Initial data identification.

Vincent Perrollaz

Institut Denis Poisson, Université de Tours

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Outline of the talk

1. Entropy solutions crash course
   - Generalities
   - Blow up
   - Riemann Problem

2. Initial data identification : statement of the problem

3. Necessity

4. Sufficiency

Conservation laws

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{d} \frac{\partial f_{i,j}(u_1, \ldots, u_n)}{\partial x_j} = 0, \qquad 1 \leq i \leq n.$$

- Many physical systems : gas dynamics, magneto-hydrodynamic, electromagnetism, shallow water theory . . .
- Scalar case : fewer physical systems (still : traffic, crowd dynamics, petroleum engineering) but first important step toward systems.
- Second order terms neglected.

Outline of the theory

- Small time solutions in H^s.
- Finite time blow up for most smooth initial data.
- Weak solutions in L^∞ but no more uniqueness.
- Entropy solutions : taking the forgotten terms into account.
- No more reversibility in time!
- No linearization!
- No fixed point!

Bibliography for the Cauchy problem

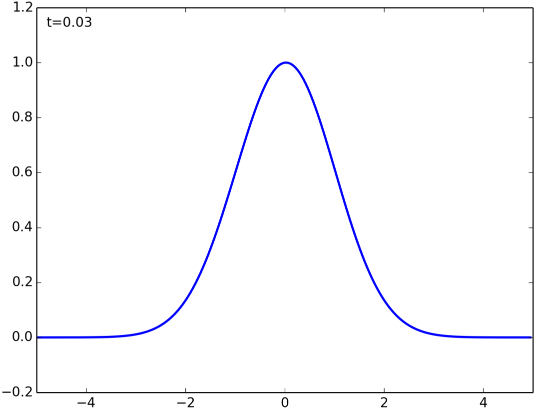
- Scalar : 1-d Oleinik (59), n-d Kruzkov (70).
- Systems 1-d : Lax (57), Glimm (65), Bressan (92. . .).
- Vanishing viscosity for systems : Bianchini-Bressan (05).
- Boundary conditions : Bardos-Leroux-Nedelec (77), Otto (95). . .
- General case : **OPEN**

Method of Characteristics : a synthetic slide

$$
\begin{aligned}
&\partial_t u + \partial_x (f(u)) = 0 \\
&\iff \partial_t u + f'(u)\partial_x u = 0 \\
&\iff \begin{cases} \frac{d}{dt}\psi(t,x) = f'(u(t,\psi(t,x))), & \psi(0,x) = x \\ \frac{d}{dt}u(t,\psi(t,x)) = 0 \end{cases} \\
&\iff \begin{cases} \frac{d}{dt}\psi(t,x) = f'(u(t,\psi(t,x))), & \psi(0,x) = x \\ u(t,\psi(t,x)) = u_0(x) \end{cases} \\
&\iff \begin{cases} \frac{d}{dt}\psi(t,x) = f'(u_0(x)), \\ u(t,\psi(t,x)) = u_0(x) \end{cases} \\
&\iff \begin{cases} \psi(t,x) = x + tf'(u_0(x)), \\ u(t,\psi(t,x)) = u_0(x) \end{cases} \\
&\iff u(t, x + tf'(u_0(x))) = u_0(x).
\end{aligned}
$$

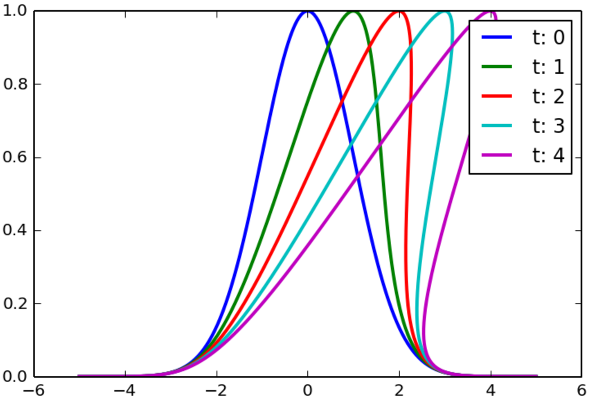
Multi valued solutions

- Let the plane evolve according to $(x, y) \mapsto (x + tf'(y), y)$.
- Look at the evolution of the graph of u.



Alternative : gradient blow up

$$\partial_t(u_x) + f'(u)\partial_x(u_x) = -(u_x)^2.$$



Riemann initial data

- Simplest discontinuity and invariance by $x \mapsto x + \eta$:

$$u_0(x) := \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

- Invariance by $(t, x) \mapsto (\lambda t, \lambda x) \Rightarrow$

$$u(t, x) = v\left(\frac{x}{t}\right)$$

- Simplest case :

$$u(t, x) = \begin{cases} u_l & \text{if } x < pt \\ u_r & \text{if } x > pt \end{cases}$$

- What is p?

Rankine Hugoniot condition for weak solution

- Integral formulation :

$$\frac{d}{dt}\int_a^b u(t,x)dx = f(u(a)) - f(u(b)).$$

- Integrating on the box between the points $(0,0)$ and $(T, pT) \Rightarrow$

$$pTu_l - pTu_r = Tf(u_l) - Tf(u_r),$$

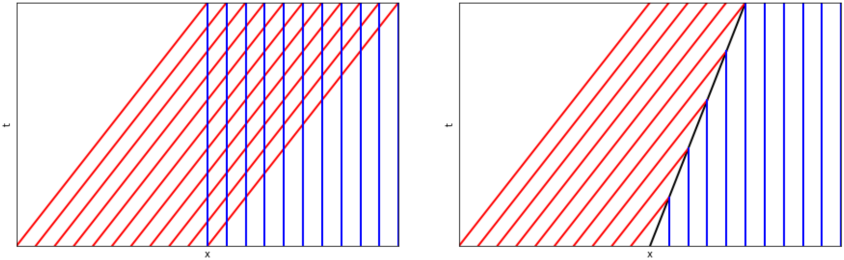
- Rankine-Hugoniot condition

$$p = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

- Can be localized with u_r and u_l smooth solutions and a moving discontinuity.

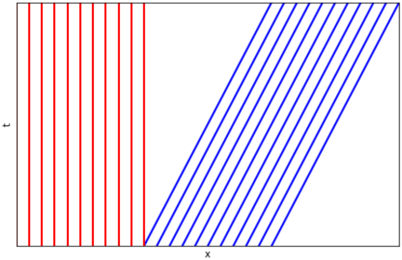
Characteristics : shock wave

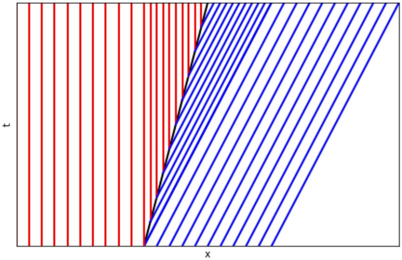
$$u_0(x) = \begin{cases} 1.0 & \text{if } x < 0 \\ 0.0 & \text{if } x > 0. \end{cases} \quad \Rightarrow \quad u(t,x) = \begin{cases} 1.0 & \text{if } x < \frac{t}{2} \\ 0 & \text{if } x > \frac{t}{2} \end{cases}$$



Characteristics : rarefaction wave

$$u_0(x) = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases} \quad \Rightarrow \quad u(t,x) = \begin{cases} 0 & \text{if } x < \frac{t}{2} \\ 1.0 & \text{if } x > \frac{t}{2} \end{cases}$$





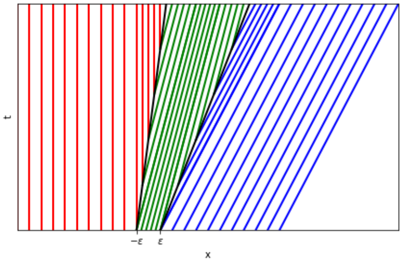
Entropy condition

- For $\epsilon > 0$ consider

$$u_0^\epsilon(x) = \begin{cases} u_l & \text{if } x < -\epsilon \\ u_m & \text{if } -\epsilon < x < \epsilon \\ u_r & \text{if } \epsilon < x \end{cases}$$

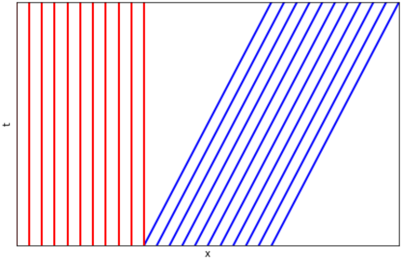
- Let $\epsilon \to 0$, admissibility condition on discontinuities :

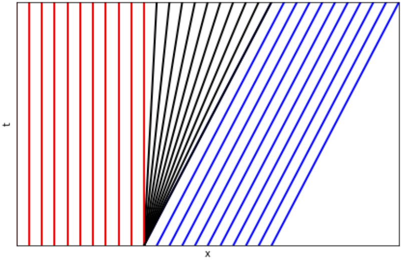
$$\forall u_m, \quad \frac{f(u_l) - f(u_m)}{u_l - u_m} \geq \frac{f(u_l) - f(u_r)}{u_l - u_r} \geq \frac{f(u_m) - f(u_r)}{u_m - u_r}$$



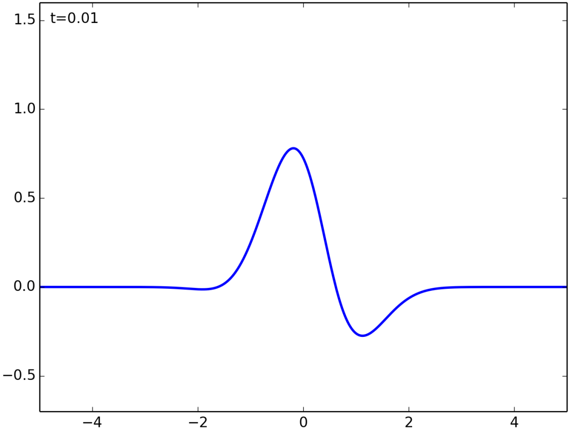
Characteristics : rarefaction wave the return

$$u_0 = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases} \quad \Rightarrow \quad u(t,x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1.0 & \text{if } x > t \end{cases}$$





Simulations



Statement

For $T > 0$, $w \in L^\infty(\mathbb{R})$ define :

$$I_T(w) := \{u_0 \in L^\infty(\mathbb{R}) \ : \ S_T^{CL} u_0 = w\}.$$

S_t^{CL} : entropy solutions semigroup.

- $I_T(w) \neq \emptyset$ iff w?
- $u_0 \in I_T(w)$ iff u_0?
- Geometric description of $I_T(w)$?

Motivation

1. Sonic boom plane design (see Gosse-Zuazua 2017)
2. Traffic flow.

A simple result

To $T > 0$, $w : \mathbb{R} \mapsto \mathbb{R}$, associate

$$p : x \mapsto x - Tf'(w(x)).$$

Theorem (Colombo, P.)

Let f be a $\mathcal{C}^2$, uniformly convex flux and $T > 0$. $I_T(w)$ is non empty iff p is nondecreasing.

Remark

- *"There exists a representative such that..."*
- *Actually just Oleinik's inequality.*
- $\Rightarrow w \in \mathrm{BV}(\mathbb{R})$.

Geometric Properties

Theorem (Colombo, P.)

Given $T > 0$ and $f \in \mathcal{C}^2$ uniformly convex. If $w \in \mathrm{SBV}(\mathbb{R})$ and $I_T(w) \neq \emptyset$ then.

- *$I_T(w)$ is a singleton iff $w \in \mathcal{C}^0(\mathbb{R})$.*
- *Otherwise $I_T(w)$ is a convex cone with the isentropic solution for only vertex.*
- *$I_T(w)$ does not have finite dimensional facets.*

In all cases $I_T(w)$ is always closed for the L^1_{loc} topology.

Necessary and sufficient condition

Theorem (Colombo, P.)

Given $T > 0$ and $f \in \mathcal{C}^2$ uniformly convex. Let w be in $\mathrm{SBV}_{loc}(\mathbb{R})$ such that p is nondecreasing. Then $U \in \mathrm{Lip}(\mathbb{R})$ satisfy $\partial_x U \in I_T(w)$ iff

- *for all point x such that p is differentiable in x and $p'(x) \neq 0$*

$$\lim_{y \to x} \frac{U(p(y)) - U(p(x))}{p(y) - p(x)} = w(x).$$

- *for all point x such that $w(x^-) \neq w(x^+)$ then $\forall y \in]p(x^-), p(x^+)[$*

$$\begin{cases} \frac{U(p(x^+)) - U(y)}{T} \leq f^* \left(\frac{x-y}{T}\right) - f^* \left(\frac{x-p(x^+)}{T}\right) \\ \frac{U(p(x^-)) - U(y)}{T} \leq f^* \left(\frac{x-y}{T}\right) - f^* \left(\frac{x-p(x^-)}{T}\right) \end{cases}$$

Tool : generalized characteristics

- Extension of the method of characteristics to low regularity (L^∞).
- Differential inclusion in the sense of Filippov (1960) (Existence but no uniqueness)
- Developped by Dafermos (1977)
- Lots of "true" characteristics $\Rightarrow$ good estimates on u.

Definition

$t \mapsto \gamma(t)$ AC generalized characteristics when

$$\dot{\gamma}(t) \in [f'(u(t,\gamma(t)^+)), f'(u(t,\gamma(t)^-))] \quad dt \ p.p.$$

A first simple case

- Target state :

$$\forall x \in \mathbb{R}, \quad w(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0. \end{cases}$$

- First obvious remark

$$\forall T > 0, \quad w \in I_T(w) \Rightarrow I_T(w) \neq \emptyset.$$

Backward propagation and key zone

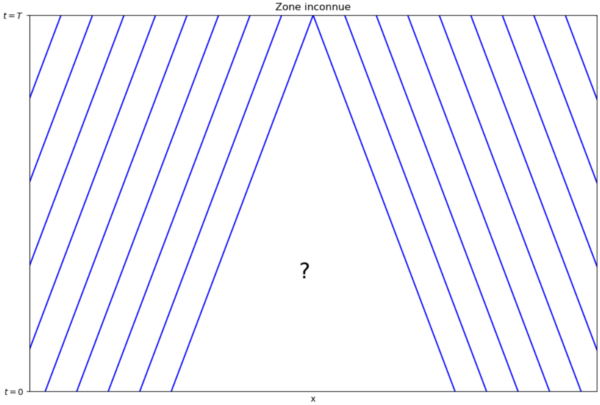
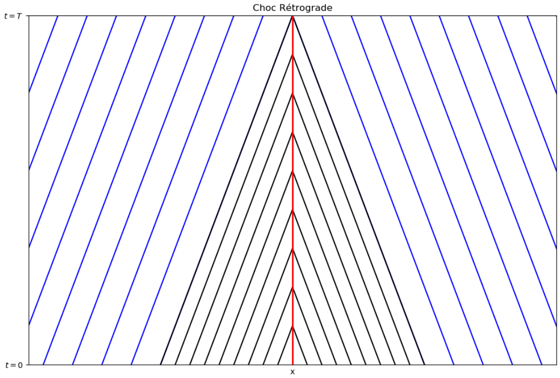
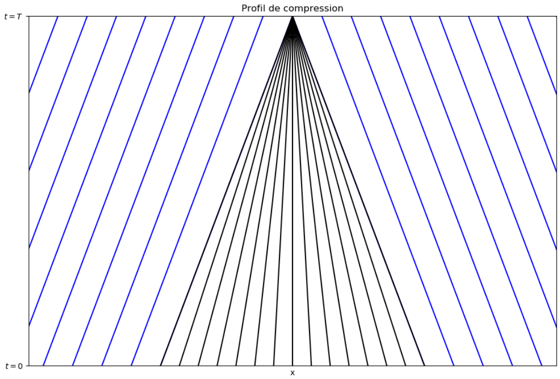


Figure :

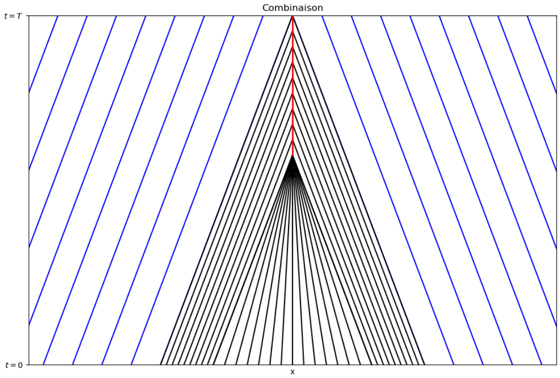
Backward shock



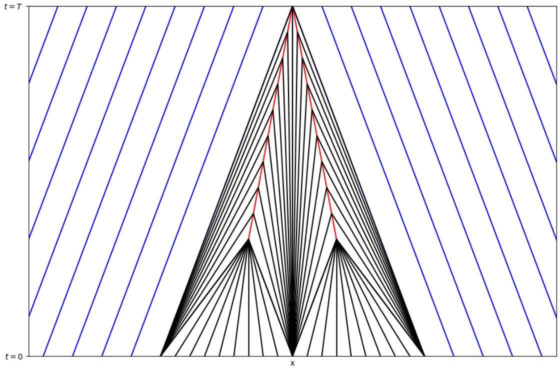
Compressive profile



"Combo"



"Double Rarefaction"



SCAM

Localization

Theorem (Colombo, P.)

If $w \in \mathrm{SBV}(\mathbb{R})$ is such that p is nondecreasing, we define

$$X_i := p\left(\{x \in \mathbb{R} \ : \ p \quad \textit{is differentiable at } x \textit{ and} \quad p'(x) \neq 0\}\right)$$

$$X_{ii} := \bigcup_{x\in\mathbb{R}}]p(x^-), p(x^+)[,$$

we have $\mathbb{R} \setminus (X_i \cup X_{ii})$ is negligible.

Lax-Hopf formula

u is an entropy solution of

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0) = u_0. \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} u(t,x) = g\left(\frac{x-y(t,x)}{t}\right) \\ y(t,x) = \operatorname{argmin}\left(tf^*\left(\frac{x-y}{t}\right) + \int_0^y u_0(z)dz\right) \end{cases}$$

- $g = f'^{-1}$ and f^* Legendre transform of f.
- f must be convex!

HJB Connection

- $U_0 \in \mathrm{Lip}(\mathbb{R})$,
- U viscosity solution of

$$\begin{cases} \partial_t U + f(\partial_x U) = 0 \\ U(0) = U_0 \end{cases} \tag{HJB}$$

- u entropy solution of

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0) = \partial_x U_0 \end{cases} \tag{CL}$$

- conclusion :

$$\forall t > 0, \quad \partial_x U(t) = u(t).$$

Perspectives

1. Numerical schemes.
2. Apriori constraints for traffic flow.
3. Space dependency and source term.
4. Nonconvex flux.
5. Multi dimensional case.
6. System case.

THANK YOU FOR YOUR ATTENTION