A semilinear hyperbolic system with space-dependent and nonlinear damping
(Part 2)

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VIII Partial differential equations, optimal design and numerics

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Joint work with Prof. Debora Amadori, Edda Dal Santo.

IBVP with space dependent damping

$$\begin{cases} \partial_t \rho + \partial_x J = 0 \\ \partial_t J + \partial_x \rho = -2k(x)g(J) \end{cases} \qquad x \in I = (0,1),\ t > 0$$

- Initial conditions: $(\rho_0, J_0) \in L^\infty(I)$
- Boundary conditions: $J(0,t) = J(1,t) = 0$

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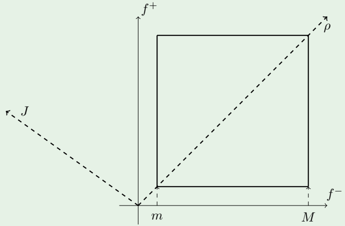
Invariant domain:



Main result (Amadori, A., Dal Santo, JMPA 2019)

Theorem Let $k(x)$ satisfy

$$0 < k_1 \leq k(x) \leq k_2 \qquad \forall\, x \in (0,1)$$

and define

$$d_1 = k_1 \min_{J \in D_J} g'(J) > 0, \qquad d_2 = k_2 \max_{J \in D_J} g'(J)$$

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$$\mathrm{e}^{2d_2} - 2d_2 < \mathrm{e}^{2d_1}$$

Then there exist $C_j > 0$, that depend only on the coefficients and on data, such that for $t \ge 0$

$$\|J_{\Delta x}(\cdot,t)\|_\infty \le C_1 \Delta x + C_2 \mathrm{e}^{-C_3 t}$$
$$\|\rho_{\Delta x}(\cdot,t)\|_\infty \le C_1 \Delta x + C_2 \mathrm{e}^{-C_3 t}\,.$$

where $(\rho_{\Delta x}, J_{\Delta x})$ are WB approximate solutions, $\Delta x = 1/N$.

Main result/Remarks

$$0 < d_1 = k_1 \min g' \leq d_2 = k_2 \max g'$$

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$$\|\rho_{\Delta x}(\cdot, t)\|_\infty \leq C_1 \Delta x + C_2 \mathrm{e}^{-C_3 t}\,.$$

- If $d_1 = d_2 = d$ (linear g, constant k) then the result holds for every $d > 0$. Moreover one has

$$C_3 = \frac{1}{2}\left|\log(1 - 2d\,\mathrm{e}^{-2d})\right| \sim d \qquad \text{as } d \to 0\,.$$

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Remark:

Surprisingly, the total variation of $J_{\Delta x}$ does not necessarily vanish at $t \to \infty$

Sketch of the proof/1

- By iteration,

$$\boldsymbol{\sigma}(t^n+) = \mathcal{B}_n \boldsymbol{\sigma}(0+)\,, \qquad \mathcal{B}_n \doteq \left[B^{(n)} B^{(n-1)} \cdots B^{(2)} B^{(1)}\right] \in M_{2N}\,.$$

- First, a proposition which relates the L^∞-norm of $J(\cdot, t^n)$, $\rho(\cdot, t^n)$ as $n \to \infty$ to the evolution of the ℓ_1–norm of the operator $\mathcal{B}_n$.

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Proposition:

There exists $\widetilde{C}_1 > 0$ independent on n, N such that for every $t \in (t^n, t^{n+1})$

$$\|J_{\Delta x}(\cdot, t)\|_\infty \leq \widetilde{C}_1\, \Delta x + \|\mathcal{B}_n \widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1}$$
$$\|\rho_{\Delta x}(\cdot, t)\|_\infty \leq \widetilde{C}_1\, \Delta x + 2\|\mathcal{B}_n \widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1}$$

where $\widetilde{\boldsymbol{\sigma}}(0+)$ is the projection of $\boldsymbol{\sigma}(0+)$ onto E_-, the $(2N-2)$-dim eigenspace related to λ_i with $|\lambda_i| < 1$.

Sketch of the proof/2

Use of Birkhoff decomposition (linear damping)

Let $k(x) = \bar{k}$, g' be constant. Then $\boldsymbol{c} = c(1, \ldots, 1)$

$$c = \frac{d\Delta x}{1 + d\Delta x}\,, \qquad d = \bar{k}g'\,, \qquad \Delta x = \frac{1}{N}\,,$$

$$B(\boldsymbol{c}) = (1 - c)B(0) + cB_1 = \left(1 + \frac{d}{N}\right)^{-1}\left[B(0) + \frac{d}{N}B_1\right]$$

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Proposition (nonlinear damping):

If $0 < d_1 = k_1 \min g'(J) \leq d_2 = k_2 \max g'(J)$, then

$$B(\boldsymbol{c}^n) \leq \left(1 + \frac{d_1}{N}\right)^{-1}\left[B(0) + \frac{d_2}{N}B_1\right] \quad \forall n$$

(inequality entrywise) .

Sketch of the proof/3

- Next, we prove that $\|\mathcal{B}_n \widetilde{\boldsymbol{\sigma}}\|_{\ell^1}$ decays exp. fast as $n \to \infty$ for every $\widetilde{\boldsymbol{\sigma}} \in E_-$.

We focus on the power $\boxed{n = 2N}$,

$$\mathcal{B}_{2N} \doteq B^{(2N)} B^{(2N-1)} \dots B^{(2)} B^{(1)}$$

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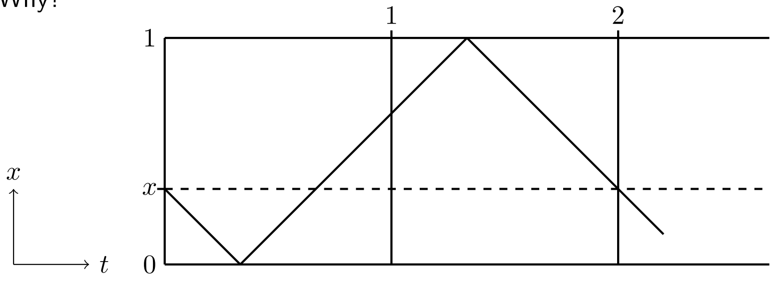
- Why?



Sketch of the proof/4: an "exponential" formula.

Theorem:

Let $d > 0$ and $N \in 2\mathbb{N}$. Let I_{2N} be the identity matrix in M_{2N}. Then

$$\left[B(0) + \frac{d}{N} B_1\right]^{2N} = I_{2N} + (2d)\widehat{P} + \sum_{j=0}^{2N-1} \zeta_{j,N} B(0)^{2j} B_2(0) + \sum_{j=1}^{2N-1} \eta_{j,N} B(0)^{2j}$$

for a suitable $\widehat{P} \in M_{2N}$ (sum of two rank-one matrices).

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Tools: hypergeometric functions, modified Bessel functions.

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Key point: The first order in d is identified + estimate on higher order in d.

A contraction estimate

Thanks to a careful decomposition of $\widetilde{\sigma} \in E_-$, we get:

Proposition:

There exists a constant $C_N = C_N(d_1, d_2)$ such that as $N \to \infty$

$$C_N \to \mathrm{e}^{-2d_1}(\mathrm{e}^{2d_2} - 2d_2) \doteq C(d_1, d_2) < 1$$

and that

$$\left\|\mathcal{B}_{2N}\widetilde{\sigma}\right\|_{\ell_1} \leq C_N \left\|\widetilde{\sigma}\right\|_{\ell_1}, \quad \widetilde{\sigma} \in E_-$$

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... then, iterate the estimate above:

For $h \geq 0$, $2h \leq t^n < 2(h+1)$, $\Delta t = N^{-1}$ one has

$$\|J_{\Delta x}(\cdot, t^n)\|_\infty \leq \frac{1}{2N} \mathrm{TV}\, \bar{J}_0 + (C_N)^h \left\|\widetilde{\boldsymbol{\sigma}}(0+)\right\|_{\ell_1}.$$

A similar estimate holds for $\|\rho_{\Delta x}(\cdot, t^n)\|_\infty$

IBVP with intermittent damping

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- On-Off damping: for some $0 < T_1 < T_2$ one has

$$\alpha(t) = \begin{cases} 1 & t \in [0, T_1), \\ 0 & t \in [T_1, T_2)\,. \end{cases}, \qquad \chi(t + T_2) = \chi(t) \quad \forall\, t > 0\,.$$

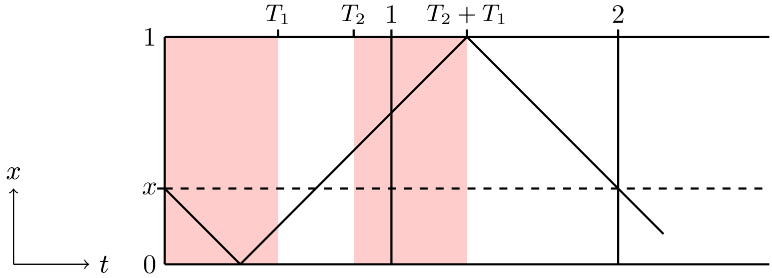

Figure: On-Off damping for some T_1 and T_2

Convergence of energy (Martinez, Vancostenoble, 2002)

Assume $k(x) \geq \bar{k} > 0$ and $g(J) = J$. Let $T_2 = qT_1$ with $2 \leq q \in \mathbb{N}$.

- If

$$T_1 \in \left\{ \frac{1}{q}, \cdots, \frac{q-1}{q} \right\}, \qquad T_1 > \frac{1}{q-1}, \quad q \geq 3\,.$$

Then there exists initial conditions for which the energy estimate remains constant with time: $E(t) = E(0) > 0$ for all $t \geq 0$.

- Otherwise the energy decays exponentially to 0 as $t \to \infty$.

Main result

Theorem Assume $T_2 - T_1$ is integer, let $k(x)$ satisfy

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and define

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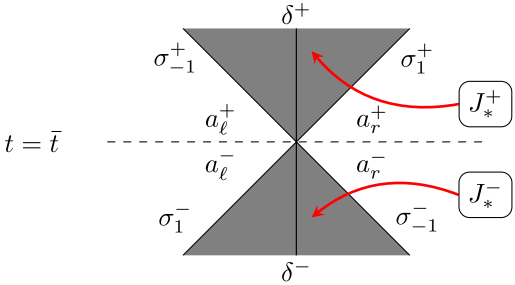
$$\|J(\cdot,t)\|_{L^\infty} \le C_1 \mathrm{e}^{-C_3 t} \left(\|J_0\|_{L^\infty} + \|\rho_0\|_{L^\infty} \right)$$
$$\|\rho(\cdot,t)\|_{L^\infty} \le C_2 \mathrm{e}^{-C_3 t} \left(\|J_0\|_{L^\infty} + \|\rho_0\|_{L^\infty} \right) .$$

where (ρ, J) are the exact solution for the problem.

Interactions

• We construct the WB approximate solutions as in the previous case for initial data in $BV(I)$.



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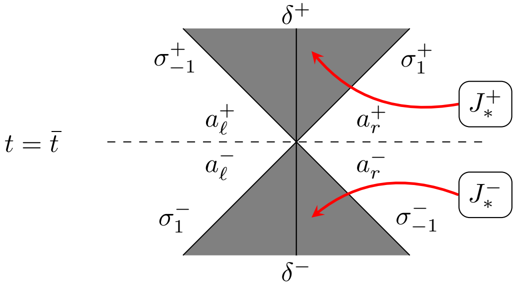

Wave sizes change: $c = \dfrac{g'(s)\delta}{g'(s)\delta + 1} \in [0,1)$

$$\begin{pmatrix}\sigma_{-1}\\ \sigma_1\end{pmatrix}^+ = \begin{pmatrix}1-c & c\\ c & 1-c\end{pmatrix}\begin{pmatrix}\sigma_{-1}\\ \sigma_1\end{pmatrix}^- + \frac{g(J_*^+)(\delta^+ - \delta^-)}{1+g'(s)\delta}\begin{pmatrix}-1\\ +1\end{pmatrix}$$

Interactions/2

Set $n_1 = 2\left[\frac{T_1}{2\Delta t}\right]$, $\quad n_2 = 2\left[\frac{T_2}{2\Delta t}\right]$, then

$$n_1 \Delta t \leq T_1 < (n_1 + 2)\Delta t\,, \qquad n_2 \Delta t \leq T_2 < (n_2 + 2)\Delta t\,.$$

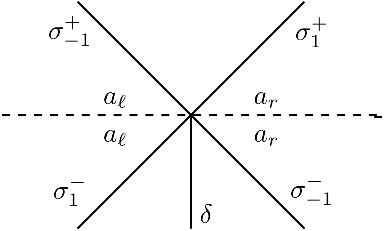

Figure: ON–OFF Interaction at time $n_1 \Delta t$.

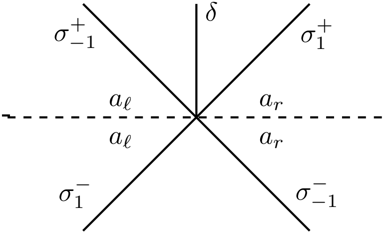

Figure: OFF–ON Interaction at time $n_2 \Delta t$.

Iteration

- Let $\boldsymbol{\sigma}(t^n+)\doteq\boldsymbol{\sigma}_n$, for $n=(h-1)n_2+i$, where $1\le h\in\mathbb{N}$ and $i=1,\cdots,n_2$, by iteration we have,

$$\boldsymbol{\sigma}_n=\begin{cases} B(\boldsymbol{c})^i\boldsymbol{\sigma}_{(h-1)n_2} & 1\le i<n_1 \\ B(\boldsymbol{c})^i\boldsymbol{\sigma}_{(h-1)n_2}+G_{(h-1)n_2+n_1} & i=n_1 \\ B(0)^{i-n_1}B(\boldsymbol{c})^{n_1}\boldsymbol{\sigma}_{(h-1)n_2}+B(0)^{i-n_0}G_{(h-1)n_2+n_1} & n_1<i<n_2 \\ B(0)^{n_2-n_1}B(\boldsymbol{c})^{n_1}\boldsymbol{\sigma}_{(h-1)n_2}+B(0)^{n_2-n_1}G_{(h-1)n_2+n_1}+G_{hn_2} & i=n_2 \end{cases}$$

where

$$G_n=\frac{(\delta^+-\delta^-)}{1+g'(s)\delta^-}\Big(0,-g(J^+_{*,1}),g(J^+_{*,1}),\cdots,-g(J^+_{*,N-1}),g(J^+_{*,N-1}),0\Big)^T.$$

Exponential formula

- The exponential formula for the matrix $B(c)^N$

Theorem:

Let $d > 0$ and $N \in 2\mathbb{N}$. Then

$$\left[B(0) + \frac{d}{N} B_1\right]^N = B(0)^N + (d)\widehat{P} + \sum_{j=0}^{N-1} \zeta_{j,N} B(0)^{2j} B_2(0) + \sum_{j=1}^{N-1} \eta_{j,N} B(0)^{2j}$$

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$$\sum_{j=0}^{N-1} \zeta_{j,N} \leq \sinh(d) - d + \frac{1}{N} f_0(d),$$

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Proposition:

For any $\boldsymbol{\sigma} \in \mathbb{R}^{2N}$ and for a suitable choice of v, the following holds true

$$\max_v \left|B(\boldsymbol{c})^N \boldsymbol{\sigma}_n \cdot v\right| \leq \frac{d}{2N}\left(1+\frac{d}{N}\right)^{-N} \|\boldsymbol{\sigma}\|_{\ell_1} + C_N(d) \max_v |\boldsymbol{\sigma}_n \cdot v| \ ,$$

where

$$C_N(d) \doteq \left(1+\frac{d}{N}\right)^{-N}\left[e^d - d \ + \ \frac{1}{N}[f_0(d)+f_1(d)]\right] \ ,$$

and $C_N(d) \to (1 - d\mathrm{e}^{-d})$ as $N \to \infty$.

Sketch of the proof/2

- **Then** by iteration, for all t^n with $n = (h-1)n_2 + i$, where $1 \leq h \in \mathbb{N}$ and $i = 1, \cdots, n_2$, there exist C_1 that depend only on the coefficients and on data, such that

$$\|J_{\Delta x}(\cdot, t^n)\|_\infty \leq C_1 \Delta x + (C_N)^h \max_v |\boldsymbol{\sigma}_0 \cdot v|$$

where

$$\max_v |\boldsymbol{\sigma}_0 \cdot v| = J_{max} \leq (\|J_0\|_{L^\infty} + \|\rho_0\|_{L^\infty})$$

- Estimates for the exact solution: we pass to the limit using density argument and this can be done for initial data in L^∞.

Work in progress

- **For the system**

$$\partial_t \rho + \partial_x J = 0\,, \quad \partial_t J + \partial_x \rho = -2k(x)g(J) \qquad x \in (0,1)$$

- Study the case of localized source: $k(x) > 0$ on a subinterval of $(0,1)$ and $k = 0$ otherwise, being $g' > 0$

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Thank you!