f'\left(w(y)\right)\Big)\,,
\end{aligned}
$$

and entirely similar equalities hold if $y < x$. Let now $x \in E$, so that p, and hence w, are differentiable at x. In the previous equality, divide by $y - x$ and pass to the limit as $y \to x$. Then, the right hand side converges and, hence, also the left hand side. We thus obtain

$$
\lim_{y \to x} \frac{U_o^\flat\left(p(y)\right) - U_o^\flat\left(p(x)\right)}{y - x} = w(x) - T\, w(x)\, w'(x)\, f''\left(w(x)\right) = w(x)\, p'(x)\,. \tag{7.17}
$$

Since p is differentiable at x, $p'(x) \neq 0$ and using (7.17), we prove the existence of the following limit, at the same time computing its value:

$$
\begin{aligned}
\lim_{y\to x} \frac{U_o^\flat\left(p(y)\right) - U_o^\flat\left(p(x)\right)}{p(y) - p(x)}
&= \lim_{y \to x} \frac{1}{p(y)-p(x)} \int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi \\
&= \lim_{y\to x} \frac{\dfrac{1}{y-x} \displaystyle\int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi}{\dfrac{p(y)-p(x)}{y-x}} \\
&= \frac{\displaystyle\lim_{y\to x} \frac{1}{y-x} \int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi}{\displaystyle\lim_{y\to x} \frac{p(y)-p(x)}{y-x}} \\
&= \frac{w(x)\, p'(x)}{p'(x)} \\
&= w(x)
\end{aligned}
$$

which completes the proof of (I♭).

Condition (II♭) holds: Still using the same point x, we consider the function Υ given by

$$
\forall v \in \Big]w(x^+), w(x^-)\Big[\,, \qquad \Upsilon(v) := \frac{1}{T} \int_{x - Tf'(w(x^-))}^{x - Tf'(v)} u_0^*(y)\,\mathrm{d}y\,. \tag{7.18}
$$

Thanks to (7.16), u_0^* is continuous on $\left]p(x^-), p(x^+)\right[$. Therefore, Υ is actually of class $\mathbf{C}^1$ on $\left]w(x^+), w(x^-)\right[$ and we have, using again (7.16),

$$\forall v \in \left]w(x^+), w(x^-)\right[, \qquad \Upsilon'(v) = -v\, f''(v). \tag{7.19}$$

Since obviously $\Upsilon(w(x^-)) = 0$ we see that for any v in $\left]w(x^+), w(x^-)\right[$,

$$\frac{1}{T}\int_{x-Tf'(w(x^-))}^{x-Tf'(v)} u_0^*(y)\,\mathrm{d}y = \Upsilon(v) = \big(f(v) - v\,f'(v)\big) - \Big(f(w(x^-)) - w(x^-)f'(w(x^-))\Big). \tag{7.20}$$

In the same way, we can also show that for any v in $\left]w(x^+), w(x^-)\right[$,

$$\frac{1}{T}\int_{x-T\,f'(v)}^{x-T\,f'(w(x^+))} u_0^*(y)\,\mathrm{d}y = \Big(f(w(x^+)) - w(x^+)f'(w(x^+))\Big) - \big(f(v) - v\,f'(v)\big)\ , \tag{7.21}$$

completing the proof. □

Proof of Theorem 3.1. Lemma 7.2 ensures that the map u_o^* defined in *(i*)* is one sided Lipschitz and satisfies $u_o^* \in I_T^{CL}(w)$. Lemma 7.3 ensures that there exists a unique u_o^* satisfying *(ii*)*. Lemma 7.4 shows that these two functions coincide, completing the proof. □

Proof of Corollary 3.2. The implication *(a)* $\Rightarrow$ *(b)* holds by [15, Theorem 11.2.1]. The converse *(b)* $\Rightarrow$ *(a)* follows from *(1*)* in Theorem 3.1, or Lemma 7.2. □

8 Proofs Related to § 4

Proof of Theorem 4.1, (necessity). Let u_o be the precise representative (5.2) of the initial datum to (1.1) such that the corresponding solution u satisfies $u(T) = w$. We now prove that conditions *(i)* and *(ii)* hold.

Since $p \in \mathbf{SBV}(\mathbb{R};\mathbb{R})$, we write below $p = p_{ac} + p_s$, with $p_{ac} \in \mathbf{AC}(\mathbb{R};\mathbb{R})$ and p_s being a sum of countably many Heaviside functions centered at the points of jump in w. Note that p, p_{ac} and p_s are all weakly increasing.

Using the notation (2.2), introduce the set

$$E := \left\{x \in \mathbb{R} \colon p \text{ is differentiable at } x \text{ and } p'(x) \neq 0\right\}\ . \tag{8.1}$$

Proof of *(i)*. Fix $x, y \in \mathbb{R}$ with $x < y$. Apply Proposition 6.2 with $a = 0$, $b = T$ and as α, respectively β, the minimal, respectively maximal, backward characteristic from (T, x), respectively (T, y). Note that our choice $u \in \mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}))$ in Definition 2.1 allows to select $t_1 = 0$ and $t_2 = T$. Then, with the notation (2.2),

$$\begin{aligned}
\int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi &= \int_x^y w(\xi)\,\mathrm{d}\xi - \int_0^t \Big(f\left(u(t,\alpha(t)-)\right) - \dot\alpha(t)\,u\left(t,\alpha(t)\right)\Big)\,\mathrm{d}t \\
&\quad + \int_0^t \Big(f\left(u(t,\beta(t)-)\right) - \dot\beta(t)\,u\left(t,\beta(t)\right)\Big)\,\mathrm{d}t \\
&= \int_x^y w(\xi)\,\mathrm{d}\xi \qquad\qquad [\text{by [15, Theorem 11.1.3]}]
\end{aligned}$$

$$-T\left(f\left(w(x)\right)-w(x)\,f'\left(w(x)\right)\right)+T\left(f\left(w(y)\right)-w(y)\,f'\left(w(y)\right)\right),$$

and similar equalities hold in the case $y < x$. Let now $x \in E$, so that p, and hence w, are differentiable at x. In the previous equality, divide by $y-x$ and pass to the limit as $y \to x$. Then, the right hand side converges and, hence, also the left hand side. We thus obtain

$$\lim_{y\to x}\frac{1}{y-x}\int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi = w(x) - T\,w(x)\,w'(x)\,f''\left(w(x)\right) = w(x)\,p'(x)\,. \tag{8.2}$$

Since p is differentiable at x, $p'(x) \neq 0$ and using (8.2), we prove the existence of the following limit, at the same time computing its value:

$$\begin{aligned}
\lim_{y\to x}\frac{1}{p(y)-p(x)}\int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi &= \lim_{y\to x}\frac{\frac{1}{y-x}\int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi}{\frac{p(y)-p(x)}{y-x}} \\
&= \frac{\lim_{y\to x}\frac{1}{y-x}\int_{p(x)}^{p(y)} u_o(\xi)\,\mathrm{d}\xi}{\lim_{y\to x}\frac{p(y)-p(x)}{y-x}} \\
&= \frac{w(x)\,p'(x)}{p'(x)} \\
&= w(x)
\end{aligned}$$

which completes the proof of (4.1).

Proof of *(ii)*. Let $x \in \mathbb{R}$ be such that $w(x-) > w(x+)$ and $v \in \left]w(x+), w(x-)\right[$. Introduce the minimal backward characteristic $\alpha(t) = x - (T-t)\,f'\left(w(x-)\right)$ and the line $\beta(t) = x - (T-t)\,f'(v)$. Apply Proposition 6.2 with $t_1 = 0$, $t_2 = T$ to obtain

$$\begin{aligned}
-\int_{\alpha(0)}^{\beta(0)} u(0,x)\,\mathrm{d}x &= \int_0^T \left(f\left(u\left(t,\alpha(t)-\right)\right) - \dot\alpha(t)\,u\left(t,\alpha(t)-\right)\right)\mathrm{d}t \\
&\quad - \int_0^T \left(f\left(u(t,\beta(t)+\right) - \dot\beta(t)\,u\left(t,\beta(t)+\right)\right)\mathrm{d}t\,.
\end{aligned}$$

To compute the first summand in the right hand side recall that u is constant along minimal backward characteristics, while the convexity of f ensures that $f(w) - f'(v)\,w \geq f(v) - f'(v)\,v$. We thus have

$$\int_{\alpha(0)}^{\beta(0)} u(0,x)\,\mathrm{d}x \geq T\,\left(v\,f'(v) - f(v)\right) - T\,\left(w(x-)\,f'\left(w(x-)\right) - f\left(w(x-)\right)\right), \tag{8.3}$$

proving the latter inequality in *(ii)*. The proof of the former one is entirely analogous. □

Proof of Lemma 4.3. The equivalence between *(i)* in Theorem 4.1 and *(I)* in Theorem 4.2 in Theorem 4.2 is immediate, thanks to the relation $u = \partial_x U$.

To prove the equivalence between *(ii)* in Theorem 4.1 and *(II)* in Theorem 4.2, note that

$$\begin{array}{rccl} \Pi_x & : & [w(x+), w(x-)] & \to & [p(x-), p(x+)] \\ & & v & \to & x - T\, f'(v) \end{array}$$

is a bijective map, since

$$y = \Pi_x(v) \iff v = g\left(\frac{x-y}{T}\right).$$

Hence, straightforward computations yield

$$\begin{aligned} v\, f'(v) - f(v) &= g\left(\frac{x-y}{T}\right) f'\left(g\left(\frac{x-y}{T}\right)\right) - f\left(g\left(\frac{x-y}{T}\right)\right) \\ &= \frac{x-y}{T}\, g\left(\frac{x-y}{T}\right) - f\left(g\left(\frac{x-y}{T}\right)\right) \\ &= f^*\left(\frac{x-y}{T}\right). \end{aligned}$$

Using the above equality at points x of jump in p, the equivalence *(ii)* $\iff$ *(II)* follows. □

Lemma 8.1. *Let (2.1) hold and let s be defined as in (7.2). If*

$$\{x_1 < x_2 \;\; \textit{and} \;\; y_1 \le y_2\} \qquad \textit{or} \qquad \{x_1 \le x_2 \;\; \textit{and} \;\; y_1 < y_2\} \tag{8.4}$$

then

$$s(T, x_1, y_1) + s(T, x_2, y_2) < s(T, x_1, y_2) + s(T, x_2, y_1). \tag{8.5}$$

Proof. Define $A := \frac{x_1-y_2}{T}$, $B := \frac{x_2-y_1}{T}$, $C := \frac{x_1-y_1}{T}$ and $D := \frac{x_2-y_2}{T}$. Using (8.4) we immediately obtain $A+B = C+D$, $A < D$, $A < C$, $C < B$ and $D < B$. So we can conclude that there exists $\vartheta \in \,]0,1[$ such that $C = \vartheta\, A + (1-\vartheta)\, B$ and $D = (1-\vartheta)\, A + \vartheta\, B$.

Denote by Δ the difference between the right side and left side of (8.5). By (7.2) we get

$$\begin{aligned} \Delta &= T\left(f^*\left(\frac{x_1-y_2}{T}\right) + f^*\left(\frac{x_2-y_1}{T}\right) - f^*\left(\frac{x_1-y_1}{T}\right) - f^*\left(\frac{x_2-y_2}{T}\right)\right) \\ &= T\left(f^*(A) + f^*(B) - f^*(C) - f^*(D)\right) \end{aligned}$$

The strict convexity of f^* now ensures that $\Delta > 0$, completing the proof. □

Proof of Theorem 4.2, (sufficiency). Let U_o be such that conditions *(I)* and *(II)* hold. Then, we prove that the solution u to (1.2) with $u(0) = \partial_x U_o$ also satisfies $u(T) = w$.

We use below Lax-Hopf Formula, i.e., Proposition 7.1. To this aim, define p as in (2.2) and the Legendre transform f^* of f as in (7.2). It is sufficient to show that for a.e. $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$

$$T\, f^*\left(\frac{x-p(x)}{T}\right) + U_o\left(p(x)\right) \le T\, f^*\left(\frac{x-y}{T}\right) + U_o(y),$$

which, by (7.2), is equivalent to

$$\forall y \in \mathbb{R}, \qquad s\left(T, x, p(x)\right) \le s(T, x, y). \tag{8.6}$$

Step 1: Let $\bar{x} \in \mathbb{R}$ be a point where p is differentiable. Then, for all $x \in \mathbb{R}$, the map $\xi \to s\left(T, x, p(\xi)\right)$ is differentiable at $\bar{x}$ and we have

$$\forall x \leq \bar{x}, \quad \frac{\mathrm{d}}{\mathrm{d}\bar{x}} s\left(T, x, p(\bar{x})\right) \geq 0\,, \qquad \text{and} \qquad \forall x \geq \bar{x}, \quad \frac{\mathrm{d}}{\mathrm{d}\bar{x}} s\left(T, x, p(\bar{x})\right) \leq 0\,. \tag{8.7}$$

Proof of Step 1: Consider first the case $p'(\bar{x}) \neq 0$, so that $p(\bar{x}) > 0$ since p is weakly increasing. Then, using the definition of s in (7.2), hypothesis *(I)* and the regularity of f^*

$$\begin{aligned}
& \frac{s\left(T, x, p(\xi)\right) - s\left(T, x, p(\bar{x})\right)}{\xi - \bar{x}} \\
= \; & T\, \frac{f^*\left(\frac{x - p(\xi)}{T}\right) - f^*\left(\frac{x - p(\bar{x})}{T}\right)}{\xi - \bar{x}} + \frac{U_o\left(p(\xi)\right) - U_o\left(p(\bar{x})\right)}{\xi - \bar{x}} \\
= \; & T\, \frac{f^*\left(\frac{x - p(\xi)}{T}\right) - f^*\left(\frac{x - p(\bar{x})}{T}\right)}{\xi - \bar{x}} + \frac{p(\xi) - p(\bar{x})}{\xi - \bar{x}}\, \frac{U_o\left(p(\xi)\right) - U_o\left(p(\bar{x})\right)}{p(\xi) - p(\bar{x})} \\
\underset{\xi \to \bar{x}}{\to} \; & -p'(\bar{x})\, g\left(\frac{x - p(\bar{x})}{T}\right) + p'(\bar{x})\, w(\bar{x}) \\
= \; & p'(\bar{x}) \left(g\left(\frac{\bar{x} - p(\bar{x})}{T}\right) - g\left(\frac{x - p(\bar{x})}{T}\right)\right),
\end{aligned}$$

because

$$w(\bar{x}) = g\left(\frac{\bar{x} - p(\bar{x})}{T}\right)$$

and since g is increasing and $p'(\bar{x}) > 0$, the present claim is proved in the case $p'(\bar{x}) \neq 0$.

Consider now the case $p'(\bar{x}) = 0$ and follow computations similar to the ones above:

$$\begin{aligned}
& \left| \frac{s\left(T, x, p(\xi)\right) - s\left(T, x, p(\bar{x})\right)}{\xi - \bar{x}} - T\, \frac{f^*\left(\frac{x - p(\xi)}{T}\right) - f^*\left(\frac{x - p(\bar{x})}{T}\right)}{\xi - \bar{x}} \right| \\
= \; & \left| \frac{U_o\left(p(\xi)\right) - U_o\left(p(\bar{x})\right)}{\xi - \bar{x}} \right| \\
\leq \; & \left\| u_o \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \left| \frac{p(\xi) - p(\bar{x})}{\xi - \bar{x}} \right| \\
\underset{\xi \to \bar{x}}{\to} \; & 0\,.
\end{aligned}$$

The proof of the present claim is completed, since $\lim_{\xi \to \bar{x}} \dfrac{f^*\left(\frac{x - p(\xi)}{T}\right) - f^*\left(\frac{x - p(x)}{T}\right)}{\xi - \bar{x}} = 0$.

Step 2: Let $\bar{x}$ be a point of jump of p. Since *(II)* holds, we have that

$$\begin{array}{lll}
\forall x \geq \bar{x}, & \forall y \in [p(\bar{x}-), p(\bar{x}+)], & s\left(T, x, p(\bar{x}+)\right) \leq s(T, x, y)\,, \\
\forall x \leq \bar{x}, & \forall y \in [p(\bar{x}-), p(\bar{x}+)], & s\left(T, x, p(\bar{x}-)\right) \leq s(T, x, y)\,.
\end{array}$$

Proof of Step 2: Consider the relations in *(II)*, which can be rewritten as

$$\forall y \in [p(\bar{x}-), p(\bar{x}+)], \qquad s\left(T, \bar{x}, p(\bar{x}\pm)\right) - s(T, \bar{x}, y) \leq 0$$

Apply Lemma 8.1 in the two cases

$$\begin{array}{lllll} x > \bar{x} & x_1 = x & x_2 = \bar{x} & y_1 = p(\bar{x}+) & y_2 = y \\ x < \bar{x} & x_1 = x & x_2 = x & y_1 = p(\bar{x}-) & y_2 = y \end{array}$$

obtaining

$$\begin{array}{lcl} s\left(T, x, p(\bar{x}+)\right) & < s(T, x, y) + s\left(T, \bar{x}, p(\bar{x}+)\right) - s(T, \bar{x}, y) & < \quad s(T, x, y) \\ s\left(T, x, p(\bar{x}-)\right) & < s(T, x, y) + s\left(T, \bar{x}, p(\bar{x}-)\right) - s(T, \bar{x}, y) & < \quad s(T, x, y) \end{array}$$

completing the proof of **Step 2**..

Step 3: For any $x \in \mathbb{R}$, the map $\xi \to s\left(T, x, p(\xi)\right)$ attains its minimum at x. Equivalently

$$\forall y \in \overline{p(\mathbb{R})} \qquad s(T, x, y) \geq s\left(T, x, p(x)\right) . \tag{8.8}$$

Proof of Step 3: Fix $x \in \mathbb{R}$ and denote $S(\xi) = s\left(T, x, p(\xi)\right)$.

Consider first the set $[x, +\infty[$. p is weakly increasing on $[x, +\infty[$, hence it is differentiable a.e. on $[x, +\infty[$. Then, by **Step 1**, S is differentiable a.e. and $S'(\xi) \geq 0$ for a.e. $\xi \in [x, +\infty[$. At all jump points $\xi \in [x, +\infty[$, by **Step 2**, we have $S(\xi-) \leq S(\xi+)$. Since the map $y \to s(T, x, y)$ is Lipschitz and p is in $\mathbf{SBV}(\mathbb{R}, \mathbb{R})$, by [3, Proposition 1.2], we have that also $S \in \mathbf{SBV}(\mathbb{R}; \mathbb{R})$. Thus, S is weakly increasing on $[x, +\infty[$.

An entirely symmetric argument shows that S is weakly decreasing on $]-\infty, x]$, completing the proof of **Step 3**.

Step 4: For any $x \in \mathbb{R}$,

$$\forall y \in \mathbb{R} \setminus \overline{p(\mathbb{R})} \qquad s(T, x, y) \geq s\left(T, x, p(x)\right) . \tag{8.9}$$

Proof of Step 4: Fix $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \overline{p(\mathbb{R})}$. By (2.2) and since $w \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, $\lim_{\xi \to \pm\infty} p(\xi) = \pm\infty$. Hence, we can define $\bar{x} = \sup\left\{\xi \in \mathbb{R} \colon p(\xi) < y\right\}$ so that, thanks to the fact that p is weakly increasing, $y \in \left]p(\bar{x}-), p(\bar{x}+)\right[$. Then, by **Step 2**,

$$\begin{array}{lcl} \bar{x} \geq x & \Rightarrow & s(T, x, y) \geq s\left(T, x, p(\bar{x}-)\right) , \\ \bar{x} \leq x & \Rightarrow & s(T, x, y) \geq s\left(T, x, p(\bar{x}+)\right) , \end{array}$$

so that, using **Step 3**, the proof of **Step 4** is completed.

The proof is completed, since (8.6) follows from (8.8) and (8.9). □

Proof of Theorem 4.4. The proof is divided into the following steps.

If $U_o \in I_T^{HJ}(W)$ then (I^{HJ}) and (II^{HJ}) hold. If $U_o \in I_T^{HJ}(W)$ then, by [22, Theorem 1.1], $\partial_x U_o \in I_T^{CL}(\partial_x W)$. Theorem 4.2 directly ensures that (I^{HJ}) holds. Concerning (II^{HJ}), consider a point $\bar x$ of jump for p and choose $y = p(\bar x-)$, respectively $y = p(\bar x+)$, in the first, respectively second, inequality in *(II)*, we obtain the equality

$$U_o\left(p(\bar x-)\right) + T\, f^*\left(\frac{\bar x - p(\bar x-)}{T}\right) = U_o\left(p(\bar x+)\right) + T\, f^*\left(\frac{\bar x - p(\bar x+)}{T}\right).$$

Call Q this value and get $U_o(y) + T\, f^*\left(\frac{\bar x - y}{T}\right) \geq Q$. We are left to prove that $Q = W(\bar x)$.

Call $U(t,x) := (S_t^{HJ} U_o)(x)$ and introduce $u_o := \partial_x U_o$, $w := \partial_x W$ and $u(t,x) := (S_t^{CL} u_o)(x)$. Let γ be the minimal backward characteristic emanating from $(T, \bar x)$, so that $\gamma(t) = \bar x + (t - T)\, f'\left(w(\bar x-)\right)$ and $u\left(t, \gamma(t)-\right) = w(\bar x-)$ by [14, Theorem 3.2 and Theorem 3.3]. Clearly, by (2.2), $p(\bar x-) = \gamma(0)$.

By Proposition 2.5 and [22, Theorem 1.1], there exists $c \in \mathbb{R}$ such that for $(t,x) \in [0,T] \times \mathbb{R}$

$$\begin{aligned} U(t,x) &= \int_{\gamma(t)}^{x} u(t,\xi)\,\mathrm{d}\xi + \int_0^t \Big(\dot\gamma(\tau)\, u\left(\tau, \gamma(\tau)-\right) - f\left(u\left(\tau,\gamma(\tau)-\right)\right)\Big)\,\mathrm{d}\tau + c \\ &= \int_{\gamma(t)}^{x} u(t,\xi)\,\mathrm{d}\xi + t\left(f'\left(w(\bar x-)\right)\, w(\bar x-) - f\left(w(\bar x-)\right)\right) + c\,. \end{aligned}$$

Direct evaluations of the latter expression above yield

$$W(\bar x) = U(T,\bar x) = T\, f^*\left(\frac{\bar x - p(\bar x-)}{T}\right) + c \quad \text{and} \quad U_o\left(p(\bar x-)\right) = U\left(0, p(\bar x-)\right) = c\,.$$

so that $W(\bar x) = Q$, completing the proof of this part.

If (I^{HJ}) and (II^{HJ}) hold, then $U_o \in I_T^{HJ}(W)$. If (I^{HJ}) holds, then clearly U_o satisfies *(I)*. Moreover, by (II^{HJ}), if $\bar x$ is a point of jump of p,

$$\begin{aligned} U_o\left(p(\bar x\pm)\right) + T\, f^*\left(\frac{\bar x - p(\bar x\pm)}{T}\right) &\leq U_o(y) + T\, f^*\left(\frac{\bar x - y}{T}\right) \\ \frac{U_o\left(p(\bar x\pm)\right) - U_o(y)}{T} &\leq f^*\left(\frac{\bar x - y}{T}\right) - f^*\left(\frac{\bar x - p(\bar x\pm)}{T}\right) \end{aligned}$$

which implies *(II)*. Then, Theorem 4.2 applies and ensures that $\partial_x U_o \in I_T^{CL}(\partial_x W)$. By [22, Theorem 1.1], it follows that $\partial_x S_T^{HJ} U_o = S_T^{CL} \partial_x U_o = \partial_x W$ and, hence, there is a constant $c \in \mathbb{R}$ such that $S_T^{CL} U - W = c$. Thus, $S_T^{HJ}(U_o + c) = S_T^{HJ} U_o + c = W$ and $(U_o + c) \in I_T^{HJ}(W)$ so that $(U_o + c)$ satisfies the equality in (II^{HJ}), as proved in the previous claim. By assumption, also U_o satisfies the equality in (II^{HJ}), hence $c = 0$, completing the proof. □

9 Proofs Related to § 5

Proof of Proposition 5.1. We prove the different parts separately.

Proof of *(T1)*: The strong $\mathbf{L}^1$ closure of $I_T^{CL}(w)$ directly follows from the strong $\mathbf{L}^1$ continuity of the semigroup generated by (1.1), see [8, Chapter 6, § 4].

Proof of *(T2)*: To prove that $I_T^{CL}(w)$ has empty interior, fix u_o in $I_T^{CL}(w)$ and use the characterization of $I_T^{CL}(w)$ provided by Theorem 4.2. Note that by Proposition 2.4, there exists an $\bar{x} \in \mathbb{R}$ such that either *(I)* or *(II)* in Theorem 4.2 holds.

Let *(I)* hold at a given $\bar{x} \in \mathbb{R}$. Define the sequence of initial data

$$u_o^n(x) := u_o(x) + \chi_{]p(\bar{x})-1/n, p(\bar{x})+1/n[}(x) .$$

Clearly, $u_o^n \to u_o$ strongly in $\mathbf{L}^1$ as $n \to +\infty$. On the other hand, with reference to (4.2) and choosing U_o^n so that $\partial_x U_o^n = u_o^n$, we have that for y sufficiently near to x,

$$\frac{U_o^n\left(p(y)\right) - U_o^n\left(p(\bar{x})\right)}{p(y) - p(\bar{x})} = \frac{U_o\left(p(y)\right) - U_o\left(p(\bar{x})\right)}{p(y) - p(\bar{x})} + 1 \underset{y \to \bar{x}}{\to} w(\bar{x}) + 1 ,$$

showing that $u_o^n \notin I_T^{CL}(w)$ by Theorem 4.2.

Let *(II)* hold at a given $\bar{x} \in \mathbb{R}$. Define the sequence of initial data

$$u_o^n(x) := u_o(x) - C\, \chi_{[p(\bar{x}-), p(\bar{x}-)+1/n]}(x) ,$$

for a sufficiently large constant C that is explicitly chosen in (9.1). We have that for $y \in \left]p(\bar{x}-), p(\bar{x}-) + \frac{1}{n}\right[$,

$$\begin{aligned}
& \frac{U_o^n(y) - U_o^n\left(p(\bar{x}-)\right)}{T} - f^*\left(\frac{\bar{x} - p(\bar{x}-)}{T}\right) + f^*\left(\frac{\bar{x} - y}{T}\right) \\
= {} & \frac{U_o(y) - U_o\left(p(\bar{x}-)\right)}{T} - f^*\left(\frac{\bar{x} - p(\bar{x}-)}{T}\right) + f^*\left(\frac{\bar{x} - y}{T}\right) - C\, \frac{y - p(\bar{x}-)}{T} \\
\leq {} & \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} - C\right) \frac{y - p(\bar{x})}{T} + f^*\left(\frac{\bar{x} - y}{T}\right) - f^*\left(\frac{\bar{x} - p(\bar{x}-)}{T}\right) \\
\leq {} & \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} - g\left(\frac{\bar{x} - p(\bar{x}-)}{T}\right) - C\right) \frac{y - p(\bar{x})}{T} + o\left(y - p(\bar{x}-)\right) \qquad \text{as } y \to p(\bar{x}-) ,
\end{aligned}$$

so that as soon as C is chosen satisfying

$$C > \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} - g\left(\frac{\bar{x} - p(\bar{x}-)}{T}\right) \tag{9.1}$$

we have

$$\frac{U_o^n(y) - U_o^n\left(p(\bar{x}-)\right)}{T} - f^*\left(\frac{\bar{x} - p(\bar{x}-)}{T}\right) + f^*\left(\frac{\bar{x} - y}{T}\right) < 0 \tag{9.2}$$

for all y sufficiently near to and larger than $p(\bar{x}-)$. But (9.2) contradicts *(II)* in Theorem 4.2, so we obtain that $u_o^n \notin I^{CL}(w)$, although $u_o^n \to u_o$ in $\mathbf{L}^1(\mathbb{R};\mathbb{R})$ and $u_o \in I^{CL}(w)$. □

The following remark provides a basic linear algebra observation of use in the subsequent proof of Proposition 5.2.

Remark 9.1. *Let V be a vector spaces, $L_\alpha, \Lambda_\alpha \colon V \to \mathbb{R}$ be linear maps and m_α, μ_α be real numbers, with α varying in a suitable set of indices I. Assume there exists a unique $v^* \in V$ such that for all $\alpha \in I$, $L_\alpha v^* = m_\alpha$ and $\Lambda_\alpha v^* = \mu_\alpha$. Then, the set $\{v \in V \colon \forall \alpha \in I \quad L_\alpha v = m_\alpha \text{ and } \Lambda_\alpha v \geq \mu_\alpha\}$ is a cone with vertex at v^*, which is its unique extremal point.*

Proof of Proposition 5.2. We split the proof in different steps.

Proof of *(G1)*: Consider the two implications separately.

If $w \in \mathbf{C}^0(\mathbb{R};\mathbb{R})$, then $I_T^{CL}(w)$ is a singleton. Let $w \in \mathbf{C}^0(\mathbb{R};\mathbb{R})$. Then, the set X_{ii} in (2.4) is empty. By Proposition 2.4, $\mathcal{L}(\mathbb{R} \setminus X_i) = 0$. For any $u_o \in I_T^{CL}(w)$ and any $\bar{x}$ in X_i,

$$\begin{aligned} w(\bar{x}) &= \lim_{y \to \bar{x}} \frac{1}{p(y) - p(\bar{x})} \int_{p(\bar{x})}^{p(y)} u_o(\xi)\, d\xi && \text{[by Theorem 4.1]} \\ &= \lim_{r \to 0} \frac{1}{r} \int_{p(\bar{x})}^{p(\bar{x})+r} u_o(\xi)\, d\xi && \text{[since } p'(\bar{x}) > 0 \text{ and } p \in \mathbf{C}^0(\mathbb{R};\mathbb{R})] \\ &= u_o\left(p(\bar{x})\right) && \text{[for the precise representative (5.2) of } u_o] \end{aligned}$$

Indeed, we used above the fact that if $p'(\bar{x}) > 0$ and $p \in \mathbf{C}^0(\mathbb{R};\mathbb{R})$, then for all $r > 0$ sufficiently small, there exists a y_r such that $r = p(y_r) - p(\bar{x})$ and $y_r \to \bar{x}$ as $r \to 0$.

If w admits a point of discontinuity $\bar{x}$, then $I_T^{CL}(w)$ is not a singleton. A first element of $I_T^{CL}(w)$ is the map u_o^* defined in Theorem 3.2. A second map can be constructed prolonging the shock at $\bar{x}$ backward to 0. To this aim, we define

$$u_o^\sharp(x) := \begin{cases} u_o^*(x) & x \leq p(\bar{x}-) \\ w(\bar{x}-) & x \in \left]p(\bar{x}-), x_\sharp\right] \\ w(\bar{x}+) & x \in \left]x_\sharp, p(\bar{x}+)\right] \\ u_o^*(x) & x > p(\bar{x}+) \end{cases} \quad \text{where} \quad \begin{array}{l} \lambda^\sharp = \dfrac{f\left(w(\bar{x}+)\right) - f\left(w(\bar{x}-)\right)}{w(\bar{x}+) - w(\bar{x}-)} \\ x_\sharp = \bar{x} - \lambda^\sharp\, T\,. \end{array}$$

We check that $u_o^\sharp$ satisfies *(ii)* in Theorem 4.1 at $\bar{x}$. To this aim, set $v^\sharp = g(\lambda^\sharp)$ and compute

$$\frac{1}{T} \int_{x - T f'(w(x-))}^{x - T f'(v)} u_o(\xi)\, d\xi = \begin{cases} \Big(f'\left(w(\bar{x}-)\right) - f'(v)\Big)\, w(\bar{x}-) & v < v^\sharp \\ \Big(f'\left(w(\bar{x}-)\right) - f'(v^\sharp)\Big)\, w(\bar{x}-) \\ \quad + \Big(f'(v^\sharp) - f'(v)\Big)\, w(\bar{x}+) & v > v^\sharp \end{cases}$$

so that, with reference to *(ii)* in Theorem 4.1, denote

$$\Delta := \frac{1}{T} \int_{\bar{x} - T f'(w(\bar{x}-))}^{\bar{x} - T f'(v)} u_o(\xi)\, d\xi - \Big(w(\bar{x}-)\, f'\left(w(\bar{x}-)\right) - f\left(w(\bar{x}-)\right)\Big) + \left(v\, f'(v) - f(v)\right)$$

and, for $v < v^\sharp$ by the convexity of f we obtain

$$\Delta = f\left(w(x-)\right) - f(v) - f'(v)\left(w(\bar{x}-) - v\right) \geq 0\,.$$

If $v > v^\sharp$, we use the identity $f'(v^\sharp)\left(w(\bar x+) - w(\bar x-)\right) = f\left(w(\bar x+)\right) - f\left(w(\bar x-)\right)$ to obtain

$$
\begin{aligned}
\Delta &= f\left(w(\bar x+)\right) - f\left(w(\bar x-)\right) + f'\left(w(\bar x-)\right) w(\bar x-) - f'(v)\, w(\bar x+) \\
&\quad - f'\left(w(\bar x-)\right) w(\bar x-) + f\left(w(\bar x)\right) + f'(v)\, v - f(v) \\
&= f\left(w(\bar x+)\right) - f(v) - f'(v)\left(w(\bar x+) - v\right) \\
&\geq 0
\end{aligned}
$$

where we used again the convexity of f.

Entirely similar computations apply to the second inequality in *(ii)*. Hence $u_o^\sharp \in I_T^{CL}(w)$ and, since $u_o^\sharp \neq u_o^*$, the proof of *(G1)* is completed.

Proof of *(G2)*: The convexity of $I_T^{CL}(w)$ directly follows from Lax-Hopf formula [25, Theorem 2.1], as well as from [19, Theorem 6.2] or by direct inspection of the conditions provided by Theorem 4.1 or Theorem 4.2.

$I_T^{CL}(w)$ is a cone with u_o^* at its vertex. (We follow the abstract reasoning sketched in Remark 9.1.) Let u_o^* be as defined in Lemma 7.3. For a $u_o \in I_T^{CL}(w) \setminus \{u_o^*\}$, and for all $\vartheta \in \mathbb{R}^+$, define $u_o^\vartheta := u_o^* + \vartheta\,(u_o - u_o^*)$ and, passing to primitives, $U_o^\vartheta = U_o^* + \vartheta\,(U_o - U_o^*)$.

If $\bar x$ is such that p is differentiable at $\bar x$ and $p'(\bar x) > 0$, then by *(I)* in Theorem 4.2 and *($I^\flat$)* in Lemma 7.3,

$$
\begin{aligned}
& \lim_{x \to \bar x} \frac{U_o^\vartheta\left(p(x)\right) - U_o^\vartheta\left(p(\bar x)\right)}{p(\bar x) - p(x)} \\
= & \lim_{x \to \bar x} \frac{U_o^*\left(p(x)\right) - U_o^*\left(p(\bar x)\right)}{p(\bar x) - p(x)} + \lim_{x \to \bar x} \vartheta \left(\frac{U_o\left(p(x)\right) - U_o\left(p(\bar x)\right)}{p(\bar x) - p(x)} - \frac{U_o^*\left(p(x)\right) - U_o^*\left(p(\bar x)\right)}{p(\bar x) - p(x)} \right) \\
= & \ w(x)\,,
\end{aligned}
$$

proving that also U_o^ϑ satisfies *(I)* in Theorem 4.2.

Choose now $x \in \mathbb{R}$ such that $p(x-) < p(x+)$ and compute:

$$
\begin{aligned}
& \frac{U_o^\vartheta\left(p(x+)\right) - U_o^\vartheta(y)}{T} \\
= & \frac{U_o^\vartheta\left(p(x+)\right) - U_o^\vartheta(y)}{T} + \vartheta \left(\frac{U_o\left(p(x+)\right) - U_o(y)}{T} - \frac{U_o^\vartheta\left(p(x+)\right) - U_o^\vartheta(y)}{T} \right) \\
\leq & \ f^*\left(\frac{x-y}{T}\right) - f^*\left(\frac{x - p(x+)}{T}\right).
\end{aligned}
$$

An entirely similar computations applies to $\frac{U_o^\vartheta(y) - U_o^\vartheta\left(p(x+)\right)}{T}$. We thus proved that also U_o^ϑ satisfies *(I)* and *(II)* in Theorem 4.2, hence it belongs to $I_t^{CL}(w)$.

If $u_o \in I_T^{CL}(w)$ is different from u_o^*, then it is not an extremal point of $I_T^{CL}(w)$. This statement directly follows from *(G3)*, which we prove independently below.

Proof of *(G3)*: Preliminary, note that *(II)* in Theorem 4.2 at x is equivalent to require that for all $x \in \mathbb{R}$ such that $w(x-) \neq w(x+)$,

$$\begin{array}{rl} & U_o\left(p(x+)\right) + Tf^*\left(\dfrac{x-p(x+)}{T}\right) = U_o\left(p(x-)\right) + Tf^*\left(\dfrac{x-p(x-)}{T}\right) \\ \forall y \in \left]p(x-), p(x+)\right[& U_o(y) + Tf^*\left(\dfrac{x-y}{T}\right) \geq U_o\left(p(x-)\right) + Tf^*\left(\dfrac{x-p(x-)}{T}\right). \end{array} \tag{9.3}$$

Fix N and $u_o \in I_T^{CL}(w)$ as in the statement of *(G3)*. Let U_o be a primitive of u_o. By *(G1)*, we may assume that there exists an $x \in \mathbb{R}$ such that (9.3) applies. If $u_o \neq u_o^*$, then for a suitable $x \in \mathbb{R}$ and $\bar{y} \in \left]p(x-), p(x+)\right[$, the strict inequality has to hold in the latter relation above computed at $\bar{y}$. Hence, there exist positive η and ε such that

$$\forall y \in]\bar{y}-\eta, \bar{y}+\eta[\quad \left\{ \begin{array}{l} y \in \left]p(x-), p(x+)\right[, \quad \text{and} \\ U_o(y) + Tf^*\left(\dfrac{x-y}{T}\right) \geq U_o\left(p(x-)\right) + Tf^*\left(\dfrac{x-p(x-)}{T}\right) + \varepsilon. \end{array} \right. \tag{9.4}$$

Define now

$$\forall k \in \{1, \ldots, N\}, \qquad y_k := \bar{y} + \frac{\eta}{N}(2k-1-N). \tag{9.5}$$

The intervals $\left]y_k - \frac{\eta}{N}, y_k + \frac{\eta}{N}\right]$ constitute a partition of $]\bar{y}-\eta, \bar{y}+\eta]$.

Define $A_1, \ldots, A_N$ by

$$A_k(y) := \begin{cases} 0, & \text{if } y \leq y_k - \frac{\eta}{N} \\ \varepsilon\left(1 + \frac{N}{\eta}(y - y_k)\right), & \text{if } y_k - \frac{\eta}{N} \leq y \leq y_k \\ \varepsilon\left(1 + \frac{N}{\eta}(y_k - y)\right), & \text{if } y_k \leq y \leq y_k + \frac{\eta}{N} \\ 0, & \text{if } y \leq y_k + \frac{\eta}{N} \end{cases} \tag{9.6}$$

and then let $V_k := U_o - A_k$ for $k = 1, \ldots, N$. Define $v_k := \partial_x V_k$ for $k = 1, \ldots, N$. It is easy to see using (9.3), (9.4) and (9.5) that $v_k \in I_T^{CL}(w)$. Finally, define $V_0 := U_o + \sum_{k=1}^N (U_o - V_k)$ and $v_0 := \partial_x V_o$. As above, $v_0 \in I_T^{CL}(w)$, since $V_k \geq U_o$ and $V_0 = U_o$ outside $[\bar{y}-\eta, \bar{y}+\eta]$. Using the definition of V_0, condition (5.1) is trivially satisfied.

Let us now consider scalars $\lambda_1, \ldots, \lambda_N$ such that

$$\sum_{k=1}^N \lambda_k \, (v_k - v_0) = 0. \tag{9.7}$$

Hence, $\sum_{k=1}^N \lambda_k \, (V_k - V_0) = C$ for a suitable $C \in \mathbb{R}$. By the definitions of $V_0, \ldots, V_N$, $\forall k \in \{1, \ldots, N\}$, $V_k - V_0 = -A_k - \sum_{j=1}^N A_j$. So now (9.7) becomes $\sum_{k=1}^N \left(\lambda_k + \sum_{j=1}^N \lambda_j\right) A_k = -C$. The A_k have compact supports, so that $C = 0$. Their supports are also disjoint, hence $\forall k \in \{1, \ldots, N\}$, $\lambda_k + \sum_{j=1}^N \lambda_j = 0$, from which it is clear that $\forall k \in \{1, \ldots, N\}$, $\lambda_k = 0$. □

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References

[1] Adimurthi, S. S. Ghoshal, and G. D. Veerappa Gowda. Exact controllability of scalar conservation laws with strict convex flux. *Math. Control Relat. Fields*, 4(4):401–449, 2014.

[2] N. Allahverdi, A. Pozo, and E. Zuazua. Numerical aspects of large-time optimal control of Burgers equation. *ESAIM Math. Model. Numer. Anal.*, 50(5):1371–1401, 2016.

[3] L. Ambrosio. A new proof of the SBV compactness theorem. *Calc. Var. Partial Differential Equations*, 3(1):127–137, 1995.

[4] L. Ambrosio and C. De Lellis. A note on admissible solutions of 1D scalar conservation laws and 2D Hamilton-Jacobi equations. *J. Hyperbolic Differ. Equ.*, 1(4):813–826, 2004.

[5] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[6] F. Ancona and A. Marson. On the attainable set for scalar nonlinear conservation laws with boundary control. *SIAM J. Control Optimization*, 36(1):290–312, 1998.

[7] S. Bianchini and L. Yu. SBV-like regularity for general hyperbolic systems of conservation laws in one space dimension. *Rend. Istit. Mat. Univ. Trieste*, 44:439–472, 2012.

[8] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[9] A. Bressan and R. M. Colombo. Decay of positive waves in nonlinear systems of conservation laws. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):133–160, 1998.

[10] C. Castro, F. Palacios, and E. Zuazua. An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks. *Math. Models Methods Appl. Sci.*, 18(3):369–416, 2008.

[11] C. Christoforou and K. Trivisa. Sharp decay estimates for hyperbolic balance laws. *J. Differential Equations*, 247(2):401–423, 2009.

[12] L. Corrias, M. Falcone, and R. Natalini. Numerical schemes for conservation laws via Hamilton-Jacobi equations. *Math. Comp.*, 64(210):555–580, S13–S18, 1995.

[13] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

[14] C. M. Dafermos. Generalized characteristics and the structure of solutions of hyperbolic conservation laws. *Indiana Univ. Math. J.*, 26(6):1097–1119, 1977.

[15] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, fourth edition, 2016.

[16] L. C. Evans. *Partial differential equations. 2nd ed.*, volume 19. Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2010.

[17] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[18] O. Glass. An extension of Oleinik's inequality for general 1D scalar conservation laws. *J. Hyperbolic Differ. Equ.*, 5(1):113–165, 2008.

[19] L. Gosse and E. Zuazua. Filtered gradient algorithms for inverse design problems of one-dimensional Burgers equation. In *Innovative algorithms and analysis*, volume 16 of *Springer INdAM Ser.*, pages 197–227. Springer, Cham, 2017.

[20] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 3:201–230, 1950.

[21] T. Horsin. On the controllability of the Burgers equation. *ESAIM Control Optim. Calc. Var.*, 3:83–95, 1998.

[22] K. H. Karlsen and N. H. Risebro. A note on front tracking and equivalence between viscosity solutions of Hamilton-Jacobi equations and entropy solutions of scalar conservation laws. *Nonlinear Anal.*, 50(4, Ser. A: Theory Methods):455–469, 2002.

[23] S. N. Kružkov. The Cauchy problem in the large for non-linear equations and for certain first-order quasilinear systems with several variables. *Dokl. Akad. Nauk SSSR*, 155:743–746, 1964.

[24] P. D. Lax. Weak solutions of nonlinear hyperbolic equations and their numerical computation. *Comm. Pure Appl. Math.*, 7:159–193, 1954.

[25] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[26] R. Lecaros and E. Zuazua. Tracking control of 1D scalar conservation laws in the presence of shocks. In *Trends in contemporary mathematics.. Selected talks based on the presentations at the INdAM day, June 18, 2014*, pages 195–219. Cham: Springer, 2014.

[27] R. Lecaros and E. Zuazua. Control of 2D scalar conservation laws in the presence of shocks. *Math. Comput.*, 85(299):1183–1224, 2016.

[28] O. A. Oleĭnik. Discontinuous solutions of non-linear differential equations. *Uspehi Mat. Nauk (N.S.)*, 12(3(75)):3–73, 1957.

[29] F. Riesz and B. Sz.-Nagy. *Functional analysis.* Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1990. Translated from the second French edition by Leo F. Boron, Reprint of the 1955 original.

[30] D. Serre. *Systems of conservation laws. 1 & 2.* Cambridge University Press, Cambridge, 1999. Translated from the 1996 French original by I. N. Sneddon.

[31] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York, second edition, 1994.