Controllability of multi-d scalar conservation laws in the entropy framework and with a simple geometrical condition.

Vincent Perrollaz

Institut Denis Poisson, Université de Tours

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Outline of the talk

1. The simplest toy example
2. Controllability to trajectories

The case of the transport equation

- Evolution :

$$\begin{cases} \partial_t y + c\partial_x y = 0, & (t,x) \in (0,T) \times (0,L) \\ y(t,0) = 0, & t \in (0,T). \end{cases}$$

- Method of characteristics $\Rightarrow$

$$y(t,x) = \begin{cases} y_0(x-ct) & \text{if } x > ct, \\ 0 & \text{otherwise.} \end{cases}$$

- Conclusion : $t > \frac{L}{c} \quad \Rightarrow \quad y(t,.) = 0.$

A Family of Lyapunov Functionals

- Definition (for $\nu > 0$) :

$$J_\nu(t) := \int_0^L y^2(t,x) e^{-\nu x} dx.$$

- Formally (at least) :

$$\begin{aligned} \dot{J}_\nu(t) &= \int_0^L 2 y_t(t,x) y(t,x) e^{-\nu x} dx \\ &= \int_0^L -2c y_x(t,x) y(t,x) e^{-\nu x} dx \\ &= [-c y^2(t,x) e^{-\nu x}]_0^L - c\nu J_\nu(t) \\ &\leq -c\nu J_\nu(t). \end{aligned}$$

- Gronwall $\Rightarrow$

$$J_\nu(t) \leq e^{-c\nu t} J_\nu(0).$$

Return to the L^2 norm

- Norm equivalence

$$\forall t \geq 0, \qquad e^{-\nu L}||y(t,.)||^2_{L^2(0,L)} \leq J_\nu(t) \leq ||y(t,.)||^2_{L^2(0,L)}.$$

- Inequality on L^2

$$||y(t,.)||^2_{L^2(0,L)} \leq e^{-\nu c(t-\frac{L}{c})}||y_0||^2_{L^2(0,L)},$$

- Conclusion :

$$t > \frac{L}{c}, \quad \nu \to +\infty \qquad \Rightarrow \qquad y(t,.) = 0$$

Remarks

- Can be adapted to general "transport" type equations.
- Good for robustness estimate and perturbation :

$$y_t + cy_x = \epsilon g(y),$$

$$y_t + cy_x = \epsilon y_{xx}.$$

(Systems and source term : Gugat, P., Rosier 2018)

- In certain cases, useful for exact controllability to trajectory.

Short tour on controllability and entropy solutions

- **Why :** Robustness, interesting trajectories, sampled controls, global results. . .
- **Results :** Ancona-Marson (98), Horsin (98), Bressan-Coclite (2002), Ancona-Coclite (2005), Glass-Guerrero (2007) Glass (2007, 2014), Leautaud (2012), Perrollaz (2012), Adimurthi-Ghoshal-Gowda (2014), Andreianov-Donadello-Ghoshal-Rasafizon (2015), Corghi-Marson (2016), Andreianov-Donadello-Marson (2017), Li-Yu (2017).
- **Techniques :**
  - No linearization possible!
  - Boundary conditions are tricky!
  - Lax-Hopf formula.
  - Generalized characteristics.
  - Wave front tracking.
  - Vanishing viscosity.
- **Restrictions :** 1d and convex flux (or genuinely nonlinear families for systems)

Kruzkov definition

Consider $\Omega \in \mathbb{R}^d$ a domain, and $f : \mathbb{R} \to \mathbb{R}^d$ a $\mathcal{C}^1$ function. A function $u \in L^\infty((0,+\infty) \times \Omega)$ is an entropy solution of

$$\begin{cases} \partial_t u + \mathrm{div}(f(u)) = 0 \\ u(0,x) = u_0(x), \end{cases} \quad , \quad x \in \Omega$$

when for any $k \in \mathbb{R}$ and any positive function $\phi \in \mathcal{C}_c^1(\mathbb{R} \times \Omega)$ we have

$$\int_{\mathbb{R}^+ \times \Omega} |u(t,x) - k| \partial_t \phi(t,x) + \mathrm{sign}(u(t,x) - k) \langle f(u(t,x)) - f(k) | \nabla \phi(t,x) \rangle dxdt$$
$$+ \int_\Omega |u_0(x) - k| \phi(0,x) dx \geq 0$$

A "simple" geometric condition

Definition

Let I be a segment of $\mathbb{R}$. We say that it satisfy **the replacement condition in time** $T > 0$ if there exists a vector $w \in \mathbb{R}^d$ and a positive number c such that

$$L := \sup_{x\in\Omega} \langle w|x\rangle - \inf_{x\in\Omega} \langle w|x\rangle < +\infty.$$

$$\forall u \in I, \qquad \langle f'(u)|w\rangle \geq c,$$

and $T = \frac{L}{c}$.

and its "simple" corresponding result

Theorem (Donadello, P.)

- *Let* $v \in L^\infty((0,+\infty)\times\Omega)$ *be an entropy solution to*

$$\partial_t u + \mathrm{div}(f(u)) = 0$$

  and u_0 *be a function in* $L^\infty(\Omega)$.
- *Suppose that both* u_0 *and* v *take value in a segment* I *such that* Ω, I *and* f *satisfy the replacement condition in time* T.
- *Then for any times* T_1 *and* T_2 *greater than* T *we have an entropy solution* u *of the previous equation satisfying*

$$u(0,x) = u_0(x), \qquad u(T_1,x) = v(T_2,x) \qquad \textit{for almost all } x\in\Omega.$$

Idea of the proof

- Kruzkov formulation with boundary condition (Amar, Carillo, Wittbold 2006).
- Trace result of Vasseur for the boundary of the reference trajectory (Vasseur 2001).
- Doubling variable trick of Kruzkov with test function (Kruzkov 1970).
- Conclusion with Lyapunov functions.

Remarks

- No technical restriction on the flux (compared with the Cauchy-Problem).
- 1-d or n-d not different.
- Many reusable steps for the proof.
- Geometric condition clearly too restrictive, see controllability of Euler for further ideas?

THANK YOU FOR YOUR ATTENTION