On the controllability of the wave equation with a second order memory term

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VIII Partial differential equations, optimal design and numerics
Benasque, 18-30 Aug, 2019

Joint work with Umberto Biccari

The wave equation

Given $\Omega \subset \mathbb{R}^N$, $\omega \subset \Omega$ open sets, $T > 0$ and (u^0, u^1), find a function $h \in L^2((0,T) \times \omega)$ such that the solution (u, u') of

$$\begin{cases} u''(t,x) - \Delta u(t,x) = \mathbf{1}_\omega h(t,x) & (t,x) \in Q = (0,T) \times \Omega \\ u(t,x) = 0 & x \in \partial\Omega,\ t \in (0,T) \\ u(0,x) = u^0(x) & x \in \Omega \\ u'(0,x) = u^1(x) & x \in \Omega, \end{cases} \tag{1}$$

verifies

$$u(T,x) = u'(T,x) = 0 \quad x \in \Omega. \tag{2}$$

- $h(t,x)$ is a localized control which acts on ω.
- The solution reaches the equilibrium at $t = T$ (and remains there if the control is zero for $t > T$).

The wave equation

Definition

Equation (1) is **observable in time** T if there exists a positive constant $C > 0$ such that the following inequality is verified

$$C \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \int_0^T \int_\omega \mid \varphi \mid^2 dxdt, \tag{3}$$

for any $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ where φ is the solution of

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,\cdot) = \varphi^0_T,\ \varphi'(T,\cdot) = \varphi^1_T & \text{in } \Omega. \end{cases} \tag{4}$$

Under hypothesis (3), equation (1) is controllable in time T.

The wave equation

In 1-d case $\Omega = \mathbb{T}$, (3) is reduced to an Ingham type inequality.

Theorem

(Ingham) Let $(\lambda_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ *and* $\gamma > 0$ *be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{5}$$

For any real T *with*

$$T > \pi/\gamma \tag{6}$$

there exists $C_1 = C_1(T, \gamma) > 0$ *such that, for any finite* $(a_n)_{n\in\mathbb{Z}}$*,*

$$C_1 \sum_{n\in\mathbb{Z}} | a_n |^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt. \tag{7}$$

A. E. Ingham, Some trigonometrical inequalities with applications in the theory of series, Math. Z., 41 (1936), 367-379.

Memory term

$$
\begin{cases}
u''(t,x) - \Delta u(t,x) \\
\quad + \int_0^t M(t,s)\Delta u(s,x)\,\mathrm{ds} = \mathbf{1}_\omega h(t,x) & (t,x) \in Q \\
u(0,x) = u^0(x) & x \in \Omega \\
u'(0,x) = u^1(x) & x \in \Omega,
\end{cases}
\tag{8}
$$

$\int_0^t M(t,s)\Delta u(s,x)\,\mathrm{ds}$ is a memory term. In a material with memory the state at a point x at time t depends on the entire history of the state at x. This type of behavior may be encountered in viscoelastic materials.

Materials with memory

- As it can be deduced from their name, viscoelastic materials possess both viscous and elastic properties in varying degrees. For a viscoelastic material, internal stress is a function not only of the instantaneous deformation, but also depends on the whole past history of deformation.
- Some examples are the polymer solutions and melts, concentrated suspensions or emulsions, asphalts, lubricating greases, muds ("soft solids" or "semi solids").
- For real materials, the most recent past history has much more influence. This is the reason why these materials may be described as having fading memory.
- Linear viscoelasticity is the simplest response of a viscoelastic material when it is submitted to deformations or stresses small enough so that its rheological functions do not depend on the value of the deformation or stress.

The controllability problem

Controllability problem: find $h \in L^2((0,T)\times\omega)$ such that

$$\begin{cases} u(T,x) = u'(T,x) = 0 & x \in \Omega \\ \displaystyle\int_0^T M(T,s)\Delta u(s,x)\,\mathrm{ds} = 0 & x \in \Omega. \end{cases} \tag{9}$$

We put $z(t,x) = \displaystyle\int_0^t M(t,s)\Delta u(s,x)\,\mathrm{ds}$ and the system is equivalently written as:

$$\begin{cases} u'(t) = v(t) \\ v'(t) = \Delta u(t) - z(t) - \mathbf{1}_\omega h(t) \\ z'(t) = M(t,t)\Delta u(t) + \displaystyle\int_0^t M_t(t,s)\Delta u(s)\,\mathrm{ds} \\ u(0) = u^0 \\ v(0) = u^1 \\ z(0) = 0 \\ u(T) = v(T) = z(T) = 0. \end{cases} \tag{10}$$

Adjoint

The adjoint problem:

$$\begin{cases} \varphi'(t) = \psi(t) \\ \psi'(t) = \Delta\varphi(t) - \Delta\zeta(t) \\ \zeta'(t) = -M(t,t)\varphi(t) + \displaystyle\int_t^T M_t(s,t)\varphi(s)\,\mathrm{ds} \\ \varphi(T) = \varphi^0 \quad \psi(T) = \varphi^1 \quad \zeta(T) = \zeta^0, \end{cases} \tag{11}$$

or, equivalently,

$$\begin{cases} \varphi''(t) - \Delta\varphi(t) & \\ \quad + \displaystyle\int_t^T M(s,t)\Delta\varphi(s)\,\mathrm{ds} + M(T,t)\Delta\zeta^0 = 0 & \text{in } Q \\ \varphi(T) = \varphi^0 \quad \varphi'(T) = \varphi^1 & \text{in } \Omega. \end{cases}$$

- P. Loreti, L. Pandolfi, and D. Sforza, *Boundary controllability and observability of a viscoelastic string*, SIAM J. Control Optim., Vol. 50 (2012), pp. 820-844.
- F. W. Chaves-Silva, L. Rosier, and E. Zuazua, *Null controllability of a system of viscoelasticity with a moving control*, J. Math. Pures Appl., Vol. 101, No. 9 (2014), pp. 198-222.
- Q. Lü, X. Zhang, and E. Zuazua, *Null controllability for wave equations with memory*, J. Math. Pures Appl., Vol. 108 (2017), pp. 500-531.
- M. Renardy, D. Maity and D. Mitra, *Lack of null controllability of viscoelastic flows*, ESAIM: COCV, (2018).
- U. Biccari and S. M., *Null-controllability properties of the wave equation with a second order memory term*, J. Diff. Eqs., Vol. 267 (2019), pp. 1376-1422.

A simplified model

To simplify, we take $N = 1$ and $M(t,s) = M =$ constant. The adjoint problem becomes:

$$\begin{cases} \varphi''(t) - \varphi_{xx}(t) + M \displaystyle\int_t^T \varphi_{xx}(s)\,\mathrm{ds} + M\zeta^0_{xx} = 0 \\ \varphi(T) = \varphi^0 \\ \varphi'(T) = \varphi^1. \end{cases} \tag{12}$$

We denote by $\mathcal{A}$ the corresponding differential operator.

Given $x_0 \in \mathbb{R}^*$, we show that there exists "approximate" solutions of (12) concentrated along a vertical ray (t, x_0), which implies that there are unobservable solutions from any ω not containing the point x_0.

Concentrated solutions

Theorem

For any $x_0 \in \mathbb{R}^*$ *and* $\varepsilon > 0$, *let* $p^\varepsilon(t,x) := \varepsilon^{\frac{7}{8}} e^{\frac{i}{\varepsilon}x - \frac{1}{\sqrt{\varepsilon}}(x-x_0)^2 + Mt - M^3\varepsilon^2 t}$.

1. *The* p^ε *are approximate solutions to* (12) *for* $q^0(x) := \frac{p^\varepsilon(0,x)}{M - M^3\varepsilon^2}$.
2. *The initial energy of* p^ε *satisfies*

$$E^\varepsilon(0) := E(p^\varepsilon(0,\cdot)) = 1 + \mathcal{O}(\sqrt{\varepsilon}). \quad (13)$$

3. *The energy is exponentially small off the vertical ray* (t, x_0):

$$\int_{|x-x_0|>\varepsilon^{\frac{1}{8}}} \left(|p_x^\varepsilon|^2 + |p_t^\varepsilon|^2\right) \mathrm{dx} = \mathcal{O}\left(e^{-2\varepsilon^{-\frac{1}{4}}}\right). \quad (14)$$

J. Ralston: Gaussian beams and the propagation of singularities, Studies in PDE, MAA Studies in Mathematics 23, 206-248 (1983)

Spectral Analysis

Theorem

If $\Omega = \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ *(one-dimensional torus) the spectrum of the adjoint operator* $\mathcal{A}$ *is* $\sigma(\mathcal{A}) = \left(\mu_n^j\right)_{n\in\mathbb{N}^*,\,1\leq j\leq 3} \cup \{M\}$*, where the eigenvalues* μ_n^j *verify*

$$\begin{cases} \mu_n^1 = M - \dfrac{M^3}{n^2} + \mathcal{O}\left(\dfrac{1}{n^4}\right), & n \in \mathbb{N}^* \\ \mu_n^2 = -\dfrac{M}{2} + \dfrac{M^3}{2n^2} + in + i\,\dfrac{3M^2}{8n} + \mathcal{O}\left(\dfrac{1}{n^3}\right), & n \in \mathbb{N}^* \\ \mu_n^3 = \bar{\mu}_n^2, & n \in \mathbb{N}^*. \end{cases} \qquad (15)$$

Spectral Analysis

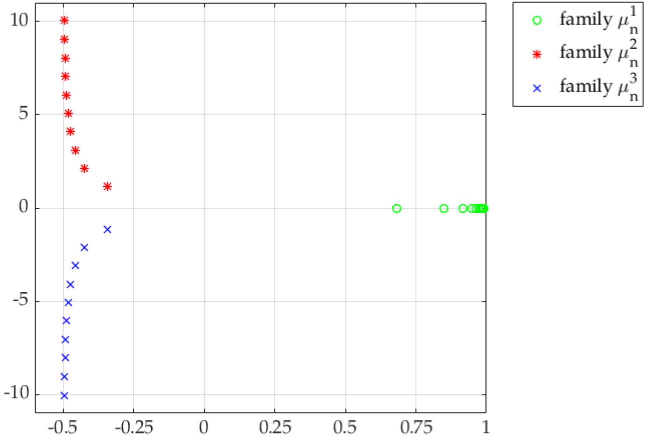

Figure: Distribution of the eigenvalues μ_n^j for $n \in \{1, \ldots, 10\}$ and $j \in \{1, 2, 3\}$, corresponding to $M = 1$. The accumulation of the family μ_n^1 zeros at M appears.

Moving control

$$\begin{cases} u''(t) - u_{xx}(t) + M \int_0^t u_{xx}(s)\,\mathrm{ds} = \mathbf{1}_{\omega(t)} h(t,x) \\ u(0) = u^0 \\ u'(0) = u^1. \end{cases} \tag{16}$$

In (16) the memory enters in the principal part, and the control is applied on an open subset $\omega(t)$ of the domain $\mathbb{T}$ where the waves propagate. The support $\omega(t)$ of the control u at time t moves in space with a constant velocity c, that is,

$$\omega(t) = \omega_0 - ct,$$

with $\omega_0 \subset \mathbb{T}$ a reference set, open and non empty. The control $h \in L^2(\mathcal{O})$ is then an applied force localized in $\omega(t)$, where

$$\mathcal{O} := \Big\{ (t,x) \,\Big|\, t \in (0,T), x \in \omega(t) \Big\}.$$

Change of variable

Let us now consider the change of variables

$$x \mapsto x + ct. \tag{17}$$

The parameter c in (17) is a constant velocity which belongs to $\mathbb{R} \setminus \{-1, 0, 1\}$. Easy computations show that (16) is equivalent to:

$$\begin{cases} \xi_{tt} - (1-c^2)\xi_{xx} + 2c\xi_{xt} + M\zeta = \mathbf{1}_{\omega_0}\tilde{h}(t,x), & (t,x) \in Q \\ \zeta_t + c\zeta_x = \xi_{xx}, & (t,x) \in Q \\ \xi(0) = u^0, \;\; \xi_t(0) = u^1 - cu^0_x, & x \in \mathbb{T} \\ \zeta(0) = 0, & x \in \mathbb{T}. \end{cases} \tag{18}$$

Remark: The control has now a fixed support ω_0.

P. Martin, L. Rosier, and P. Rouchon, *Null controllability of the structurally damped wave equation with moving control*, SIAM J. Control Optim., Vol. 51 (2013), pp. 660-684.

Spectral analysis of the new operator

Theorem

Let $c \in \mathbb{R} \setminus \{-1, 0, 1\}$ *and* $\mathcal{S} := \{(n, j) \,:\, j \in \{1, 2, 3\},\ n \in \mathbb{Z}^*\}$.
The spectrum of the adjoint operator $\mathcal{A}_c$ *is*

$$\sigma(\mathcal{A}_c) = \left(\lambda_n^j\right)_{(n,j)\in\mathcal{S}}, \tag{19}$$

where the eigenvalues λ_n^j *are defined as follows*

$$\lambda_n^j = icn + \mu_{|n|}^j, \qquad (n, j) \in \mathcal{S}. \tag{20}$$

and $\mu_{|n|}^j$ *are given in Theorem 2.*

Spectral analysis of the new operator

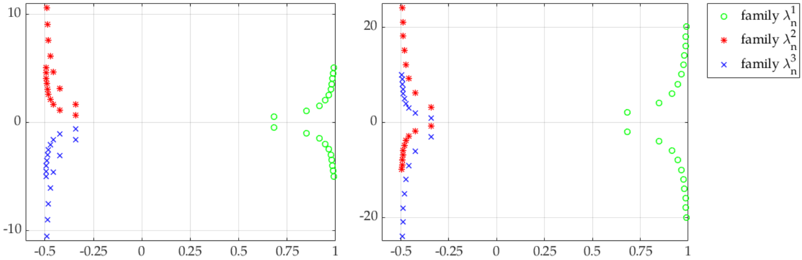

Figure: tiny Distribution of the eigenvalues λ_n^j for $n \in \{1, \ldots, 10\}$ and $j \in \{1, 2, 3\}$, corresponding to $M = 1$ and $c = 0.5$ (left) and $c = 2$ (right).

Spectral analysis of the new operator

Theorem

Each eigenvalue $\lambda_n^j \in \sigma(\mathcal{A}_c)$ *has an associated eigenvector*

$$\Psi_n^j = \begin{pmatrix} 1 \\ -\lambda_n^j \\ \dfrac{1}{\lambda_n^j - icn} \end{pmatrix} e^{inx}, \qquad (n,j) \in \mathcal{S}. \tag{21}$$

For any $\sigma \geq 0$, *the set* $\left(|n|^\sigma \Psi_n^j\right)_{(n,j)\in\mathcal{S}}$ *forms a Riesz basis in the spaces* $X_{-\sigma} := H_p^{-\sigma}(\mathbb{T}) \times H_p^{-\sigma-1}(\mathbb{T}) \times H_p^{-\sigma}(\mathbb{T})$.

$$L_p^2(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} \,\middle|\, \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{dx} < +\infty, \int_{-\pi}^{\pi} f(x)\,\mathrm{dx} = 0 \right\},$$

$$H_p^\sigma(\mathbb{T}) := \left\{ f \in H_p^{\sigma-1} \,\middle|\, \int_{-\pi}^{\pi} |f^{(\sigma)}(x)|^2 \,\mathrm{dx} < +\infty \right\}, \qquad H_p^{-\sigma}(\mathbb{T}) = \left(H_p^\sigma(\mathbb{T})\right)'.$$

Spectral analysis of the new operator

All the elements of the spectrum $\sigma(\mathcal{A}_c) = \left(\lambda_n^j\right)_{(n,j)\in S}$ are well separated one from another, i. e.

$$\left|\lambda_n^j - \lambda_m^k\right| \geq \gamma > 0,$$

except for the following special cases:

1. If $c \in (0,1)$, for each m large enough there exists n_m such that eigenvalues λ_m^2 and $\lambda_{-n_m}^2$ have a distance of order $\frac{1}{m^2}$ between them and a similar relation holds for λ_m^3 and $\lambda_{-n_m}^3$.
2. If $c \in (1,\infty)$, for each m large enough there exists n_m such that eigenvalues λ_m^2 and $\lambda_{n_m}^3$ have a distance of order $\frac{1}{m^2}$ between them and a similar relation holds for λ_{-m}^3 and $\lambda_{-n_m}^2$.
3. If $c \in \mathcal{V} := \left\{\sqrt{1+3\left(\frac{\mu_n^1}{2n}\right)^2} \ : \ n \in \mathbb{N}^*\right\}$, there exists a unique double eigenvalues $\lambda_{-n_c}^2 = \lambda_{n_c}^3$.

Even though the asymptotic gap between the elements of the spectrum is zero, the fact that we know the velocities with which the distances between these eigenvalues tend to zero allow us to estimate the norm of the biorthogonal to the family of exponential functions $\left(e^{-\lambda_n^j t}\right)_{(n,j)\in S}$ and prove the following result:

Theorem

Let $c \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $T > \pi\left(\frac{1}{|c|} + \frac{1}{|c-1|} + \frac{1}{|c+1|}\right)$. For any finite sequence of scalars $(a_n^j)_{(n,j)\in S} \subset \mathbb{C}$, it holds the inequality

$$\sum_{(n,j)\in S} \frac{\left|a_n^j\right|^2}{n^4} \leq C \left\| \sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t} \right\|^2_{L^2(-T,T)}. \qquad (22)$$

Proof's ideas (I)

- Consider the infinite product

$$P(z) = z^3 \prod_{(n,j)\in S} \left(1 + \frac{z}{i\overline{\lambda}_n^j}\right), \tag{23}$$

define the entire function

$$\widehat{\theta}_m^j(z) = \frac{P(z)}{P'(-i\overline{\lambda}_m^j)}, \tag{24}$$

and let θ_m^j be its inverse Fourier transform,

$$\theta_m^j = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\theta}_m^j(x) e^{ixt} \, \mathrm{dx}. \tag{25}$$

- P is an entire function of exponential type $T' = \frac{\pi}{|c|} + \frac{\pi}{|c-1|} + \frac{\pi}{|c+1|}$, thus $\theta_m^j \in L^2(-T', T')$ is a biorthogonal sequence to the family $\left(e^{-\lambda_n^j t}\right)_{(n,j)\in S}$ in $L^2\left(-T', T'\right)$ such that

$$\|\theta_m^j\|_{L^2(-T',T')} \leq C\, m^2 \qquad ((m,j)\in S).$$

Proof's ideas (II)

- For $T > T'$, construct a new biorthogonal sequence $\left(\theta_m^k\right)_{(m,k)\in S}$ to the family $\left(e^{-\lambda_n^j t}\right)_{(n,j)\in S}$ in $L^2(-T,T)$ with the property that

$$\left\| \sum_{(m,k)\in S} \beta_m^k \theta_m^k \right\|_{L^2(-T,T)}^2 \leq C \sum_{(m,k)\in S} m^4 \left|\beta_m^k\right|^2, \qquad (26)$$

for any finite sequence of complex numbers $\left(\beta_m^k\right)_{(m,k)\in S}$.

J.-P. Kahane, *Pseudo-périodicité et séries de Fourier lacunaires*, Ann. Sci. École Norm. Sup., Vol. 79, No. 3 (1962), pp. 93-150.

Proof's ideas (III)

- The orthogonality properties of $\left(\theta_m^k\right)_{(m,k)\in S}$ and (26) imply that

$$\sum_{(n,j)\in S} \frac{\left|a_n^j\right|^2}{n^4} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{\left(\sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t}\right)} \left(\sum_{(m,k)\in S} \frac{a_m^k}{m^4}\theta_m^k(t)\right) \mathrm{dt}$$

$$\leq \left\|\sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t}\right\|_{L^2(-T,T)} \left\|\sum_{(m,k)\in S} \frac{a_m^k}{m^4}\theta_m^k\right\|_{L^2(-T,T)}$$

$$\leq \sqrt{C}\left\|\sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t}\right\|_{L^2(-T,T)} \sqrt{\sum_{(n,j)\in S} \frac{\left|a_n^j\right|^2}{n^4}}.$$

The main controllability result

Theorem

Let $M \neq 0$, $c \in \mathbb{R} \setminus \{-1, 0, 1\}$, $T > 2\pi \left(\frac{1}{|c|} + \frac{1}{|1-c|} + \frac{1}{|1+c|} \right)$, ω_0 *a nonempty open set in* $\mathbb{T}$ *and*

$$\omega(t) = \omega_0 - ct, \qquad t \in [0, T].$$

For each initial data $(u^0, u^1) \in H_p^3(\mathbb{T}) \times H_p^2(\mathbb{T})$ *there exists a control* $h \in L^2(\mathcal{O})$ *verifying*

$$\int_{-\pi}^{\pi} \mathbf{1}_{\omega(t)} h(t, x) \, \mathrm{dx} = 0 \qquad t \in (0, T), \tag{27}$$

such that the solution (u, u') *of* (16) *satisfies*

$$\begin{cases} u(T, x) = u'(T, x) = 0 & x \in \mathbb{T} \\ \int_0^T u_{xx}(s, x) \, \mathrm{ds} = 0 & x \in \mathbb{T}. \end{cases} \tag{28}$$

The end

Thank you very much for your attention!