Non local models based on hyperelasticity.

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Index

Classical non linear elasticity

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Let $u(x, t)$ be the position occupied by the material point $x \in \Omega$ at time t. $u : \Omega \times (0, T) \to \mathbb{R}^3$.

The deformation gradient is the differential of u with respect to x, $F = Du$. In components, $F_{i\alpha} = u_{i,\alpha} = \frac{\partial u_i}{\partial x_\alpha}$.

Cauchy's equation of motion

After imposing second Newton's law, it is obtained:

$$(\rho_R \dot{v}) = Div\ T_R + \rho_R b$$

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Density · acceleration.

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Piola-Kirchhoff
stress tensor.

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External forces.

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- Hyperelastic materials: There exists an energy function $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that $T_R = D_F W(F)$.
- Equilibrium equations ($\star$) as Euler-Lagrange equations.
- Deformation can be searched as a minimizer of the functional

$$I(u) = \int_\Omega [W(Du) - Bu]\,dx.$$

Direct method of Calculus of Variations

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1. **Coercivity:** $\lim_{||u||\to\infty} I(u) = +\infty$.
2. **Weak lower semi-continuity:** For every $u_j \rightharpoonup u$ (weakly), we have the following inequality
$$I(u) \le \liminf I(u_j).$$

Polyconvexity

- Scalar case ($n = 1$ or $m = 1$): The s.w.l.s.c. of I is obtained through the convexity of $W(x, u, \cdot)$.

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Definition: Polyconvexity

$W : \mathbb{R}^{n\times n} \to \mathbb{R}$ is said to be polyconvex iff there exists a convex function $h : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R}$ such that

$$W(A) = h(A, \mathrm{cof}A, \det A).$$

Polyconvexity of W $+$

Polyconvexity of W $+$ Weak continuity of $\det Du$

$$\Downarrow$$

Weak lower semi-continuity of the functional $I = \int W(x, u, Du)$

Piola Identity

$$\mathrm{div}(\mathrm{cof} Du) = 0$$

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Material symmetry: Isotropy

Isotropic material

W is isotropic if $SO(3) \subset \mathcal{S}$, i.e.,

$$W(F) = W(FR) \qquad \forall R \in SO(3).$$

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In this case, the **Rivlin-Ericksen theorem** establishes that there exists $\tilde{h} : (0, +\infty)^3 \to \mathbb{R}$ such that

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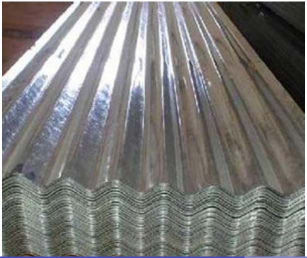


Existence theorem

Theorem (John Ball)

- There exists a convex function $\hat{W} : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times (0, \infty) \to (0, \infty)$ such that

$$W(F) = \hat{W}(F, \mathrm{cof}F, \det F) \qquad \forall F \in \mathbb{R}^{3\times 3}_{+}.$$

- $W(F) \to \infty$ when $\det F \to 0$.

- $W(F) \geq c_1(|F|^p + |\mathrm{cof}F|^q + (\det F)^r) - c_2$.

Then there exists a minimizer of the functional

$$I[u] := \int_{\Omega} W(x, u, Du)dx.$$

Isotropic models

- **Money Rivlin materials**

$$W(F) = \alpha |F|^2 + \beta |\text{cof} F|^2 + J(\det F),$$

with $\lim_{t \to \infty} J(t) = +\infty$. $\alpha, \beta > 0$.

- **Neo-Hookean materials** in the case $\beta = 0$.
Curious note: Pixar's characters simulation: "Stable Neo-Hookean Flesh Simulation".

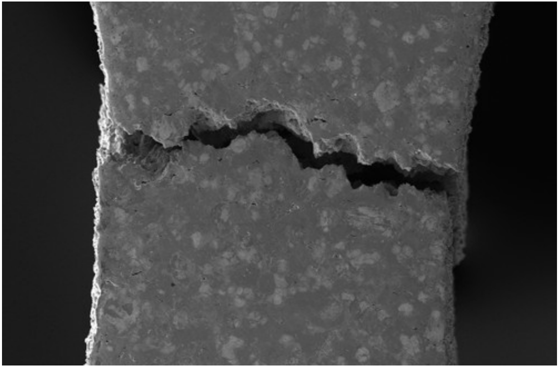
- **Odgen materials**

Index

Why a fractional model of hyperelasticity ?

Problem

It is no valid any more when the functions stop being continuous and some singularities arise, like fractures.



Motivation

- Lavrentiev phenomenon: minimizers may change as the functional space changes.

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- When a solid is subjected to great loads, singularities may appear such as fracture and cavitation (the sudden formation of voids in the material).

- $W^{1,p}$ with $p > n$ forces functions to be continuous.

Previous approach

$$I = \int \int w(x, x', u(x), u(x'))dxdx'$$

- Existence of solution and Γ- convergence were studied (Bellido, Mora-Corral 2014; Bellido, Mora-Corral, Pedregal 2015).
- Not suitable in hyperelasticity.

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$$I = \int W \left(\int \cdots \right)$$

Functional space $H^{s,p}$

$$D^s u(x) := c_{n,s}\, \mathrm{p.\,v.}_x \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy.$$

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- $\widehat{D^s u}(\xi) = \frac{1}{|2\pi\xi|^{1-s}} \widehat{Du}$
  $= \frac{2\pi i \xi}{|2\pi\xi|^{1-s}} \hat{u}(\xi)$

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- $D^s u = (\frac{c_{n,s}}{|x|^{n-(1-s)}}) * Du$
- $(-\Delta)^s u = -\sum_{j=1}^N \frac{\partial^s}{\partial x_j^s} \frac{\partial^s}{\partial x_j^s} u$
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  $= \frac{2\pi i \xi}{|2\pi\xi|^{1-s}} \hat{u}(\xi)$
- $D^s u \to Du$

Functional space $H^{s,p}$

Our goal was to prove the existence of minimizer of the functional

$$I(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x))dx,$$

where we have substituted the gradient by the so-called Riesz s-fractional gradient

$$D^s u(x) := c_{n,s}\, \mathrm{p.\,v.}_x \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy.$$

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And so, the new space we are going to search the minimizers in is

$$H^{s,p}_g(\Omega) := \{u \in H^{s,p}(\mathbb{R}^n) : u = g \text{ in } \Omega^c\},$$

where

$$H^{s,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) : D^s u \in L^p(\mathbb{R}^n; \mathbb{R}^n)\},$$

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Functional space $H^{s,p}$

Proposition

a) $C_c^\infty(\mathbb{R}^n)$ is dense in $H^{s,p}(\mathbb{R}^n)$.

b) $H^{s,p}(\mathbb{R}^n)$ is reflexive.

c) If $s < t < 1$ and $1 < q \leq p \leq \frac{nq}{n-(t-s)q}$, then $H^{t,q}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$.

d) If $0 < \mu \leq s - \frac{n}{p}$, then $H^{s,p}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n)$.

e) If $p = 2$, then $H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ with equivalence of norms.

f) If $0 < s_1 < s < s_2 < 1$ then $H^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s_1,p}(\mathbb{R}^n)$.

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Functional space $H^{s,p}$

Theorem (Shieh-Spector 2015)

Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists $C = C(|\Omega|, n, p, s) > 0$ such that

$$||u||_{L^q(\Omega)} \leq C||D^s u||_{L^p(\mathbb{R}^n)}$$

for all $u \in H^{s,p}(\mathbb{R}^n)$, and any q satisfying

$$\begin{cases} q \in [1, p^*] & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n. \end{cases}$$

$p* = \frac{np}{n-sp}$.

Functional space $H^{s,p}$

Theorem (Shieh–Spector 2017)

Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $g \in H^{s,p}(\mathbb{R}^n)$. Then for any sequence $\{u_j\}_{j\in\mathbb{N}} \subset H^{s,p}_g(\Omega)$ such that

$$u_j \rightharpoonup u \quad \text{in } H^{s,p}(\mathbb{R}^n),$$

for some $u \in H^{s,p}(\mathbb{R}^n)$, one has $u \in H^{s,p}_g(\Omega)$ and

$$u_j \to u \quad \text{in } L^q(\mathbb{R}^n),$$

for every q satisfying

$$\begin{cases} q \in [1, p^*) & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty) & \text{if } sp > n. \end{cases}$$

Singularities: fracture and cavitation

$H^{s,p}$ may include functions with singularities forbidden in Sobolev spaces of interest in a pure mathematical point of view as well as an applied one.

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- **Fracture**: Let $Q = (0,1)^n$ and $\varphi_2, \dots, \varphi_n \in C_c^\infty(\mathbb{R}^n)$. Define $u = (\chi_Q, \varphi_2, \dots, \varphi_n)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{1}{s}, \qquad \text{and} \qquad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{1}{s}.$$

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- **Cavitation**: Let $\varphi \in C_c^\infty([0,\infty))$ be such that $\varphi(0) > 0$, and $u(x) = \frac{x}{|x|}\varphi(|x|)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{n}{s} \qquad \text{and} \qquad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{n}{s}.$$

Cavitation:

The sudden formation of voids in a material.

Index

Theorem

Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ *satisfy the following conditions:*

a) $W(x, y, \cdot)$ *is polyconvex.*

b) *There exists* $h : [0,\infty) \to [0,\infty)$ *such that* $\lim_{t\to\infty} \frac{h(t)}{t} = \infty$ *and*

$$\begin{cases} W(x,y,F) \geq a(x) + c\,|F|^p + c\,|\mathrm{cof} F|^q + h(|\det F|), & \text{if } sp < n, \\ W(x,y,F) \geq a(x) + c\,|F|^p\,, & \text{if } sp \geq n, \end{cases}$$

for $a \in L^1$ *and some* $q > \frac{p^*}{p^*-1}$.

Let Ω *be a bounded open subset of* $\mathbb{R}^n$ *and* $u_0 \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$.
Then there exists a minimizer of I *in* $H^{s,p}_{u_0}(\Omega, \mathbb{R}^n)$.

How do we do it ?

Index

Fractional divergence

The definition of the s-fractional divergence is given in order to fulfil an integration by parts/Divergence Theorem.

$$\operatorname{div}^s \varphi(x) = -c_{n,s} \, \mathrm{p.\,v.}_x \int_{\mathbb{R}^n} \frac{\varphi(x) + \varphi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy.$$

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Theorem: Integrations by parts

Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $D^s u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, then for all $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int D^s u(x) \cdot \phi(x)\, dx = -\int u(x) \mathrm{div}^s \phi(x)\, dx.$$

Polyconvexity of W $+$

Polyconvexity of W $+$ Weak continuity of $\det D^s u$

$\Downarrow$

Weak lower semi-continuity of the functional $I = \int W(x, u, D^s u)$

Piola Identity

$$\mathrm{div}^s(\mathrm{cof} D^s u) = 0$$

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Weak lower semi-continuity of the functional $I = \int W(x, u, D^s u)$

s-fractional divergence of the product

Lemma

Let $g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ *and* $\varphi \in C_c^1(\mathbb{R}^n)$. *Then* $\varphi g \in H^{s,p(\mathbb{R}^n),\mathbb{R}^n}$ *and for a.e.* $x \in \mathbb{R}^n$,

$$div^s(\varphi g)(x) = \varphi(x) div^s g(x) + K_{\varphi(x)}(g^T)(x),$$

where the operator $K_\varphi : L^q(\mathbb{R}^n, \mathbb{R}^{k\times n}) \to L^p(\mathbb{R}^n, \mathbb{R}^k)$ *defined as*

$$K_\varphi(U)(x) = c_{n,s} \int \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} U(y) \frac{x-y}{|x-y|} dy, \qquad a.e.\ x \in \mathbb{R}^n,$$

is linear and bounded for all $p \in [1, q]$.

Index

Piola Identity

Fractional Piola Identity

Let $u \in C_c^\infty(\mathbb{R}^n)$, $s \in (0,1)$. Then

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In the case of the first row:

$$-c_{n,s} pv_x \int_{\mathbb{R}^n} \frac{(cof\,(D^s u))_1(x')}{|x-x'|^{n+s}} \cdot \frac{x-x'}{x-x'} dx' = 0$$

Special attention had to be paid to the limits in the singularities!

Index

Integration by parts of the determinant

Classical result

$$\int \det(Du)(x)\varphi(x)\,dx = -\frac{1}{n}\int u(x)\cdot\nabla\varphi \operatorname{cof}(Du)(x)\,dx$$

$$\Uparrow$$

$$\det(Du) = \operatorname{div}\left(u_k(\operatorname{cof} Du)_k\right) \quad \forall k = 1, \ldots, n$$

Integration by parts of the determinant

Classical result

$$\int \det(Du)(x)\varphi(x)\,dx = -\frac{1}{n}\int u(x)\cdot\nabla\varphi\mathrm{cof}(Du)(x)\,dx$$

Lemma

For every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have that $u \cdot K_\varphi(cofD^s u) \in L^1(\mathbb{R}^n)$ and

$$\int \det(D^s u)(x)\,\varphi(x)\,dx = -\frac{1}{n}\int u(x)\cdot K_\varphi(cof(D^s u))(x)\,dx.$$

Weak continuity of the determinant

Weak continuity of the minors

Let $u_j \rightharpoonup u$ in $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ and $M : \mathbb{R}^{n\times n} \to \mathbb{R}$ a minor of order r, then

$$M(D^s u_j) \rightharpoonup M(D^s u)$$

in $L^{\frac{p}{r}}(\mathbb{R}^n)$.

Index

Γ-convergence

Convergence of the fractional gradients

For $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ we have that

$$D^s u \to Du$$

as s goes to 1^-, strongly in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times m})$.

Recovering the classical model

Γ-convergence in fractional hyperelasticity

Let $W : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ such that $W(x, u, \cdot)$ is quasiconvex for a.e. $\mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Let

$$I_s(u) = \int_{\mathbb{R}^n} W(x, u, D^s u)\, dx$$

be defined in $H^{s,p}_g(\Omega; \mathbb{R}^m)$, and let

$$I(u) = \int_{\mathbb{R}^n} W(x, u, Du)\, dx$$

be defined on $W^{1,p}_g(\Omega; \mathbb{R}^m)$. Then

I_s Γ-converges to I.

Γ-convergence

Fractional Mean Value Theorem ($p = 1$ Comi-Stefani 2019)

Let $u \in H^{s,p}(\mathbb{R}^n)$. Then, for every $s_0 > 0$ there exists a constant $C > 0$ such that for every s, $s_0 \leq s < 1$ and for every $h \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p dx \leq C|h|^{sp} ||D^s u||^p_{L^p(\mathbb{R}^n)}.$$

Proposition

Let $\{u_s\}_{s\in(0,1)}$ where each $u_s \in H^{s,p}_g(\Omega)$. If

$$||D^s u_s||_{L^p(\mathbb{R}^n)} \leq C,$$

then there exists $u \in W^{1,p}(\mathbb{R}^n)$ such that $D^s u_s \rightharpoonup Du$ in $L^p(\mathbb{R}^n)$.

Index

New model on bounded domains

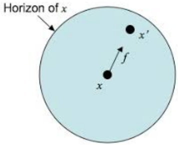
Motivated by applications, we would like to obtain similar results but over bounded domains and including different kinds of boundary conditions.

$$I[u] = \int_\Omega W(G_\rho u(x))dx$$

where $G_\rho u$ is a non-local gradient,

$$G_\rho(u) = \int_\Omega \frac{u(x) - u(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho(|x - y|)dy,$$

and $W(F) = \hat{W}(F, \mathrm{cof} F, \det F)$,
with $\hat{W}$ convex.



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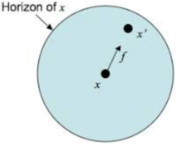
$\rho(|x-y|) = \frac{1}{|x-y|^{n-(1-s)}} \chi_{B(0,1-s)}$



Thank you very much for your attention.

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