Spectral analysis of discrete elliptic operators and applications in control theory

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Thematic session '*Numerics and control*'

Joint work with D. Allonsius & F. Boyer

The problem under study

- One dimensional Sturm Liouville operator on $(0,1)$

$$\mathcal{A} = -\partial_x(\gamma(x)\partial_x \bullet) + q(x)\bullet,$$

where $q \in C^0([0,1];\mathbb{R})$ and $\gamma \in C^1([0,1];\mathbb{R})$ with $\gamma_{min} := \inf_{x\in[0,1]} \gamma(x) > 0$.

- Finite difference scheme with N points

$$(\mathcal{A}^h U)_j = -\frac{1}{h}\left(\gamma_{j+1/2}\frac{u_{j+1}-u_j}{h} - \gamma_{j-1/2}\frac{u_j - u_{j-1}}{h}\right) + q_j u_j,$$

where $U = (u_j)_{1\le j\le N}$.

- The associated parabolic control problems

$$\begin{cases} (y^h)'(t) + \mathcal{A}^h y^h(t) = \mathbf{1}_\omega v^h(t), \\ y_0^h(t) = y_{N+1}^h(t) = 0, \\ y^h(0) = y^{0,h}, \end{cases} \qquad \begin{cases} (y^h)'(t) + \mathcal{A}^h y^h(t) = 0, \\ y_0^h(t) = 0,\ y_{N+1}^h(t) = v^h(t), \\ y^h(0) = y^{0,h}, \end{cases}$$

with **uniformly bounded** controls (with respect to h).

Some previous results

Uniform null controllability.

- Lopez & Zuazua (1998) for the 1D Laplace operator.

A weaker controllability notion ($\phi(h)-$null controllability).

- Labbé & Trélat (2006).
- Boyer, Hubert & Le Rousseau (2010).
- Present work: controllability via the moment method.
  Application to cascade systems of equations, for distributed and boundary controls.
  Limitation to 1D.
  $\longrightarrow$ need a careful spectral analysis of $\mathcal{A}^h$.

1. The continuous problem
   - The moment method
   - A strategy for spectral analysis

2. Spectral analysis of the discrete problem
   - The discrete setting
   - A rough estimate
   - A refined estimate

3. Application to controllability

1 The continuous problem
- The moment method
- A strategy for spectral analysis

2 Spectral analysis of the discrete problem

3 Application to controllability

Setting

$$\begin{cases} \partial_t y(t) + \mathcal{A}y(t) = \mathbf{1}_\omega(x)v(t,x), \\ y(t,0) = y(t,1) = 0, \\ y(0) = y^0. \end{cases}$$

- Eigenelements: $\mathcal{A}\varphi_k = \lambda_k \varphi_k$.
  Complete family of normalized eigenvectors in $L^2(0,1)$.
- Solution by transposition.

$$\langle y(t), z\rangle - \langle y_0, e^{-t\mathcal{A}^*} z\rangle = \int_0^t \langle v(\tau), e^{-(t-\tau)\mathcal{A}^*} z\rangle_{L^2(\omega)} \, \mathrm{d}\tau,$$

for any $t \in [0,T]$, and any $z \in L^2(0,1)$.

The moment problem

- For any $k \geq 1$,

$$\langle y(T), \varphi_k \rangle - \langle y_0, e^{-\lambda_k T} \varphi_k \rangle = \int_0^T e^{-\lambda_k (T-t)} \langle v(t), \varphi_k \rangle_{L^2(\omega)} \, \mathrm{d}t.$$

- The moment problem: $y(T) = 0$ if and only if

$$\int_0^T e^{-\lambda_k (T-t)} \langle v(t), \varphi_k \rangle_{L^2(\omega)} \, \mathrm{d}t = -e^{-\lambda_k T} \langle y_0, \varphi_k \rangle, \qquad \forall k \geq 1.$$

- Definition of a biorthogonal family: $(q_j)_{j \geq 1} \in L^2(0, T; \mathbb{R})$ such that

$$\int_0^T e^{-\lambda_k t} q_j(t) \, \mathrm{d}t = \delta_{k,j}.$$

Existence of such biorthogonal family $\iff$ $\displaystyle\sum_{k \geq 1} \frac{1}{\lambda_k} < +\infty.$

$\longrightarrow$ Restriction to 1D setting.

Resolution of the moment problem

$$\int_0^T e^{-\lambda_k(T-t)}\langle v(t),\varphi_k\rangle_{L^2(\omega)}\,\mathrm{d}t = -e^{-\lambda_k T}\langle y_0,\varphi_k\rangle, \qquad \forall k\geq 1.$$

- Lagnese (1983). Look for a control v in the form

$$v(t,x)=\sum_{k\geq 1}\alpha_k q_k(T-t)(\mathbf{1}_\omega\varphi_k)(x)$$

leads to the formal solution

$$v(t,x)=-\sum_{k\geq 1}e^{-\lambda_k T}\langle y_0,\varphi_k\rangle q_k(T-t)\frac{(\mathbf{1}_\omega\varphi_k)(x)}{\|\varphi_k\|^2_{L^2(\omega)}}.$$

Resolution of the moment problem

$$\int_0^T e^{-\lambda_k(T-t)} \langle v(t), \varphi_k \rangle_{L^2(\omega)} \, \mathrm{d}t = -e^{-\lambda_k T} \langle y_0, \varphi_k \rangle, \qquad \forall k \geq 1.$$

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- In the case of a boundary control

$$v(t) = \sum_{k \geq 1} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\gamma(1)\varphi_k'(1)} q_k(T-t).$$

Resolution of the moment problem

$$\int_0^T e^{-\lambda_k(T-t)}\langle v(t),\varphi_k\rangle_{L^2(\omega)}\,\mathrm{d}t = -e^{-\lambda_k T}\langle y_0,\varphi_k\rangle, \qquad \forall k\geq 1.$$

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- In the case of a boundary control

$$v(t)=\sum_{k\geq 1} e^{-\lambda_k T}\frac{\langle y_0,\varphi_k\rangle}{\gamma(1)\varphi_k'(1)}q_k(T-t).$$

- To prove it rigorously, need of spectral analysis of eigenelements of $\mathcal{A}$
  - existence and estimate of the biorthogonal family;
  - lower bound on $\|\varphi_k\|^2_{L^2(\omega)}$ and on $|\varphi_k'(1)|$.

Uniform bounds on biorthogonal family

Fattorini & Russell (1974)

Definition (A particular class of sequences)

Let $\rho > 0$ and $\mathcal{N} : (0, +\infty) \to \mathbb{N}$. Let $\Lambda = (\lambda_k)_{k\geq 1} \subset \mathbb{R}^+$ an increasing sequence satisfying $\sum_{k\geq 1} \frac{1}{\lambda_k} < +\infty$. We say that $\Lambda \in \mathcal{L}(\rho, \mathcal{N})$ if

- $\lambda_{k+1} - \lambda_k > \rho$, for any $k \geq 1$;
- we have for any $\varepsilon > 0$,

$$\sum_{k\geq \mathcal{N}(\varepsilon)} \frac{1}{\lambda_k} < \varepsilon.$$

Uniform bound on biorthogonal sequences

Let $T > 0$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \to \mathbb{N}$. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $\Lambda \in \mathcal{L}(\rho, \mathcal{N})$, there exists $(q_j)_{j\geq 1} \subset L^2(0, T; \mathbb{R})$ such that

$$\int_0^T e^{-\lambda_k t} q_j(t) \mathrm{d}t = \delta_{k,j}, \qquad \forall k, j \geq 1,$$

and

$$\|q_j\|_{L^2(0,T;\mathbb{R})} \leq C_\varepsilon e^{\varepsilon \lambda_k}, \qquad \forall j \geq 1.$$

Summary

$$v(t,x) = -\sum_{k\geq 1} e^{-\lambda_k T} \langle y_0, \varphi_k \rangle q_k(T-t) \frac{(\mathbf{1}_\omega \varphi_k)(x)}{\|\varphi_k\|^2_{L^2(\omega)}},$$

$$v(t) = \sum_{k\geq 1} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\gamma(1)\varphi_k'(1)} q_k(T-t).$$

- Gap condition on Λ + asymptotic behaviour $\Longrightarrow \|q_k\|_{L^2(0,T;\mathbb{R})} \leq C_\varepsilon e^{\varepsilon \lambda_k}$
- Lower bounds on $\|\varphi_k\|_{L^2(\omega)}$ and $|\varphi_k'(1)|$ to be compared to $e^{-\lambda_k T}$.

The Laplace operator

Assume that $\gamma = 1$ and $q = 0$ i.e. $\mathcal{A} = -\partial_{xx}$.

$$\lambda_k = k^2\pi^2 \qquad \text{and} \qquad \varphi_k(x) = \sqrt{2}\sin(k\pi x).$$

- Asymptotic behaviour: $\sum_{k\geq 1} \frac{1}{\lambda_k} < +\infty$.
- Gap property:
$$\lambda_{k+1} - \lambda_k = (k+1)^2\pi^2 - k^2\pi^2 \geq C\sqrt{\lambda_k}.$$
- Normal derivative: $|\varphi_k'(1)| = C\sqrt{\lambda_k}$.
- Localization of eigenvectors:
$$\int_a^b \varphi_k^2(x)\mathrm{d}x \underset{k\to+\infty}{\longrightarrow} b - a.$$

Eigenelements not explicitely known but same results.

Proposition

- Gap property: $\lambda_{k+1} - \lambda_k \geq Ck$, for every $k \geq 1$.
- Normal derivative: $|\varphi_k'(1)| \geq Ck$, for every $k \geq 1$.
- Localization of eigenvectors: there exists $C(\omega) > 0$ such that

$$\|\varphi_k\|_{L^2(\omega)} \geq C(\omega).$$

A useful change of variables

$$\mathcal{A}u(x) = \lambda u(x) + f(x).$$

Let

$$U(x) := \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}} u'(x) \end{pmatrix}.$$

Then,

$$U'(x) = \underbrace{\begin{pmatrix} 0 & \sqrt{\frac{\lambda}{\gamma(x)}} \\ -\sqrt{\frac{\lambda}{\gamma(x)}} & 0 \end{pmatrix}}_{\text{evolution operator } M(x)} U(x) + \underbrace{\begin{pmatrix} 0 & 0 \\ \frac{q(x)}{\sqrt{\lambda\gamma(x)}} & \sqrt{\gamma(x)} \left(\frac{1}{\sqrt{\gamma}}\right)'(x) \end{pmatrix}}_{\text{remainder } Q(x)} U(x) + \underbrace{\begin{pmatrix} 0 \\ -\frac{f(x)}{\sqrt{\gamma(x)\lambda}} \end{pmatrix}}_{F(x)}$$

- The resolvant operator associated with M is $S(y,x) = \exp\left(\int_x^y M(s)\mathrm{d}s\right)$ and satisfies $\|S(y,x)\| = 1$.
- The remainder contains bounded terms in λ.

$$\|U(y)\| \leq C \left(\|U(x)\| + \left| \int_x^y \|F(s)\| \mathrm{d}s \right| \right).$$

Let $u := \varphi_k$.

$$U(x) := \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}}u'(x) \end{pmatrix}, \qquad \|U(y)\| \leq C\|U(x)\|.$$

Then,

$$|\varphi_k(y)|^2 \leq \|U(y)\|^2 \leq C\|U(x)\|^2 = C\left(|\varphi_k(x)|^2 + \frac{\gamma(x)}{\lambda_k}|\varphi_k'(x)|^2\right).$$

Integrating for $y \in (0,1)$ and using $\|\varphi_k\|_{L^2(0,1)} = 1$ it comes that

$$|\varphi_k(x)|^2 + \frac{\gamma(x)}{\lambda_k}|\varphi_k'(x)|^2 \geq C, \qquad \forall x \in [0,1].$$

Normal derivative and localization

Let $u := \varphi_k$.

$$U(x) := \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}} u'(x) \end{pmatrix}, \qquad \|U(y)\| \leq C\|U(x)\|.$$

$$|\varphi_k(x)|^2 + \frac{\gamma(x)}{\lambda_k}|\varphi_k'(x)|^2 \geq C, \qquad \forall x \in [0,1]. \tag{$*$}$$

- Normal derivative. Taking $x = 1$ in $(*)$ implies

$$|\varphi_k'(1)| \geq C\sqrt{\lambda_k}, \qquad \forall k \geq 1.$$

- Localization. Caccioppoli-like inequality: for $\omega_0 \subset\subset \omega$,

$$\int_{\omega_0} \gamma(x)|\varphi_k'(x)|^2 \mathrm{d}x \leq C\lambda_k\|\varphi_k\|_\omega^2 + \frac{C}{\lambda_k}\|\varphi_k'\|_\omega^2,$$

  Integrating $(*)$ for $x \in \omega_0$ leads to

$$\|\varphi_k\|_\omega \geq C, \qquad \forall k \geq 1.$$

The gap property

$$U(x) := \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}} u'(x) \end{pmatrix}, \qquad \|U(y)\| \leq C \left(\|U(x)\| + \left| \int_x^y \|F(s)\| \mathrm{d}s \right| \right).$$

- Let $u(x) := \varphi_k'(1)\varphi_{k+1}(x) - \varphi_{k+1}'(1)\varphi_k(x)$. Thus $U(1) = 0$.
- We have $\mathcal{A}u = \lambda_{k+1}u + \varphi_{k+1}'(1)\left(\lambda_{k+1} - \lambda_k\right)\varphi_k$. We have for any $y \in [0,1]$

$$\begin{aligned} \|U(y)\| &\leq C \int_y^1 \|F(s)\| \mathrm{d}s \leq C \int_0^1 \|F(s)\| \mathrm{d}s \\ &\leq \frac{C}{\sqrt{\gamma_{min}}} \left(\frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}}} |\varphi_{k+1}'(1)| \right) \int_0^1 |\varphi_k(s)| \, \mathrm{d}s \end{aligned}$$

The gap property

$$U(x) := \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}} u'(x) \end{pmatrix}, \qquad \|U(y)\| \leq C \left(\|U(x)\| + \left| \int_x^y \|F(s)\| \mathrm{d}s \right| \right).$$

- Let $u(x) := \varphi_k'(1)\varphi_{k+1}(x) - \varphi_{k+1}'(1)\varphi_k(x)$. Thus $U(1) = 0$.
- We have $\mathcal{A}u = \lambda_{k+1} u + \varphi_{k+1}'(1)\left(\lambda_{k+1} - \lambda_k\right)\varphi_k$. We have for any $y \in [0,1]$

$$\|U(y)\| \leq \frac{C}{\sqrt{\gamma_{min}}} \left(\frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}}} |\varphi_{k+1}'(1)| \right) \int_0^1 |\varphi_k(s)| \, \mathrm{d}s$$

Using $\|\varphi_k\|_{L^2(0,1)} = 1$ and the expression of U,

$$\left|\varphi_k'(1)\varphi_{k+1}(y) - \varphi_{k+1}'(1)\varphi_k(y)\right|^2 \leq \frac{C}{\gamma_{min}} \left(\frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}}} |\varphi_{k+1}'(1)| \right)^2.$$

Integrating for $y \in (0,1)$ and using $\int_0^1 \varphi_k(y)\varphi_{k+1}(y)\mathrm{d}y = 0$ we get

$$|\varphi_{k+1}'(1)|^2 \leq |\varphi_{k+1}'(1)|^2 + |\varphi_k'(1)|^2 \leq \frac{C}{\gamma_{min}} \left(\frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}}} |\varphi_{k+1}'(1)| \right)^2,$$

Thus, $\lambda_{k+1} - \lambda_k \geq C\sqrt{\lambda_{k+1}}$.

1 The continuous problem

2 Spectral analysis of the discrete problem
- The discrete setting
- A rough estimate
- A refined estimate

3 Application to controllability

Form of the controls

$$h = \frac{1}{N+1}, \qquad \mathcal{A}^h \varphi_k^h = \lambda_k^h \varphi_k^h, \quad \forall k \in [\![1, N]\!], \qquad \partial_r \varphi_k^h := \frac{0 - \varphi_{k,N}^h}{h}.$$

Distributed control

$$v^h(t) = -\sum_{k=1}^{N} e^{-\lambda_k^h T} \langle y^{0,h}, \varphi_k^h \rangle \, q_k^h(T-t) \, \frac{(\mathbf{1}_\omega \varphi_k^h)}{\|\mathbf{1}_\omega \varphi_k^h\|_h^2} \in L^2(0,T;\mathbb{R}^N).$$

Boundary control

$$v^h(t) = \sum_{k=1}^{N} e^{-\lambda_k^h T} \, \frac{\langle y^{0,h}, \varphi_k^h \rangle}{\gamma_{N+1/2} \partial_r \varphi_k^h} \, q_k^h(T-t) \in L^2(0,T;\mathbb{R}).$$

No problem of convergence of sums BUT we aim to design bounded controls !

- Class $\mathcal{L}(\rho, \mathcal{N})$ ensures uniform bounds for q_k. Uses a uniform gap

$$\lambda_{k+1}^h - \lambda_k^h \geq C, \quad \forall h > 0, \forall k \in [\![1, N]\!].$$

- Estimate (not necessarily uniformly) the spectral quantities

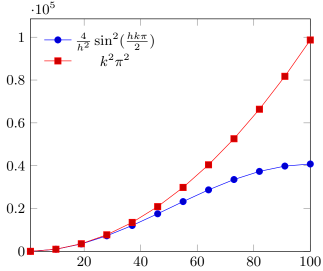
$$\|\mathbf{1}_\omega \varphi_k^h\|_h^2, \qquad \partial_r \varphi_k^h.$$

Comparison with the continuous setting

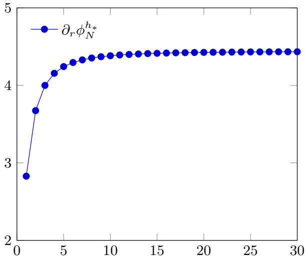
- Lopez & Zuazua (1998). The Laplace operator ($\gamma = 1$ and $q = 0$):

$$\mu_k^h = \frac{4}{h^2}\sin^2\left(\frac{hk\pi}{2}\right), \qquad \phi_k^h = \left(\sqrt{2}\sin(k\pi jh)\right)_{j\in[\![1,N]\!]}.$$

Uniform lower bounds on the considered spectral quantities.



(a) Discrete and continuous eigenvalues.



(b) $N \mapsto \partial_r\phi_N^h$.

- Different behaviour from the continuous setting (linear gap, growth of the normal derivative). More similar results to be expected for low frequencies

Discrete spectral analysis

Theorem [Allonsius, Boyer, Morancey (2018)]

There exists $C > 0$ such that

$$\frac{1}{\sqrt{\lambda_k^h}} \left|\partial_r \varphi_k^h\right| \geq C e^{-C\sqrt{\lambda_k^h}}, \quad \forall k \in [\![1, N]\!],$$

and

$$\|\mathbf{1}_\omega \varphi_k^h\|_h^2 \geq C e^{-C\sqrt{\lambda_k^h}}, \quad \forall k \in [\![1, N]\!],$$

for any $h > 0$ sufficiently small.
Moreover, if $\gamma \in C^3([0,1];\mathbb{R})$ and $q \in C^2([0,1];\mathbb{R})$, there exists $C, \alpha > 0$ such that

$$\lambda_{k+1}^h - \lambda_k^h \geq Ck, \quad \forall k \in [\![1, \alpha N^{2/5}]\!].$$

- Lower bounds for eigenelements on the whole spectrum. Suitable for control

$$v^h(t) = -\sum_{k=1}^{N} e^{-\lambda_k^h T} \langle y^{0,h}, \varphi_k^h \rangle \, q_k^h(T-t) \, \frac{(\mathbf{1}_\omega \varphi_k^h)}{\|\mathbf{1}_\omega \varphi_k^h\|_h^2} \in L^2(0,T;\mathbb{R}^N).$$

- Uniform gap-property but only for low frequencies.
- Extension to non-uniform meshes with *a-priori* bound on $\dfrac{\max_{i \in [\![0,N]\!]} h_{i+1/2}}{\min_{i \in [\![0,N]\!]} h_{i+1/2}}$.

Back to the change of variables

$$\mathcal{A}^h u^h = \lambda u^h + f^h.$$

Let

$$U_j^h := \begin{pmatrix} u_j^h \\ \sqrt{\frac{\gamma_{j-1/2}}{\lambda}} \frac{u_j^h - u_{j-1}^h}{h} \end{pmatrix}.$$

Then,

$$U_{j+1}^h = \left(\mathrm{I} + hM_j^h\right) U_j^h + hQ_j^h U_j^h + hF_j^h,$$

where

$$M_j^h := \begin{pmatrix} -h\frac{\lambda}{\gamma_{j+1/2}} & \sqrt{\frac{\lambda}{\gamma_{j+1/2}}} \\ -\sqrt{\frac{\lambda}{\gamma_{j+1/2}}} & 0 \end{pmatrix}, \text{ and } F_j^h := \begin{pmatrix} -h\frac{f_j^h}{\gamma_{j+1/2}} \\ -\frac{f_j^h}{\sqrt{\gamma_{j+1/2}\lambda}} \end{pmatrix},$$

and

$$Q_j^h := \begin{pmatrix} h\frac{q_j}{\gamma_{j+1/2}} & h\sqrt{\lambda}\sqrt{\frac{\gamma_{j-1/2}}{\gamma_{j+1/2}}} \frac{\frac{1}{\sqrt{\gamma_{j+1/2}}} - \frac{1}{\sqrt{\gamma_{j-1/2}}}}{h} \\ \frac{q_j}{\sqrt{\lambda\gamma_{j+1/2}}} & \sqrt{\gamma_{j-1/2}} \frac{\frac{1}{\sqrt{\gamma_{j+1/2}}} - \frac{1}{\sqrt{\gamma_{j-1/2}}}}{h} \end{pmatrix}.$$

A rough estimate

$$\mathcal{A}^h \varphi_k^h = \lambda_k^h \varphi_k^h.$$

$$\Phi_{k,j}^h := \begin{pmatrix} \varphi_{k,j}^h \\ \sqrt{\frac{\gamma_{j-1/2}}{\lambda}} \frac{\varphi_{k,j}^h - \varphi_{k,j-1}^h}{h} \end{pmatrix}, \qquad \Phi_{k,j+1}^h = \left(\mathrm{I} + hM_j^h + hQ_j^h\right) \Phi_{k,j}^h.$$

- Min-max principle and $\mu_k^h = \frac{4}{h^2}\sin^2(\frac{hk\pi}{2})$: $h\sqrt{\lambda_k^h}$ is bounded.
- $\|Q_j^h\| \leq C$ and $\|M_j^h\| \leq C\sqrt{\lambda_k^h}$.
- Then,

$$\left\| \left(\mathrm{I} + hM_j^h + hQ_j^h\right)^{\pm 1} \right\| \leq \exp\left(Ch\sqrt{\lambda_k^h}\right).$$

  This gives the lower bound on $|\partial_r \varphi_k^h|$ and $\|\mathbf{1}_\omega \varphi_k^h\|_h^2$ but cannot lead to a uniform gap-property...
- Discrete gap-property: uses the continuous gap property and the estimate

$$|\lambda_k^h - \lambda_k| \leq Ch^2\lambda_k^3.$$

  Uses extra-regularity of γ and q.

A uniform bound for a part of the spectrum

Let

$$k^h_{max,\varepsilon} := \max\left\{ k \in [\![1,N]\!] \; ; \; \lambda^h_k < \frac{4}{h^2}\gamma_{min}(1-\varepsilon) \right\} \geq \alpha N \sqrt{1-\varepsilon}.$$

Theorem [Allonsius, Boyer, Morancey (2018)]

There exists $C > 0$ such that

$$\frac{1}{\sqrt{\lambda^h_k}} \left| \partial_r \varphi^h_k \right| \geq C, \quad \forall k \in [\![1, k^h_{max,\varepsilon}]\!],$$

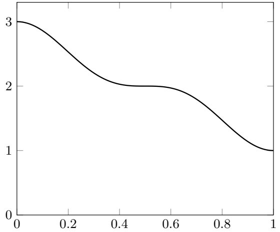
$$\|\mathbf{1}_\omega \varphi^h_k\|^2_h \geq C, \quad \forall k \in [\![1, k^h_{max,\varepsilon}]\!],$$

and

$$\lambda^h_{k+1} - \lambda^h_k \geq C\sqrt{\lambda^h_{k+1}}, \quad \forall k \in [\![1, k^h_{max,\varepsilon} - 1]\!],$$

for any $h > 0$ sufficiently small.

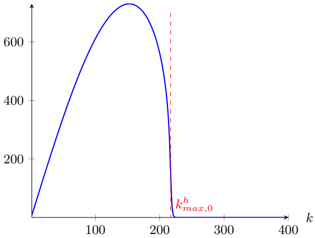
- Uniform bound similar to the continuous case only for a portion of the spectrum (numerically optimal). Hold for the whole spectrum if γ is constant.



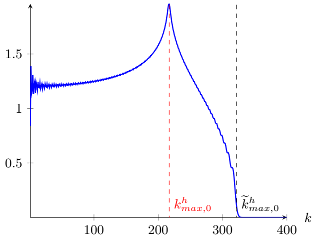
(c) The diffusion coefficient γ

$\widetilde{k}^h_{max,0}$: same definition as $k^h_{max,0}$ with

$$\gamma_{min} \leftarrow \inf_{x\in[0,1]\setminus\omega} \gamma(x).$$

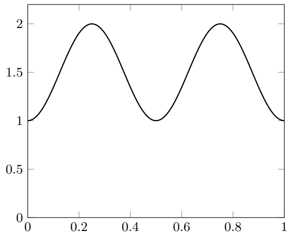


(d) Normal derivative

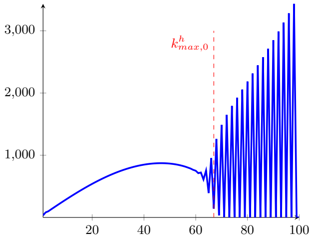


(e) $L^2(\omega)$ norm

On the optimality of the uniform bound: the gap property



(f) The diffusion coefficient γ



(g) $k \mapsto |\lambda^h_{k+1} - \lambda^h_k|$

Comparison between continuous and discrete setting

Continuous setting.

$$\mathcal{A}u(x) = \lambda u(x) + f(x)$$

rewritten as

$$U'(x) = M(x)U(x) + Q(x)U(x) + F(x).$$

Evolution operator

$$M(x) = \begin{pmatrix} 0 & \sqrt{\frac{\lambda}{\gamma(x)}} \\ -\sqrt{\frac{\lambda}{\gamma(x)}} & 0 \end{pmatrix}.$$

Associated resolvant operator

$$S(y,x) = \exp\left(\int_x^y M(s)\mathrm{d}s\right)$$

satisfying $\|S(y,x)\| = 1$.

Comparison between continuous and discrete setting

Continuous setting.

$$\mathcal{A}u(x) = \lambda u(x) + f(x)$$

rewritten as

$$U'(x) = M(x)U(x) + Q(x)U(x) + F(x).$$

Evolution operator

$$M(x) = \begin{pmatrix} 0 & \sqrt{\frac{\lambda}{\gamma(x)}} \\ -\sqrt{\frac{\lambda}{\gamma(x)}} & 0 \end{pmatrix}.$$

Associated resolvant operator

$$S(y,x) = \exp\left(\int_x^y M(s)\mathrm{d}s\right)$$

satisfying $\|S(y,x)\| = 1$.

Discrete setting.

$$\mathcal{A}^h u^h = \lambda u^h + f^h$$

rewritten as

$$U_{j+1}^h = \left(\mathrm{I} + hM_j^h\right) U_j^h + hQ_j^h U_j^h + hF_j^h.$$

$S(x_j, x_{j+1})$ replaced by $(\mathrm{I} + hM_j^h)$ with

$$M_j^h := \begin{pmatrix} -h\frac{\lambda}{\gamma_{j+1/2}} & \sqrt{\frac{\lambda}{\gamma_{j+1/2}}} \\ -\sqrt{\frac{\lambda}{\gamma_{j+1/2}}} & 0 \end{pmatrix}.$$

Discrete resolvant $S_{i\leftarrow j,k}^h$ defined by

$$\begin{cases} \left(\mathrm{I} + hM_{i-1,k}^h\right)\cdots\left(\mathrm{I} + hM_{j,k}^h\right) & \text{for } i > j, \\ \mathrm{I} & \text{for } i = j, \\ (S_{j\leftarrow i,k}^h)^{-1} & \text{for } i < j. \end{cases}$$

Goal: estimate the norm of $S_{i\leftarrow j,k}^h$.

Uniform estimate of the discrete resolvant operator

Recall that $k^h_{max,\varepsilon} := \max \left\{ k \in [\![1, N]\!] \; ; \; \lambda^h_k < \frac{4}{h^2} \gamma_{min}(1-\varepsilon) \right\}$.

$$\|S^h_{i \leftarrow j,k}\| \leq C \exp\left(\frac{C}{\varepsilon}\right), \qquad \forall i,j \in [\![1, N+1]\!], \quad \forall k \in [\![1, k^h_{max,\varepsilon}]\!].$$

- $S^h_{i \leftarrow j,k} U^h_j = U^h_i$. Let $U^h_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$.
- Define the hamiltonian

$$H_i := x_i^2 + y_i^2 - \frac{h\sqrt{\lambda}}{\sqrt{\gamma_{i+1/2}}} x_i y_i.$$

  - equivalent to the norm of U_i if $i \in [\![1, k^h_{max,\varepsilon}]\!]$:

$$\frac{\varepsilon}{2}\left(x_i^2 + y_i^2\right) \leq H_i \leq C\left(x_i^2 + y_i^2\right),$$

  - conserved ($H_{i+1} = H_i$) if γ is constant. Otherwise

$$|H_{i+1}| \leq |H_i| + Ch\left(x_i^2 + y_i^2\right) \leq \exp\left(h\frac{C}{\varepsilon}\right)|H_i|.$$

1 The continuous problem

2 Spectral analysis of the discrete problem

3 Application to controllability

A uniform null controllability result

- Uniform null controllability of

$$\begin{cases} (y^h)'(t) + \mathcal{A}^h y^h(t) = \mathbf{1}_\omega v^h(t), \\ y_0^h(t) = y_{N+1}^h(t) = 0, \\ y^h(0) = y^{0,h}, \end{cases} \qquad \begin{cases} (y^h)'(t) + \mathcal{A}^h y^h(t) = 0, \\ y_0^h(t) = 0,\ y_{N+1}^h(t) = v^h(t), \\ y^h(0) = y^{0,h}, \end{cases}$$

if γ is constant (and the mesh is uniform).

$$v^h(t) = -\sum_{k=1}^{N} e^{-\lambda_k^h T} \langle y^{0,h}, \varphi_k^h \rangle \, q_k^h(T-t) \, \frac{(\mathbf{1}_\omega \varphi_k^h)}{\|\mathbf{1}_\omega \varphi_k^h\|_h^2} \in L^2(0,T;\mathbb{R}^N).$$

$$v^h(t) = \sum_{k=1}^{N} e^{-\lambda_k^h T} \frac{\langle y^{0,h}, \varphi_k^h \rangle}{\gamma_{N+1/2} \partial_r \varphi_k^h} \, q_k^h(T-t) \in L^2(0,T;\mathbb{R}).$$

- Application of uniform lower bounds on the whole spectrum and uniform bounds on the biorthogonal family associated with

$$\tilde{\Lambda}^h := \begin{cases} \lambda_k^h \text{ for } k \in [\![1, N]\!], \\ \lambda_N^h + 4\gamma k^2 \text{ for } k \geq N+1. \end{cases}$$

A weaker controllability notion

Definition of $\phi(h)$-null controlability

Let $\phi : (0, +\infty) \to (0, +\infty)$ be a function such that $\lim_{h \to 0} \phi(h) = 0$.
We say that we have uniform $\phi(h)$-null controllability if there exists $C > 0$, such that, for any h small enough and any $y^{0,h} \in \mathbb{R}^N$, we can find a control $v^h \in L^2(0, T; \mathbb{R}^N)$ (resp. $v^h \in L^2(0, T; \mathbb{R})$) that satisfies

$$\left(\int_0^T \|v_h\|_h^2 \mathrm{d}t \right)^{1/2} \leq C \|y^{0,h}\|_h, \qquad \left(\text{resp. } \|v^h\|_{L^2(0,T;\mathbb{R})} \leq C \|y^{0,h}\|_h \right),$$

and such that the associated solution y^h satisfies

$$\|y^h(T)\|_h^2 \leq C\phi(h) \|y^{0,h}\|_h^2.$$

- Different weakening of the uniform null controllability property than the filtering process.

Uniform $\phi(h)$-null controlability results

- Uniform mesh. We have uniform $\phi(h)$-null controllability for

$$\phi(h) \underset{h\to 0}{\sim} C_1 \exp\left(-\frac{C_2 T}{h^2}\right).$$

- Quasi-uniform mesh. We have uniform $\phi(h)$-null controllability for

$$\phi(h) \underset{h\to 0}{\sim} C_1 \exp\left(-\frac{C_2 T}{h^{2/5}}\right).$$

Use the (possibly non-uniform) lower bounds on eigenelements and the uniform bound on biorthogonal families associated with

$$\tilde{\Lambda}^h_\varepsilon := \begin{cases} \lambda^h_k \text{ for } k \in [\![1, k^h_{max,\varepsilon}]\!], \\ \lambda^h_{k^h_{max,\varepsilon}} + 4\gamma_{min} k^2 \text{ for } k \geq k^h_{max,\varepsilon} + 1. \end{cases}$$

or

$$\tilde{\Lambda}^h := \begin{cases} \lambda^h_k \text{ for } k \in [\![1, \alpha N^{2/5} - 1]\!], \\ \lambda^h_k + 4\gamma_{min} k^2 \text{ for } k \geq \alpha N^{2/5}. \end{cases}$$

to set to 0 the considered frequencies and then use dissipation.

Direct extension to systems of coupled equations in cascade form

$$\begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} \mathbf{1}_\omega v^h(t) \\ 0 \end{pmatrix}, \\ y^h(0) = y^{h,0} \in (\mathbb{R}^N)^2, \\ y_0^h(t) = 0, \end{cases}$$

or

$$\begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = 0, \\ y^h(0) = y^{h,0} \in (\mathbb{R}^N)^2, \\ y_0^h(t) = 0,\ y_{N+1}^h(t) = \begin{pmatrix} v^h(t) \\ 0 \end{pmatrix}. \end{cases}$$

Conclusion and perspectives

Conclusion:

- Optimal spectral analysis of discretized 1D elliptic operators
- $\phi(h)$-null controllability results for 1D semi-discretized parabolic problems including systems of coupled equations with distributed or boundary control.

Perspectives:

- More general systems. Not in cascade form ? With a minimal time on the continuous problem ?
- Higher dimension.

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Thank you for your attention