On composite elastic plate and Hashin-Shtrikman bounds

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[VIII PDEs, optimal design and numerics]

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Motivation - optimal design problems

PDE with b.c.

$$\mathcal{P}(\mathbf{M})u = f \quad \text{in } \Omega,$$

where $\mathcal{P}(\mathbf{M})$ is a partial differential operator with coefficient $\mathbf{M}$.

Examples: $\mathcal{P}(\mathbf{M})u = \operatorname{div}(\mathbf{M}\nabla u)$, $\mathcal{P}(\mathbf{M})u = \operatorname{div}(\mathbf{M}e(u))$

$\mathbf{M}$ describes properties of material that occupies Ω, and we assume

$$\mathbf{M}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{A} + (1-\chi(\mathbf{x}))\mathbf{B}, \quad \mathbf{x} \in \Omega,$$

for some constant $\mathbf{A}$ and $\mathbf{B}$, and $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$ such that $\int_\Omega \chi = q_{\mathbf{A}}$.

Nonlinear optimal control problem: $J(\chi) := \int_\Omega g(u) \longrightarrow \min$

Problem: solution does not exist!

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Motivation - optimal design problems cont.

Relaxation by the homogenization method (Tartar, Murat):

$$\chi \longrightarrow (\theta, \mathbf{M}^*) = \lim_n (\chi^n, \chi^n \mathbf{A} + (1-\chi^n)\mathbf{B})$$

right topologies for above limit: L^∞ weak $*$ and H-topology

- existence of a solution

- robust numerical methods (optimality criteria method)

Done for stationary diffusion equation and system of linearized elasticity.

Motivation - optimal design problems cont.

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Kirchhoff-Love plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

P. G. Ciarlet, *Mathematical Elasticity, volume II: Theory of Plates,* Elsevier Science, Amsterdam, 1997.

- $\Omega \subseteq \mathbb{R}^d$ bounded domain ($d = 2 \ldots$ plate)
- $f \in H^{-2}(\Omega)$ external load
- $u \in H_0^2(\Omega)$ vertical displacement of the plate
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\ \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e. } \mathbf{x}\}$ describes properties of material of the given plate

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Antonić, Balenović, 1999.

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

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Properties

K. Burazin, J. Jankov, M. Vrdoljak, *Homogenization of elastic plate equation*, Mathematical Modelling and Analysis **23(2)**,190-204, 2018.

K. Burazin, J. Jankov, *Small-amplitude homogenization of elastic plate equation, Applicable Analysis, 2019, DOI: 10.1080/00036811.2019.1634255*

V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, Kha T'en Ngoan, Averaging and G-convergence of differential operators, Russian Math. Surveys **34:5**, 69–147, 1979.

- Locality of the H-convergence
- Irrelevance of boundary conditions
- Energy convergence
- Ordering property
- Metrizability
- Corrector results
- H-limit in periodic case
- Smooth dependence of a parameter
- Small-amplitude homogenization

Definition

Let $\chi^n \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *be a sequence of characteristic functions and* $(\mathbf{M}^n)$ *be a sequence of tensors defined by*

$$\mathbf{M}^n(\mathbf{x}) = \chi^n(\mathbf{x})\mathbf{A} + (1-\chi^n(\mathbf{x}))\mathbf{B},$$

where **A** *and* **B** *are assumed to be positive definite fourth order tensors. Assume that there exist* $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$ *and* $\mathbf{M}^* \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ *such that*

$$\chi^n \xrightarrow{*} \theta \text{ in } \mathrm{L}^\infty(\Omega, [0,1]),$$

$$\mathbf{M}^n \xrightarrow{H} \mathbf{M}^* \text{ in } \mathfrak{M}_2(\alpha, \beta; \Omega).$$

The H-limit $\mathbf{M}^*$ *is said to be the homogenized tensor of a two-phase composite material obtained by mixing* **A** *and* **B** *in proportions* θ *and* $(1-\theta)$, *respectively, with a microstructure defined by the sequence* (χ^n).

Laminated materials

Theorem

Let **A** *and* **B** *be two constant tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and* $\chi_n(\mathbf{x} \cdot \mathbf{e})$ *be a sequence of characteristic functions that converges to* $\theta(\mathbf{x} \cdot \mathbf{e})$ *in* $\mathrm{L}^\infty(\Omega; [0,1])$ *weakly-*$*$*. Then, sequence* $(\mathbf{M}^n)$ *of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, defined as*

$$\mathbf{M}^n(\mathbf{x} \cdot \mathbf{e}) = \chi_n(\mathbf{x} \cdot \mathbf{e})\mathbf{A} + (1 - \chi_n(\mathbf{x} \cdot \mathbf{e}))\mathbf{B}$$

H-converges to

$$\mathbf{M}^* = \theta\mathbf{A} + (1-\theta)\mathbf{B} - \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{A}-\mathbf{B})^T(\mathbf{e} \otimes \mathbf{e})}{(1-\theta)\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e}) + \theta\mathbf{B}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})},$$

which also depends only on $\mathbf{x} \cdot \mathbf{e}$*.*

Sequential laminates

Corollary

If $(\mathbf{A}-\mathbf{B})$ *is an invertible fourth order tensor, the lamination formula is equivalent to*

$$\theta(\mathbf{M}^*-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}).$$

If we repeat iterative process of lamination p times, in lamination directions $(\mathrm{e}_i)_{1\le i\le p}$ and proportions $(\theta_i)_{1\le i\le p}$, we obtain a rank-p sequential laminate with matrix **B** and core **A**, which is defined by the following formula:

$$\left(\prod_{j=1}^{p}\theta_j\right)(\mathbf{A}_p^*-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \sum_{i=1}^{p}\left((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\right)\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}.$$

Sequential laminates

Corollary

If $(\mathbf{A} - \mathbf{B})$ is an invertible fourth order tensor, the lamination formula is equivalent to

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G-closure problem

...identify all possible composite materials (done for conductivity, an open problem for 3D elasticity)

For fixed $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$, find

$$G_\theta := \Big\{ \text{H-limit of } \chi^n(\mathbf{x})\mathbf{A} + (1-\chi^n(\mathbf{x}))\mathbf{B} : \chi^n \in \mathrm{L}^\infty(\Omega; \{0,1\}) \text{ and } \chi^n \overset{*}{\rightharpoonup} \theta \text{ in } \mathrm{L}^\infty(\Omega; [0,1]) \Big\}.$$

Theorem

Let $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$. Then $\mathbf{M} \in G_\theta$ if and only if

$$\mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a. e. } \mathbf{x} \in \Omega.$$

. . . an open problem!

Bounds on effective energy

Definition

The function $f^-(\theta, \mathbf{A}, \mathbf{B}; \cdot) : \mathrm{Sym} \to \mathbf{R}$ (respectively, $f^+(\theta, \mathbf{A}, \mathbf{B}; \cdot) : \mathrm{Sym} \to \mathbf{R}$), which is real-valued, is said to be a lower bound (respectively, an upper bound) if for any $\mathbf{A}^* \in G_\theta$

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq f^-(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi}), \ \forall \boldsymbol{\xi} \in \mathrm{Sym} \quad (\text{respectively},$$
$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq f^+(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi}), \ \forall \boldsymbol{\xi} \in \mathrm{Sym}).$$

The lower bound $f^-(\theta, \mathbf{A}, \mathbf{B}; \cdot)$ (respectively, the upper bound $f^+(\theta, \mathbf{A}, \mathbf{B}; \cdot)$) is said to be optimal if for any $\boldsymbol{\xi} \in \mathrm{Sym}$ there exists $\mathbf{A}^* \in G_\theta$ such that

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} = f^-(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi}) \quad (\text{respectively}, \ \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} = f^+(\theta, \mathbf{A}, \mathbf{B}; \boldsymbol{\xi})).$$

Hashin-Shtrikman bounds

Theorem

For any $\boldsymbol{\xi} \in \mathrm{Sym}$, *the effective energy of a composite material* $\mathbf{A}^* \in G_\theta$ *satisfies the following bounds:*

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + (1-\theta) \max_{\boldsymbol{\eta}\in\mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta g(\boldsymbol{\eta})],$$

where $g(\boldsymbol{\eta}) := \sup_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}$, *and*

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{B}\boldsymbol{\xi} : \boldsymbol{\xi} + \theta \min_{\boldsymbol{\eta}\in\mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} + (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta) h(\boldsymbol{\eta})],$$

where $h(\boldsymbol{\eta}) := \inf_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}$.

These bounds are called Hashin-Shtrikman bounds, they are optimal and optimality is achieved by a finite-rank sequential laminate.

Explicit Hashin-Shtrikman bounds for mixtures of two isotropic materials in dimension $d = 2$

$$\mathsf{A} = 2\mu_1 \mathsf{I}_4 + (\kappa_1 - \mu_1)\mathbf{I}_2 \otimes \mathbf{I}_2$$

$$\mathsf{B} = 2\mu_2 \mathsf{I}_4 + (\kappa_2 - \mu_2)\mathbf{I}_2 \otimes \mathbf{I}_2$$

$$0 < \mu_1 \leq \mu_2,\ 0 < \kappa_1 \leq \kappa_2$$

$$g(\boldsymbol{\eta}) = \max_{\mathbf{e} \in S^{d-1}} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \boldsymbol{\eta}|^2}{\mathsf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}$$

$$h(\boldsymbol{\eta}) = \min_{\mathbf{e} \in S^{d-1}} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \boldsymbol{\eta}|^2}{\mathsf{B}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}$$

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Explicit expressions for g and h

Lemma

If we label the eigenvalues of $\boldsymbol{\eta}$ by η_1 and η_2, then for the isotropic phase **A** *the function g equals*

$$g(\boldsymbol{\eta}) = \frac{1}{\mu_1 + \kappa_1} \begin{cases} \eta_1^2, \text{ if } & |\eta_1| \geq |\eta_2| \\ \eta_2^2, \text{ if } & |\eta_2| \geq |\eta_1| \end{cases}, \tag{3.1}$$

while for the isotropic phase **B** *the function h equals*

$$h(\boldsymbol{\eta}) = \frac{1}{\mu_2 + \kappa_2} \begin{cases} \eta_2^2, \text{ if } & \eta_1 \leq \eta_2 \leq 0 \text{ or } 0 \leq \eta_2 \leq \eta_1 \\ 0, \text{ if } & \eta_1 < 0 < \eta_2 \text{ or } \eta_2 < 0 < \eta_1 \\ \eta_1^2, \text{ if } & \eta_2 \leq \eta_1 \leq 0 \text{ or } 0 \leq \eta_1 \leq \eta_2 \end{cases}. \tag{3.2}$$

Theorem (Explicit lower Hashin-Shtrikman bound)

Denoting by ξ_1 and ξ_2 the eigenvalues of $\boldsymbol{\xi}$, $\theta_1 := \theta$ and $\theta_2 := 1-\theta$, the explicit formula for the lower Hashin-Shtrikman bound is

$$(\theta_1 \mathbf{A} + \theta_2 \mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} - \theta_1\theta_2 \frac{[(\kappa_2-\kappa_1)|\xi_1+\xi_2| + (\mu_2-\mu_1)|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)},$$

$$\text{if } \theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2| < (\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2| \ \& \ (\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2| > \theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|;$$

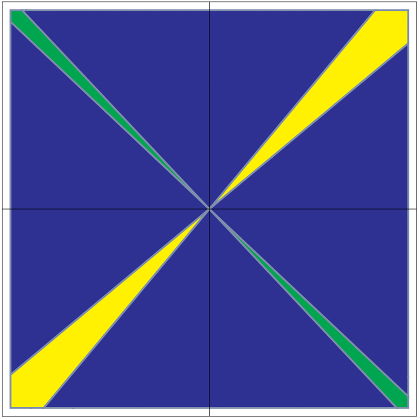
$$\mu_1(\xi_1-\xi_2)^2 + \frac{\kappa_1\kappa_2+\mu_1(\theta_1\kappa_1+\theta_2\kappa_2)}{\theta_1\kappa_2+\theta_2\kappa_1+\mu_1}(\xi_1+\xi_2)^2,$$

$$\text{if } \theta_1(\kappa_2-\kappa_1)|\xi_1+\xi_2| \geq (\theta_1\kappa_2+\theta_2\kappa_1+\mu_1)|\xi_1-\xi_2|;$$

$$\kappa_1(\xi_1+\xi_2)^2 + \frac{\mu_1\mu_2+\kappa_1(\theta_1\mu_1+\theta_2\mu_2)}{\theta_1\mu_2+\theta_2\mu_1+\kappa_1}(\xi_1-\xi_2)^2,$$

$$\text{if } (\theta_1\mu_2+\theta_2\mu_1+\kappa_1)|\xi_1+\xi_2| \leq \theta_1(\mu_2-\mu_1)|\xi_1-\xi_2|.$$

This cases are disjoint and their union is the set $\mathbf{R}^2$.



Domain splitting for $\theta_1 = 0.4, \mu_1 = 10, \mu_2 = 12, \kappa_1 = 16, \kappa_2 = 23$.

Theorem (Explicit upper Hashin-Shtrikman bound)

The explicit formula for the upper Hashin-Shtrikman bound is

$$(\theta_1 \mathbf{A} + \theta_2 \mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi} - \theta_1\theta_2 \frac{[(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| - (\mu_2 - \mu_1)|\xi_1 - \xi_2|]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)},$$

$$\text{if } \theta_2(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| < (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2)|\xi_1 - \xi_2| \ \& \ (\kappa_2 - \kappa_2)|\xi_1 + \xi_2| \geq (\mu_2 - \mu_1)|\xi_2 - \xi_1|;$$

$$(\theta_1 \mathbf{A} + \theta_2 \mathbf{B})\boldsymbol{\xi} : \boldsymbol{\xi},$$

$$\text{if } (\kappa_2 - \kappa_1)|\xi_1 + \xi_2| < (\mu_2 - \mu_1)|\xi_2 - \xi_1|;$$

$$\mu_2(\xi_1 - \xi_2)^2 + \frac{\kappa_1\kappa_2 + \mu_2(\theta_2\kappa_2 + \theta_1\kappa_1)}{\mu_2 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2,$$

$$\text{if } \theta_2(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| \geq (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2)|\xi_1 - \xi_2|.$$

This cases are disjoint and their union is the set $\mathbf{R}^2$*.*

Hashin-Shtrikman bounds on complementary energy

Theorem

For any $\boldsymbol{\sigma} \in \mathrm{Sym}$*, a homogenized tensor* $\mathbf{A}^* \in G_\theta$ *satisfies*

$$\mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \theta \max_{\boldsymbol{\eta}\in\mathrm{Sym}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta) g^c(\boldsymbol{\eta})\right]$$

$$\leq \mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \leq$$

$$\mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + (1-\theta) \min_{\boldsymbol{\eta}\in\mathrm{Sym}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} + (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta h^c(\boldsymbol{\eta})\right]$$

where $g^c(\boldsymbol{\eta}) := \mathbf{B}\boldsymbol{\eta} : \boldsymbol{\eta} - \min_{\mathbf{e}\in S^{d-1}} \dfrac{|(\mathbf{e}\otimes\mathbf{e}) : \mathbf{B}\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}$*, and*

$$h^c(\boldsymbol{\eta}) := \mathbf{A}\boldsymbol{\eta} : \boldsymbol{\eta} - \max_{\mathbf{e}\in S^{d-1}} \frac{|(\mathbf{e}\otimes\mathbf{e}) : \mathbf{A}\boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}.$$

Bounds are optimal and optimality is achieved by a finite-rank sequential laminate.

Theorem (Explicit lower bound on complementary energy)

Denoting by σ_1 and σ_2 the eigenvalues of $\boldsymbol{\sigma}$, $\theta_1 := \theta$ and $\theta_2 := 1-\theta$, the explicit formula for the lower complementary bound is

$$(\theta_1 \mathbf{A}^{-1} + \theta_2 \mathbf{B}^{-1})\boldsymbol{\sigma} : \boldsymbol{\sigma} - \theta_1\theta_2 \frac{[\mu_1\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| + \kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]}$$

$$\text{if } \theta_2\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2|,$$
$$(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| \geq (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_1-\sigma_2|;$$

$$\mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{\theta_1}{4}\left[\frac{(\mu_2-\mu_1)(\sigma_1-\sigma_2)^2}{\mu_2(\theta_1\mu_1+\theta_2\mu_2)} + \frac{(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)^2}{\kappa_2(\theta_1\kappa_1+\theta_2\kappa_2)}\right],$$

$$\text{if } (\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| < (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_1-\sigma_2|;$$

$$\mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{\theta_1(\kappa_2-\kappa_1)(\mu_2+\kappa_2)(\sigma_1+\sigma_2)^2}{4\kappa_2[\kappa_1(\mu_2+\kappa_2)+\mu_2(\kappa_2-\kappa_1)\theta_2]},$$

$$\text{if } \mu_2\theta_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2|.$$

This cases are disjoint and their union is the set $\mathbf{R}^2$.

Theorem (Explicit upper bound on complementary energy)

Denoting by σ_1 and σ_2 the eigenvalues of $\boldsymbol{\sigma}$, $\theta_1 := \theta$ and $\theta_2 := 1 - \theta$, the explicit formula for the upper complementary bound is

$$(\theta_1 \mathbf{A}^{-1} + \theta_2 \mathbf{B}^{-1})\boldsymbol{\sigma} : \boldsymbol{\sigma} - \theta_1\theta_2 \frac{[\mu_1\mu_2(\kappa_1 - \kappa_2)|\sigma_1 + \sigma_2| + \kappa_1\kappa_2(\mu_2 - \mu_1)|\sigma_1 - \sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2 + \kappa_2) + \mu_2\kappa_2\theta_2(\kappa_1 + \mu_1)]}$$

$$\text{if } \theta_1\mu_1(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| < (\theta_2\mu_1\kappa_2 + \theta_1\mu_1\kappa_1 + \kappa_1\kappa_2)|\sigma_2 - \sigma_1|,$$

$$\theta_1\kappa_1(\mu_2 - \mu_1)|\sigma_2 - \sigma_1| < (\theta_1\mu_1\kappa_1 + \theta_2\mu_2\kappa_1 + \mu_1\mu_2)|\sigma_1 + \sigma_2|;$$

$$\mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{\theta_2(\mu_1 + \kappa_1)(\kappa_1 - \kappa_2)(\sigma_1 + \sigma_2)^2}{4\kappa_1[\kappa_2(\mu_1 + \kappa_1) + \theta_1\mu_1(\kappa_1 - \kappa_2)]},$$

$$\text{if } \theta_1\mu_1(\kappa_2 - \kappa_1)|\sigma_1 + \sigma_2| \geq (\theta_2\mu_1\kappa_2 + \theta_1\mu_1\kappa_1 + \kappa_1\kappa_2)|\sigma_2 - \sigma_1|,$$

$$\mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{\theta_2(\mu_1 - \mu_2)(\mu_1 + \kappa_1)(\sigma_1 - \sigma_2)^2}{4\mu_1[\mu_2(\mu_1 + \kappa_1) + \theta_1\kappa_1(\mu_1 - \mu_2)]},$$

$$\text{if } \theta_1\kappa_1(\mu_2 - \mu_1)|\sigma_2 - \sigma_1| \geq (\theta_1\mu_1\kappa_1 + \theta_2\mu_2\kappa_1 + \mu_1\mu_2)|\sigma_1 + \sigma_2|.$$

This cases are disjoint and their union is the set $\mathbf{R}^2$.

Now what?

- Calculate derivatives of Hashin-Shtrikman bounds
- G-closure problem (in low-contrast regime)
- Small-amplitude homogenization - non-periodic case
- Optimal design of plates

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