Controllability properties of a magnetic microswimmer

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PDEs, optimal design and numerics, Benasque,
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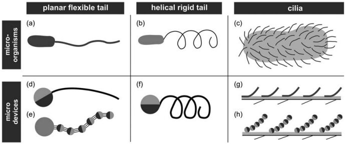
Summary

Microswimming

Definition

Swimming is the ability of moving in a fluid with suitable body deformation.

- At microscale, many natural organisms are able to swim (bacterias, spermatozoids...).
- Try to mimic the form and motion of them : Biomimetics.
- Medical applications : drug delivery, minimized invasive microchirurgical operations.
- One "non-intrusive" method : magnetized robot that deforms itself under the application of an exterior magnetic field.



Low Reynolds number and time-reversibility

The Navier-Stokes equation

$$\rho\left(\partial_t u + (u.\nabla)u\right) - \nu\Delta u + \nabla p = 0, \ \text{div}\ u = 0.$$

- Size of robots : $\simeq 1\mu m$.
- Water viscosity : $\simeq 1m^2/s$.
- Water density : $\simeq 1kg/m^2$.
- Characteristic speed : $\simeq 10\mu m/s$.
- $\Rightarrow$ Reynolds Number $\simeq 10^{-6}$ at this scale, very low.

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Fluide-structure interaction given by the Stokes equation, which is time-reversible, leading to some different phenomena than usual at the macroscopic scale.

Low Reynolds number and time-reversibility (2)

Time-reversibility of the Stokes equation

Life at low Reynolds number

Obstructions to swimming because of the time-reversibility : the scallop theorem (Purcell'77).

The Scallop Theorem

A self-propelled micro-swimmer with one degree of freedom cannot move, because it only makes time-reversible movements !

Not true anymore as soon as the swimmer can do non-time reversible movements.

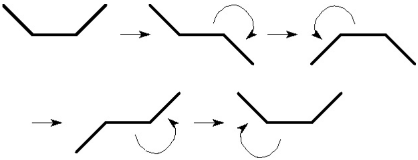
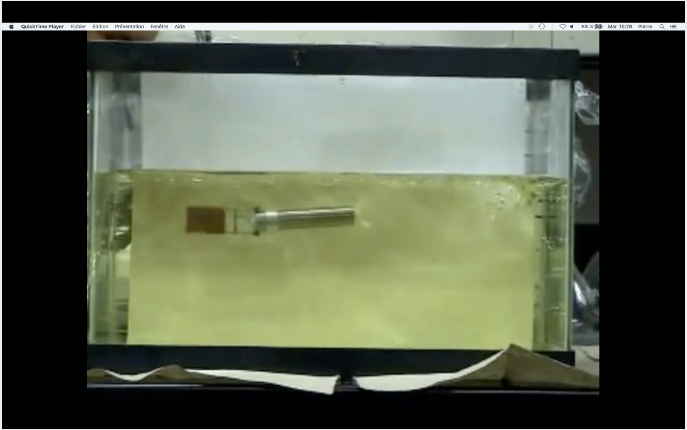


Figure – Non-reversible movement

Life at low Reynolds number (2)

The scallop theorem



Magnetic Microswimmers

Scallop Theorem not true anymore for magnetic microswimmers (Giraldi-Pomet'17, IEEE TAC).

- Swimmer which is made of 2 magnetized segments, subject to a uniform magnetic field, with elastic joint (2-link magnetic swimmer).
- One can move it and even control it locally around its equilibrium states (straight positions).

Main goal of the talk

Study a 2-link and a 3-link magnetic swimmer.

Long-term goal

Study a N-link magnetic swimmer, with N "very large" (discretization of a continuous model), pass to the limit.

(Incomplete) state of the art

- Dreyfus et al.'05 (Nature) : study of artificial swimmer that possesses a head plus a magnetic flexible tail. Control of velocity and position, numerical study.
- Gadelha'13 (Reg. and Chao. Dyn.) : numerical study of the optimal form of a magnetic head plus elastic tail system.
- Gutman-Or'14 (Phys. Rev. E) : study of a two-link model. Optimal controls to maximize displacement per cycle and average speed.
- Alouges et al.'15 (Soft Rob.) : discretization of the filament into magnetized segments. Prescription of a direction by sinusoidal magnetic field, numerical study.
- Giraldi-Pomet'17 (IEEE TAC) : theoretical study fo the 2-link swimmer. Proof of a "weak" STLC result.
- Alouges et al.'17 (IFAC) : focus on the Purcell (3-link) swimmer. Prescription of a direction by sinusoidal magnetic field, theoretical study (asymptotic analysis).

Parametrization

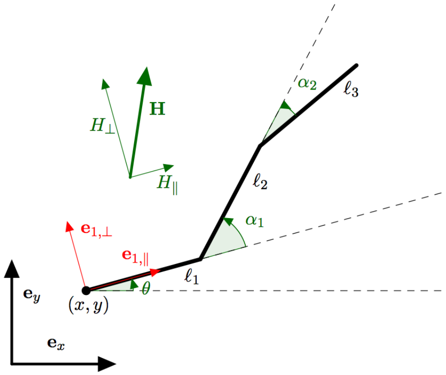


Figure – Parametrization of the 3-link microswimmer

Computation of the net force (1)

Elastic forces

- Torque $\mathbf{T}_2^{\mathrm{el}}$ on S_2 given by $\mathbf{T}_2^{\mathrm{el}} = \kappa\alpha_1\mathbf{e}_z$;
- Torque $\mathbf{T}_3^{\mathrm{el}}$ on S_3 given by $\mathbf{T}_3^{\mathrm{el}} = \kappa\alpha_2\mathbf{e}_z$;

Steady states : $(x, y, \theta, 0, 0)$ with $(x, y, \theta) \in \mathbf{R}^3$.

Magnetic forces

- Uniform magnetic field $H(t)$.
- Magnetic torque on S_i :

$$\mathbf{T}_i^m = M_i \mathbf{e}_{i,\|} \times \mathbf{H}.$$

Magnetic moments M_i assumed to be nonzero.

Computation of the net force (2)

Hydrodynamic effects

- Hydrodynamic coefficients ξ_i and η_i.
- Approximation : Resistive Force Theory (Gray-Hancock'55, Journal of Experimental Biology).
- Force $\mathbf{F}_i^h$ on S_i :

$$\mathbf{F}_i^h = \int_{S_i} \mathbf{f}_i(s) ds,$$

where

$$\mathbf{f}_i(s) = -\xi_i u_{i,\parallel} \mathbf{e}_{i,\parallel} - \eta_i u_{i,\perp} \mathbf{e}_{i,\perp}.$$

- Torque generated by S_i at point $\mathbf{x}_0$:

$$\mathbf{T}_{i,\mathbf{x}_0}^h = \int_{S_i} (\mathbf{x}_i(s) - \mathbf{x}_0) \times \mathbf{f}_i(s) ds.$$

Equations of the model

- We apply the second Newton law successively to $\{S_1 + S_2 + S_3\}$, $\{S_2 + S_3\}$ and $\{S_3\}$:

$$\left\{\begin{array}{llllll}
\underbrace{\begin{array}{l}\mathbf{F}_1^h + \mathbf{F}_2^h + \mathbf{F}_3^h \\ \mathbf{T}_{1,\mathbf{x}}^h + \mathbf{T}_{2,\mathbf{x}}^h + \mathbf{T}_{3,\mathbf{x}}^h \\ \mathbf{T}_{2,\mathbf{x}_2}^h + \mathbf{T}_{3,\mathbf{x}_2}^h \\ \mathbf{T}_{3,\mathbf{x}_3}^h\end{array}}_{\text{hydrodynamic terms}} &
\begin{array}{l} \\ + \\ + \\ + \end{array} &
\underbrace{\begin{array}{l} \\ \mathbf{T}_1^m + \mathbf{T}_2^m + \mathbf{T}_3^m \\ \mathbf{T}_2^m + \mathbf{T}_3^m \\ \mathbf{T}_3^m\end{array}}_{\text{magnetic terms}} &
\begin{array}{l} \\ \\ + \\ + \end{array} &
\underbrace{\begin{array}{l} \\ \\ \mathbf{T}_2^{\text{el}} \\ \mathbf{T}_3^{\text{el}}\end{array}}_{\text{elastic terms}} &
\begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = 0\end{array}
\end{array}\right.$$

Equations of the model (2)

We denote by $Z = \begin{pmatrix} x & y & \theta & \alpha_1 & \alpha_2 \end{pmatrix}^T$. System can then be rewritten as

$$M(\alpha_1, \alpha_2) R_{-\theta} \dot{Z} = Y,$$

with

$$R_\theta = \left(\begin{array}{cc|c} \cos\theta & \sin\theta & \multirow{2}{*}{0} \\ -\sin\theta & \cos\theta & \\ \hline \multicolumn{2}{c|}{0} & I_3 \end{array}\right)$$

and

$$Y = \begin{pmatrix} 0 \\ 0 \\ H_{\|}(M_2 \sin\alpha_1 + M_3 \sin(\alpha_1+\alpha_2)) - H_{\perp}(M_1 + M_2 \cos\alpha_1 + M_3 \cos(\alpha_1+\alpha_2)) \\ -\kappa\alpha_1 + H_{\|}(M_2 \sin\alpha_1 + M_3 \sin(\alpha_1+\alpha_2)) - H_{\perp}(M_2 \cos\alpha_1 + M_3 \cos(\alpha_1+\alpha_2)) \\ -\kappa(\alpha_2) + H_{\|} M_3 \sin(\alpha_1+\alpha_2) - H_{\perp} M_3 \cos(\alpha_1+\alpha_2) \end{pmatrix}.$$

Equations of the model (3)

- R_θ depends only on the shape parameters (α_1, α_2).
- Equilibrium states when there is no control : $(x, y, \theta, 0, 0)$ with $(x, y, \theta) \in \mathbf{R}^3$.

M is invertible, hence we obtain a control system of the form

$$R_{-\theta}\dot{Z} = \mathbf{F}_0(\alpha_1, \alpha_2) + H_\perp(t)\mathbf{F}_1(\alpha_1, \alpha_2) + H_{||}(t)\mathbf{F}_2(\alpha_1, \alpha_2)$$

F_0,F_1 and F_2 : linear combinations of the last three columns of M^{-1} ($\mathbf{X}_3$, $\mathbf{X}_4$ and $\mathbf{X}_5$).

Equations of the model (4)

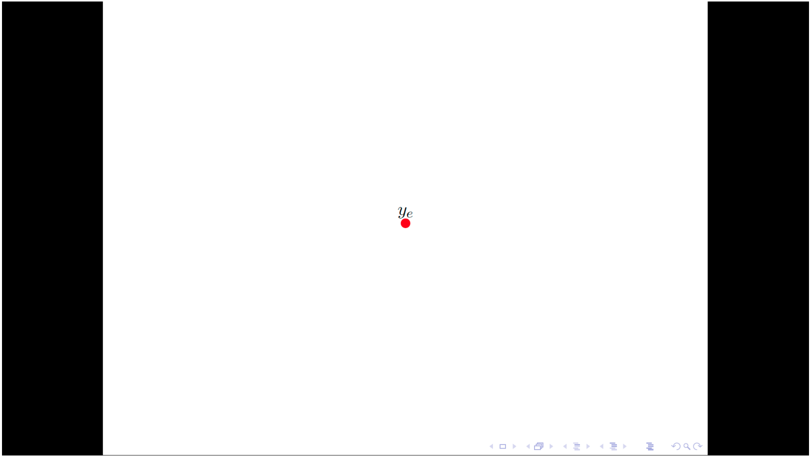
$$\begin{aligned}
\mathbf{F}_0 &= -\kappa(\alpha_1 \mathbf{X}_4 + (\alpha_2)\mathbf{X}_5) \\
\mathbf{F}_1 &= -M_1\mathbf{X}_3 - (M_2 \cos\alpha_1 + M_3\cos(\alpha_1+\alpha_2))(\mathbf{X}_3+\mathbf{X}_4) - M_3\cos(\alpha_1+\alpha_2)\mathbf{X}_5 \\
\mathbf{F}_2 &= (M_2 \sin\alpha_1 + M_3\sin(\alpha_1+\alpha_2))(\mathbf{X}_3+\mathbf{X}_4) + M_3\sin(\alpha_1+\alpha_2)\mathbf{X}_5.
\end{aligned}$$

- $H_{||}$ and $H_{\perp}$ are the controls.
- 2 controls for 5 states $(x, y, \theta, \alpha_1, \alpha_2)$. The controls does not appear in the two first equations (indirect controllability).
- Affine control system with drift.
- We have $F_2(0) = 0$. Hence, the parallel control acts "less" that the orthogonal control.

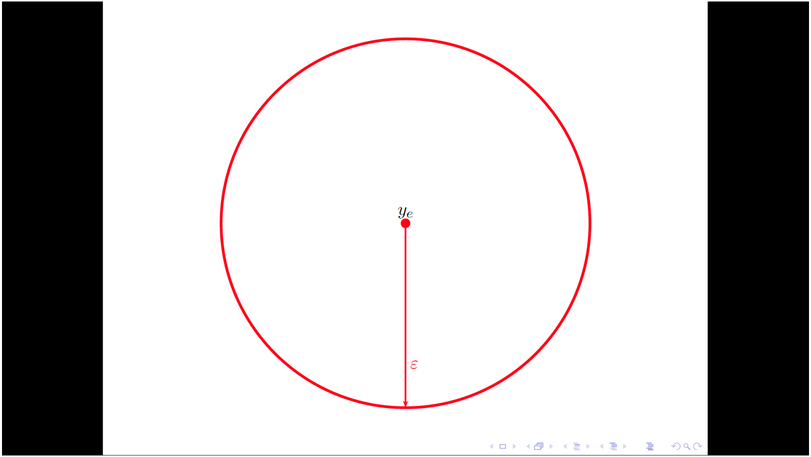
Question

Can we prove a positive controllability result ?

Small-time local controllability



Small-time local controllability



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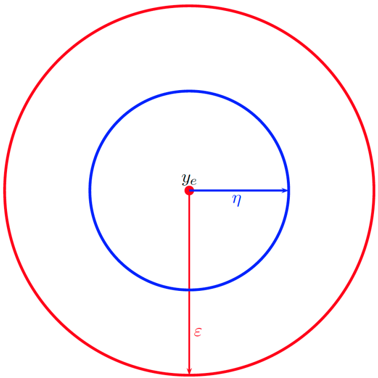

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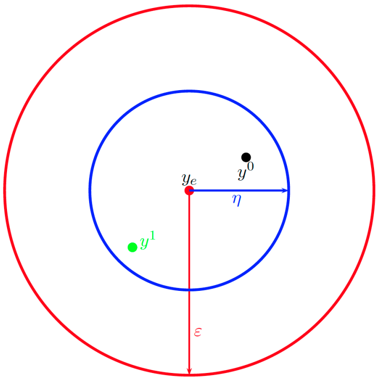

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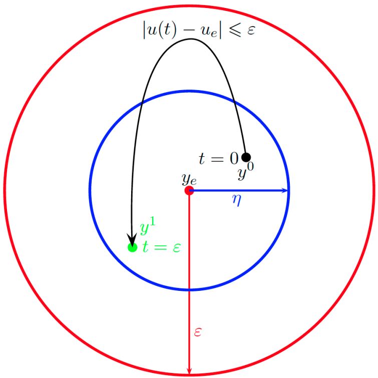

Small time local controllability for non-linear systems

Definition

Let $(y^e, u^e) \in \mathbb{R}^n \times \mathbb{R}^m$ an equilibrium of the control system $\dot{y} = f(y, u)$. This system is small time locally controllable around the equilibrium (y^e, u^e) (STLC) if for any $\epsilon > 0$, there exists $\eta > 0$ such that for any $(y^0, y^f) \in B_\eta(y^e) \times B_\eta(y^e)$, there exists a L^∞ function $u : [0, \epsilon] \to \mathbf{R}^m$ such that

$$\begin{array}{ll} \text{(i)} & \forall t \in [0, \epsilon], |u(t) - u^e| \leq \epsilon; \\ \text{(ii)} & \dot{y} = f(y, u), (y(0) = y^0 \Rightarrow (y(\epsilon) = y^f). \end{array}$$

Here, we assume that we have smallness in time and in control (as in Coron'07).
STLC is ensured for instance by the linear test (Kalman rank condition).

What happens for the 2-link swimmer ?

Giraldi-Pomet'17 (IEEE TAC) : same modelization with two links.
Goal : obtain local controllability results around the equilibriums **without smallness assumptions on the control** (even for small displacements !)

- We cannot control with only one of the controls.
- The Kalman rank condition at the equilibrium points does not hold. Cannot use the standard linearization method.
- The Sussman conditions on the bad and good Lie brackets (1987) do not hold.

$\Rightarrow$ Use of the return method of Coron'92, MCSS.

A remark

The control created does not lead STLC ! Indeed, the control $H_\perp$ can be as small as we want, but $H_{||}$ in this construction is such that

$$||H_{||}||_\infty \geqslant \frac{2\kappa|M_1 + M_2|}{|M_1 M_2|}.$$

This leads to the following definition.

Definition (STLC(q))

Let $q \geqslant 0$. The control system $\dot{y} = f(y, u)$ is STLC(q) at (y_e, u_e) if and only if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every y_0, y_1 in the ball centered at y_e with radius η, there exists a solution $(y(\cdot), u(\cdot)) : [0, \varepsilon] \to \mathbb{R}^{n+m}$ such that $y(0) = z_0$, $y(\varepsilon) = z_1$, and, for almost all t in $[0, \varepsilon]$,

$$\|u(t) - u_e\| \leq q + \varepsilon\,.$$

A strange behaviour

Theorem (Giraldi-Lissy-Moreau-Pomet'18 (IEEE TAC))

*Assume $M_1 + M_2 \neq 0$. If $\xi \neq \eta$, $M_1 \neq M_2$ and $M_1 + M_2 \neq 0$, then the two-link swimmer is not STLC at **O** (but it is STLC(q) for some $q > 0$).*

Proof : "by hand", using a contradiction argument.
In fact, we have now an optimal result.

Theorem (Moreau'19, IEEE L-CSS, Giraldi-Lissy-Moreau-Pomet'19)

Assume $M_1 \neq 0$, $M_2 \neq 0$, $M_1 \pm M_2 \neq 0$. Then, the two-link swimmer is STLC$\left(\frac{2\kappa|M_1+M_2|}{|M_1 M_2|}\right)$ but not STLC(q) for $q < \frac{2\kappa|M_1+M_2|}{|M_1 M_2|}$.

The positive result can be proved by making an adequate translation in time of the system and using already known criterium.

Lie Brackets

Definition

Let
$X = (X^1, \dots, X^n) \in \mathcal{C}^\infty(\Omega, \mathbb{R}^n)$, $Y = (Y^1, \dots, Y^n) \in \mathcal{C}^\infty(\Omega, \mathbb{R}^n)$
The j-th component of the Lie Bracket $[X, Y]$ *is*

$$[X, Y]^j := \sum_{k=1}^{n} \left(\partial_{x_k} X^j\right) Y^k - X^k \left(\partial_{x_k} Y^j\right).$$

Principal interest from control theory point of view : enables to reach new directions. For affine control systems without drift

$$x' = \sum_{i=1}^{d} u_i f_i(x),$$

we have the Chow-Rashevskii-Hörmander Theorem : we have STLC if (and only if, in case of analytic vector fields) $Lie(f_1, \dots f_d) = R^n$.

Lie brackets for $\dot{x} = u_1 f_1(x) + u_2 f_2(x)$

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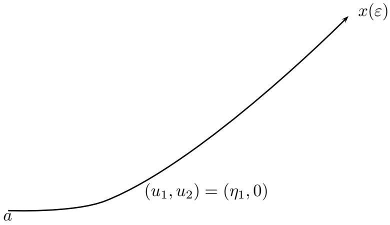

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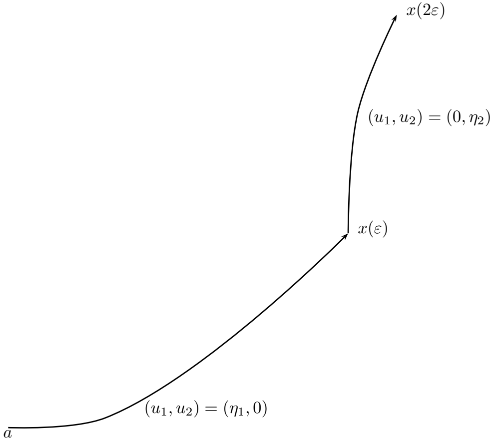

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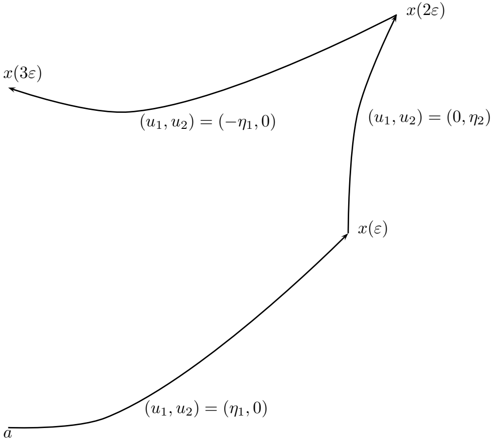

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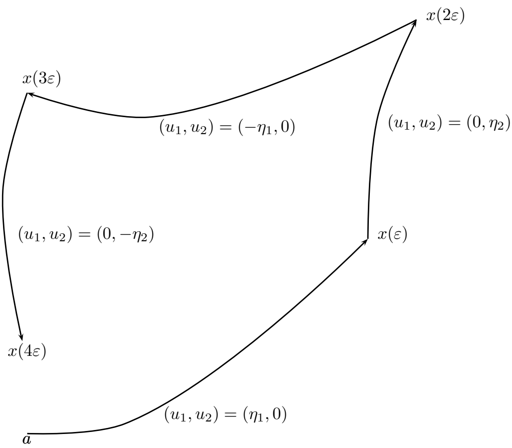

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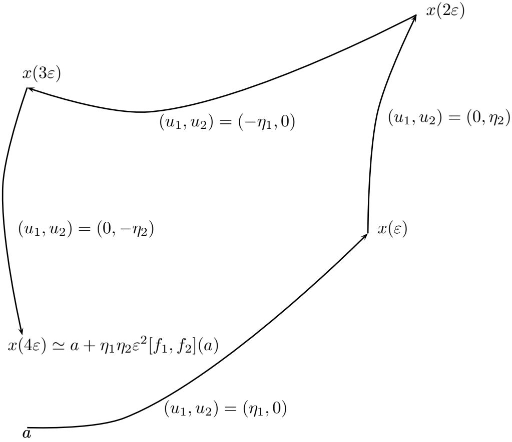

Necessary conditions of STLC with 1 controls (1)

Let us recall a well-known necessary condition for affine control for affine control system with scalar control of the form

$$y' = f_0(y) + u_1 f_1(y). \tag{Affine-1-Cont}$$

Let $k \in \mathbb{N}^*$. We introduce S_k the span of the Lie brackets of f_0 and f_1 that contains only f_1 less that k times, and $S_k(0)$ its value at $t = 0$.

Theorem (Sussman, 1983 (SICON))

Assume $f_0(0) = 0$ *and* $[f_1, [f_0, f_1]](0) \notin S_1(0)$*. Then* (Affine-1-Cont) *is not STLC(q) for no* $q \geqslant 0$*.*

Necessary conditions of STLC with 1 controls (2)

This is exactly the first obstruction on the following sufficient condition :

Theorem (Sussman, 1983 (SICON))

Assume $f_0(0) = 0$, $Lie(f_0, f_1)(0) = \mathbb{R}^n$ *and* $S_{2k+2}(0) \subset S_{2k+1}(0)$ *for any* $k \in \mathbb{N}$*. Then* (Affine-1-Cont) *is STLC.*

Other works by Sussman, Kawski, Krastanov, Stefani, Beauchard-Marbach'17 JDE (in fact, non-STLC in $W^{-1,\infty}$ norm, higher obstructions in higher Sobolev spaces).

Natural question

Find necessary conditions for STLC with 2 controls ?

Necessary conditions of STLC with 2 controls (1)

We introduce R_1 the span of Lie brackets of f_0, f_1, f_2 where f_1 appears only one time. $R_1(0)$: value at 0.

Theorem

Assume that $f_0(0) = 0$, $f_2(0) = 0$ *(so that* $(0, 0, u_2^{eq})$ *is an equilibrium for all* u_2^{eq}*), and* $[f_1, [f_0, f_1]](0) \notin R_1(0)$*. Then, if* $f_1, [f_0, f_1]](0) \in Span(R_1(0), f_1, [f_2, f_1]](0)$ *and* $\beta \in \mathbb{R}$ *is such that*

$$[f_1, [f_0, f_1]](0) + \beta[f_1, [f_2, f_1]](0) \in R_1(0),$$

system is not STLC at $(0, 0, u_2^{eq})$ *for* $u_2^{eq} \neq \beta$*. Notably, the system is not STLC(q) for* $q < |\beta|$ *around* $(0, 0, 0)$*.*

Necessary conditions of STLC with 2 controls (2)

Proof : based on Chen-Fliess series (Chen'57 (Annals), Fliess'78 (CRAS)) in the spirit of Sussman'83 (SICON). Choose of a good ϕ, real-valued, such that $\phi(0) = 0$ and $\phi(x(T)) \geqslant 0$ for all control. This prevents controllability of states x^T verifying $\phi(x^T) < 0$. We have to ensure that such states exist (it is the case if $d\phi(0) \neq 0$).

Necessary conditions of STLC with 2 controls (3)

We can write

$$\phi(x(T)) = \sum_I \left(\int_0^T u_I \right) (f_I \varphi)(0),$$

I : multi-index $(i_1, \dots i_k)$ with $k \in \mathbb{N}^*$, $j \in \{1, \dots, k\}$, $i_j \in \{0, 1, 2\}$.
$\int_0^T u_I$: iterated integral
$\int_0^T \int_0^{\tau_k} \dots \int_0^{\tau_1} u_{i_k}(\tau_k) \dots u_{i_1}(\tau_1) d\tau_k \dots d\tau_1$. f_I : $f_{i_1} f_{i_2} \dots f_{i_k}$. The product to be understood in terms of composition of differential operators associated to the $f_i = (f_i^1, \dots, f_i^n)$:

$$f_i \phi(x) = \sum_{k=1}^n f_i^k(x) \partial_{x_k} \phi(x).$$

One has to understand how to choose ϕ, isolate 6 different types of terms in the series, find the dominant one, and compare the others.

Back to the three-link

We have many examples of application of this result, notably the 3-link swimmer (and also the 2-link swimmer).

Theorem (Moreau'19, IEEE L-CSS, Giraldi-Lissy-Moreau-Pomet'19)

Under some assumptions on the coefficients, the three-link swimmer is STLC at $(0_{\mathbb{R}^5}, (0, \gamma))$ *with*

$$\gamma = \kappa \frac{17m - 16M}{-7M_2^2 + 9M_2 m - 5M_1 M_3}$$

but not STLC at $(0_{\mathbb{R}^5}, (q, 0))$ *for* $q \neq \gamma$*. Notably, it is not STLC(q) for* $q < |\gamma|$ *around* $(0_{\mathbb{R}^5}, (0, 0))$.

The positive result is obtained through a clever change of unknowns and applications of positive results by Sussman.

Going further

In the spirit of Beauchard-Marbach, we also investigated higher Lie Brackets, in order to prove some non-STLC results in higher Sobolev norms. Many technical difficulties appear in the treatment of the Chen-Fliess series, preventing us to obtain similar results as in their article. Still, we are able to obtain non STLC-results in $W^{1,\infty}$ norm for the first control and L^∞ norm for the second control.

Perspectives

- Replacing the Resistive Force Theory by the Stokes equation ? (coupling between ODEs and PDEs)
- More links ? Convergence to a continuous model ?
- Other shapes of microswimmers ?

References

- L. Giraldi, , P. Lissy, C. Moreau, J.-B. Pomet, *Addendum to "Local Controllability of the Two-Link Magneto-Elastic Micro-Swimmer"*, IEEE TAC 63 (2018), no. 7, 2303–2305.
- L. Giraldi, , P. Lissy, C. Moreau, J.-B. Pomet, *A necessary condition of local controllability for systems with two scalar controls*, submitted.

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Thank you for your attention.