Fluid-Structure models arising in blood-flow models

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Outline

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4. Future direction of work

Outline

Problem Description

Motivation : Blood flow in large arteries. Viscous fluid interacts with a thin elastic structure located on one part of the fluid domain.

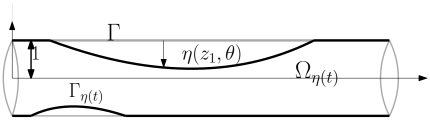

Figure: Reference and deformed configuration of the domain

- The fluid domain depends on the structure displacement. We have a free boundary value problem.

Problem Description

- The reference configuration:

$$\Omega = \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1 \in (0, L), \sqrt{z_2^2 + z_3^2} \leqslant 1 \right\}$$

Γ_s is the lateral boundary, which is deformable. Γ_{in} and Γ_{0ut} are inflow and outflow boundaries respectively.

- Current configuration : Let $\overrightarrow{d}(t, \cdot)$ displacement of the shell from the reference configuration Γ_s. Displacement is only in the radial direction. Thus $\overrightarrow{d}(t, z_1, \theta) = \eta(t, z_1, \theta) e_r(\theta)$.

$$\Omega_{\eta(t)} = \left\{ (z_1, x, y) \in \mathbb{R}^3 \mid z_1 \in (0, L), \sqrt{x^2 + y^2} \leqslant 1 + \eta(t, \cdot) \right\},$$

$$\Gamma_{\eta(t)} == \left\{ (z_1, x, y) \in \mathbb{R}^3 \mid z_1 \in (0, L), \sqrt{x^2 + y^2} = 1 + \eta(t, \cdot) \right\}.$$

Governing equations

- Fluid equation : The fluid is Newtonian, viscous and incompressible. The fluid velocity u and pressure p satisfy

$$\rho_f(\partial_t u + u\cdot\nabla u) - \mathrm{div}\sigma(u,p) = 0, \qquad \mathrm{div}\ u = 0 \quad \text{in } (0,T)\times\Omega_{\eta(t)},$$

where $\sigma(u,p) = (\nabla u + \nabla u)^\top - pI$.

- Boundary conditions :

$$\sigma(u,p)n = 0 \text{ on } \Gamma_{in}\cup\Gamma_{out}.$$

- Structure equation : η satisfies viscoelastic cylindrical nonlinear Koiter shell equation :

$$\partial_{tt}\eta + \mathcal{L}_{mem}\eta + \Delta_s^2\eta - \beta_2\Delta_s\partial_t\eta = \mathcal{H}(u,p,\eta) \text{ on } \Gamma_s,$$

- $$\eta = \frac{\partial\eta}{\partial n} = 0 \text{ on } \partial\Gamma_{in}\cup\partial\Gamma_{out}.$$

Interface conditions

Coupling between the fluid and the structure is expressed through the kinematic and dynamic lateral boundary conditions:

- Continuity of the velocity (the no-slip condition) at the interface Γ_η

$$u = \partial_t \eta e_r \text{ on } \Gamma_\eta$$

- Balance of the contact forces at the interface

$$\mathcal{H}(u, p, \eta) = -J(\sigma(u, p)\widetilde{n})|_{\Gamma_\eta} \cdot e_r,$$

  $\widetilde{n}$ is the unit normal to Γ_η.

Goal : To study existence and uniqueness of strong solutions in L^2 framework.

State of the art

- Strong Solution and $2D/1D$ model :
  - Structure equation : $\partial_{tt}\eta + \alpha\partial_{xxxx}\eta - \beta\partial_{xx}\eta - \gamma\partial_{txx}\eta = \mathcal{H}$.
  - fluid boundary conditions at the inlet and outlet.

- Local in time existence : Lequeurre (11 and 13), Casanova (18), Grandmont, Hilariet and Lequeurre (18), Badra and Takahashi (19), Djebour and Takahashi (19),...
  - $\gamma = 0$ and periodic boundary condition at the inlet/outlet.
  - $\gamma > 0$, Dirichlet / pressure boundary conditions.
- Global in time existence : Grandmont and Hilariet (2016) , $\gamma > 0$, $\alpha > 0$ and periodic boundary conditions.

Our result : Local in time existence with $\gamma > 0$, $\alpha > 0$ and Neumann boundary condition at the inlet/outlet.

Outline

Monolithic approach:

- Rewrite the system in the fixed domain : Lagrangian or Geometric change of variables.
- System in fixed domain

$$z'(t) = \mathcal{A}_{FS} z(t) + \mathcal{N}(z), \quad z(0) = z_0.$$

- Linearized FSI system in a suitable space $\mathcal{X}$:

$$z'(t) = \mathcal{A}_{FS} z(t) + f(t), \quad z(0) = z_0.$$

- Regularity of linear system.
- Fixed point argument. (local in time or global in time for small initial data)

Linearized problem in 2D/1D setting

$\Omega = (0, L) \times (0, 1)$, $\Gamma_s = (0, L) \times \{1\}$, $\Gamma_{in} = \{0\} \times (0, 1)$ and $\Gamma_{out} = \{L\} \times (0, 1)$

$$
\begin{cases}
\partial_t u - \Delta u + \nabla p = f, \mathrm{div} u = 0, & \text{in } \Omega, \\
u = \partial_t \eta e_2 & \text{on } \Gamma_s, \\
\sigma(u, p) n = 0 & \text{on } \Gamma_{in} \cup \Gamma_{out}, \\
\partial_{tt} \eta + \partial_{xxxx} \eta - \partial_{txx} \eta = p|_{\Gamma_s} + h & \text{in } \Gamma_s
\end{cases}
$$

- The linear fluid-structure operator generates an analytic semigroup.
- The fluid operator (with homogeneous BC) and the structure operator generates analytic semigroup.
- The coupling can be seen as compact perturbation.

- Remove pressure from the fluid and structure equation.
- Use Leray projector to remove the pressure from fluid equation. $\partial_t \mathcal{P} u = A_F \mathcal{P} u + B \partial_t \eta$.
- The pressure can be written as

$$\Delta p = 0, \quad \frac{\partial p}{\partial n} = -\partial_{tt}\eta + \Delta u \cdot n \text{ on } \Gamma_s, \quad p = \varepsilon(u) n \cdot n \text{ on } \Gamma_{in/out}.$$

  Thus $p = N_0(\partial_{tt}\eta) + N_1(u)$.
- The structure equation becomes:

$$(I + \gamma_s N_0)\partial_{tt}\eta - A_s = \gamma_s N_1(u).$$

- The operator $(I + \gamma_s N_0)$ is known as "added mass" operator and is invertible in $L^2(\Gamma_s)$.

The fluid-structure operator

The system can be written as

$$\frac{d}{dt}\begin{pmatrix}\mathcal{P}u\\ \eta_1\\ \eta_2\end{pmatrix} = \mathcal{A}_{FS}\begin{pmatrix}\mathcal{P}u\\ \eta_1\\ \eta_2\end{pmatrix} + \text{ source term }.$$

$$\mathcal{A}_{FS} = \begin{pmatrix} I & & \\ & I & \\ & & (I+\gamma_s N_0)^{-1}\end{pmatrix}\begin{pmatrix} A_F & 0 & B\\ 0 & 0 & I\\ N_1(u) & -\Delta^2 & \Delta\end{pmatrix}$$

- $\mathcal{X} = L^2_\sigma(\Omega)\times H^2(\Gamma_s)\times L^2(\Gamma_s)$
- $\mathcal{D}(\mathcal{A}_{FS}) \sim H^{3/2+\epsilon_0}\times H^4(\Gamma_s)\times H^2(\Gamma_s)$
- Loss of regularity for fluid due to mixed boundary condition and the angle of Dirichlet-Neumann junction is $\pi/2$.
- Study the weak form of $N_1(u)$ and to show it is a compact operator.
- $(u,\eta_1,\eta_2)\in L^2(0,T;\mathcal{D}(\mathcal{A}_{FS}))\cap H^1(0,T;\mathcal{X})$.

Outline

Main result

Theorem (DM, J.-P. Raymond, A. Roy)

Let $\eta(0) = 0$, $(u_0, \partial_t \eta(0)) \in H^1(\Omega) \times H^1(\omega)$ with compatibility conditions. Then there exists a $T > 0$, depending only on the initial data such that the system admits a strong solution

$$u \in L^2(0, T; H^{3/2+\varepsilon_0}(\Omega_{\eta(\cdot)})) \cap H^1(0, T; L^2(\Omega_{\eta(\cdot)})) \cap C([0, T]; H^1(\Omega_{\eta(\cdot)})),$$
$$p \in L^2(0, T; H^{1/2+\varepsilon_0}(\Omega_{\eta(\cdot)})), \quad \mathrm{div}\ \sigma(u, p) \in L^2(\widetilde{Q}_T),$$
$$\eta \in L^2(0, T; H^4(\omega)) \cap H^2(0, T; L^2(\omega)),$$
$$1 + \eta(t, \cdot) > 0, \quad , t \in [0, T]$$

for some $\varepsilon_0 \in (0, 1/2)$.

$L^p - L^q$ regularity

- We look for solutions of fluid and structure in $L^p(0, T; L^q)$.
- The idea is the same : $L^p - L^q$ regularity of fluid and structure with compactness of the fluid-structure coupling.
- $L^p - L^q$ regularity is no longer characterised by analyticity of the linear semigroup. We need to show $\mathcal{R}$-sectoriality of the resolvent operator.

Theorem (DM, T. Takahashi)

The reference domain is smooth. Let us assume that $\frac{1}{p} + \frac{n}{2q} < \frac{3}{2}$. For suitable initial data with compatibility conditions, we have local in time existence of strong solutions :

$$u \in L^p(0, T; W^{2,q}) \cap W^{1,p}(0, T; L^q)$$

$$\eta \in L^p(0, T; W^{4,q}) \cap W^{2,p}(0, T; L^q).$$

Outline

Future direction of work

- In the 3D case, can we remove the viscosity of the structure.
- Wave or damped wave.
- Global existence in 2D, without the damping term.
- Other fluid models : Compressible Navier-Stokes-Fourier.

Thank you very much.